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Isobaric flows of an ideal fluid

Z. PERADZYŃSKI (WARSZAWA)

THE EQUATIONS of isobaric flows (i.e., such that p = const) are investigated. We prove that the solutions are global in time, are of rank 2 (as the mappings $\mathsf{R}^4 \to \mathsf{R}^3$) and that their gradients are increasing, at the most, linearly with time along the trajectories of particles.

W pracy rozważane są równania przepływów izobarycznych płynu doskonałego. Pokazano, że ich rozwiązania traktowane jako odwzorowania z R^4 do R^3 mają dwuwymiarowe obrazy, istnieją globalnie w czasie, a ich gradienty narastają (wzdłuż trajektorii cząstek) co najwyżej liniowo w czasie.

В работе рассматриваются уравнения изобарических течений идеальной жидкости. Показано, что их решения, трактованные как отображения из R^4 в R^3, имеют двумерные образы, существуют глобальным образом во времени, а их градиенты нарастают (вдоль траектории частиц) по крайней мере линейно во времени.

1. Statement of the problem

ONE OF THE FUNDAMENTAL problems of hydrodynamics concerns the question of global existence, in time, of solutions of the Euler equations

$$\frac{\partial v}{\partial t}+(v\cdot\nabla)v+\nabla p=0, \qquad \text{div}v=0, \qquad n\cdot v|_{\partial\Omega}=0. \tag{1.1}$$

This problem is investigated here for particular classes of solutions of Eqs. (1.1). For example, the constraint rot $v=0$, compatible with Eqs. (1.1) is preserved during the evolution, therefore the search for potential solutions (i.e., such that $v=\nabla\phi$) reduces the problem (1.1) to the linear problem

$$\Delta\phi=0 \tag{1.2}$$

and the nonlinear relation determining the pressure

$$\phi_t+\frac{1}{2}(\nabla\phi)^2+p=0. \tag{1.3}$$

Obviously, in the case of potential flows the global existence can be easily established. Therefore, when one is interested in finding counterexamples of global existence of solutions of Eqs. (1.1), one should look for solutions with nonzero vorticity.

Isobaric flows form one of the simplest classes of flows with nonvanishing (in general) vorticity, what makes them interesting for us. As shown below, the general solution of the linear counterpart of Eqs. (1.1)

$$\begin{aligned}(1.4)\qquad &\frac{\partial v}{\partial t}+(v_0\cdot\nabla)v+\nabla p=0,\\ &\operatorname{div} v=0, \qquad\qquad v_0=\text{const},\end{aligned}$$

is composed of two components $v = u+w$, where u is rotation-free and w is the rotational isobaric solution(1). Thus we have

$$\begin{aligned}(1.5)\qquad &\operatorname{rot} u=0, \quad \operatorname{div} u=0,\\ &\frac{\partial u}{\partial t}+\nabla[v_0\cdot u+p]=0\end{aligned}$$

and

$$\begin{aligned}(1.6)\qquad &\frac{\partial w}{\partial t}+(v_0\cdot\nabla)w=0,\\ &\operatorname{div} w=0, \qquad\qquad p=\text{const}.\end{aligned}$$

This splitting is related to the fact that the Fourier transform of Eqs. (1.4)

$$(1.7)\qquad \begin{cases}(\xi_0+v_0\cdot\bar{\xi})\hat{v}+\bar{\xi}\hat{p}=0,\\ \bar{\xi}\cdot v=0, \quad \bar{\xi}=(\xi_1,\xi_2,\xi_3)\end{cases}$$

has the characteristic determinant

$$(1.8)\qquad W(\xi)=(\xi_0+v_0\cdot\bar{\xi})^2|\bar{\xi}|^2$$

in the form of a product of two polynomials. The irrotational solutions defined by Eqs. (1.5) are related to the "elliptic branch" $|\xi|^2$ of $W(\xi)$. The linear branch $\xi_0+v_0\cdot\bar{\xi}=0$ of multiplicity 2 leads to $\bar{\xi}\hat{p}=0$ and thus $\hat{p}=p_0\,\delta(\xi)$, which indeed implies that the pressure is constant: $p(x)=p_0$.

As far as v_0 is constant, both systems, i.e. Eqs. (1.5) and (1.6) are compatible. That is, for given initial conditions u_0 (or w_0) satisfying the constraints rot $u_0=0$, div$u_0=0$ in case of Eqs. (1.5), or div $w_0=0$ in case of Eqs. (1.6), there exists a unique solution of the Cauchy problem.

Equations (1.5) and (1.6) are examples of systems generated by components (branches) of the characteristic polynomial (see [1]). In the quasi-linear case such systems are connected with the appropriate decomposition of the tangent mapping of the solution. This decomposition has a bearing on the decomposition of the space of values of the solutions into eigenspaces related to the branches of $W(\xi)$.

In this way in the nonlinear case one obtains

$$\begin{aligned}(1.9)\qquad &\frac{\partial v}{\partial t}+(v\cdot\nabla)v=0,\\ &\operatorname{div} v=0, \qquad\qquad p=\text{const}\end{aligned}$$

instead of Eqs. (1.6). One can see immediately that every solution of the system (1.9) satisfies also Eqs. (1.1). However, contrary to the case of Eqs. (1.6), the system (1.9) is incompatible, i.e., the constraint div$v(0,x)=0$ for the initial condition $v(0,x)$ is not

(1) This splitting is not necessarily unique.

sufficient to guarantee the solubility of the Cauchy problem. There are certain extra conditions which must be satisfied in order to guarantee the existence of the solution. The compatibility conditions for Eqs. (1.9) lead to further constraints for $v(0, x)$. In order to obtain them, let us denote $D = \left(\frac{\partial}{\partial x^j} v^i\right)$. We have from Eqs. (1.9)

$$\frac{\partial}{\partial t} D+(v \cdot \nabla)D = -D^2 \tag{1.10}$$

and hence

$$\frac{d}{dt} D^n = -nD^{n+1}, \tag{1.11}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla$. By taking the trace of Eq. (1.11), we have

$$\frac{d}{dt} \mathrm{Tr} D^n = -n \mathrm{Tr} D^{n+1}. \tag{1.12}$$

Let us note that $\mathrm{Tr} D = \operatorname{div} v$. Thus $\operatorname{div} v = 0$ implies that $\mathrm{Tr} D^2 = 0$. This implies $\mathrm{Tr}\, D^3 = 0$ and so on. On the other hand, for any 3×3 matrix A the vanishing of $\mathrm{Tr}\, A$, $\mathrm{Tr}\, A^2$, $\mathrm{Tr}\, A^3$ implies that the eigenvalues of A are vanishing and thus $\mathrm{Tr}\, A^k = 0$ for any $k \geqslant 1$. As follows from Eq. (1.12), if $v(0, x)$ satisfies $\mathrm{Tr}\, D = 0$, $\mathrm{Tr}\, D^2 = 0$, $\mathrm{Tr}\, D^3 = 0$ then these relations are preserved automatically at later times. Since the eigenvalues of D are vanishing, the matrix D has one of the possible canonical forms

$$D \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad D \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.13}$$

and not only $\mathrm{Tr}\, D^3 = 0$, but also $D^3 = 0$. Therefrom, as the hodographs of isobaric flows are degenerated, these flows are either simple or double waves [1].

2. Global existence in time

As we have noted, Eqs. (1.9) imply two other consequences $\mathrm{Tr}\, D^2 = 0$, $\mathrm{Tr}\, D^3 = 0$ which, in fact, are constraints for the initial conditions. Thus the full, compatible set of equations of isobaric flows are the following:

$$\begin{cases} \mathrm{Tr} D = 0, \quad \mathrm{Tr} D^2 = 0, \quad \mathrm{Tr} D^3 = 0, \\ \frac{\partial v}{\partial t} + (v \cdot \nabla) v = 0, \quad p = \text{const}. \end{cases} \tag{2.1}$$

The first three conditions

$$\mathrm{Tr} D = 0, \quad \mathrm{Tr} D^2 = 0, \quad \mathrm{Tr} D^3 = 0 \tag{2.2}$$

can be treated as constraints for the initial conditions. To see how restrictive they are, and to estimate the freedom of the general solution, let us demonstrate that Eqs. (2.2) can be written in the Cauchy form. In order to prove that, we will look for noncharacteristic surfaces for Eqs. (2.2). Taking $D \to D+\varepsilon G$, $G = (u^{\alpha}_{,\beta})$ and linearizing Eqs. (2.2), one obtains the following linear equations for G:

$$\mathrm{Tr}\,G = 0, \quad \mathrm{Tr}(DG) = 0, \quad \mathrm{Tr}(D^2G) = 0. \tag{2.3}$$

The characteristic determinant for this system is

$$\det\|\bar{\xi}, D\bar{\xi}, D^2\bar{\xi}\| = 0,$$

where $D\bar{\xi} = (v^{\alpha}_{,\beta}\xi_{\alpha})$. Therefore the characteristic covectors of Eqs. (2.3) and thus also Eqs. (2.2) satisfy $D^2\bar{\xi} = 0$

$$\bar{\xi} \text{ is characteristic} \Leftrightarrow \bar{\xi} \in \mathrm{Ker}\,D^2.$$

Let us now take a point $x_0 \in \mathrm{R}^3$, say $x_0 = (0,0,0)$ and a matrix $\tilde{D} = (\tilde{D}^i_{\nu})$ satisfying $\mathrm{Tr}\,\tilde{D}^k = 0$, $k = 1,2,3$. Let rank $\tilde{D} = 2$, then $\dim\{\mathrm{Ker}\,\tilde{D}^2\} = 2$, and there exists a vector, say $\tilde{\xi}$, which is not characteristic, $\tilde{\xi} \notin \mathrm{Ker}\,\tilde{D}^2$. Without loosing generality, we may assume that $\bar{\xi}$ is perpendicular to the plane $x_3 = 0$. In such a case, the plane $x_3 = 0$ would be noncharacteristic and for values of $(v^{\mu}_{,\alpha})$ from some neighbourhood of $\tilde{D}$, Eqs. (2.2) can be solved for $v^{\alpha}_{,3}$. Then the Cauchy–Kowalewska theorem can be applied to state that the general (analytic) solution of Eqs. (2.2) in the vicinity of x_0 depends on three functions $\tilde{v}_1(x^1, x^2)$, $\tilde{v}^2(x^1, x^2)$, $\tilde{v}^3(x^1, x^2)$ of two variables. We have assumed here that

$$\left.\frac{\partial v^i}{\partial x_{\mu}}\right|_{x=0} = D^i_{\mu}.$$

Suppose now that the initial condition $v(0, x) \in C^1(\Omega)$ is defined in some domain $\Omega \subset \mathrm{R}^3$ where it satisfies Eqs. (2.2). As it follows from Eqs. (2.1), the velocity v is constant along the characteristics of the second Eqs. (2.1) which are defined by

$$\frac{dx}{dt} = v(t, x), \qquad x(0) = x_0. \tag{2.4}$$

Therefore these characteristics are straight lines defined by

$$x = v(0, x_0)t + x_0. \tag{2.5}$$

To find the velocity field in the coordinates t, x one must solve Eq. (2.5) for x_0, $x_0 = x_0(t, x)$ and substitute it for x_0 in $v(0, x_0)$

$$v(t, x) := v(0, x_0(t, x)).$$

It appears that the transformation (2.5) is nonsingular for all t. Indeed, we have

$$\frac{\partial x}{\partial x_0} = D_0 t + I, \tag{2.6}$$

where $\dfrac{\partial x}{\partial x_0} = \left(\dfrac{\partial x^{\nu}}{\partial x_0^{\mu}}\right)$ is the Jacobi matrix of transformation (2.5) and where $D_0 = \left.\dfrac{\partial v(0, x_0)}{\partial x_0}\right|_{t=0}$. Since D_0 is nilpotent, we have $\det\|D_0 t + I\| = 1$ and

$$(D_0 t + I)^{-1} = I - D_0 t + D_0^2 t^2. \tag{2.7}$$

Thus Eq. (2.4) is invertible, which implies that the solution of Eqs. (2.1) is defined for all t in the region of space-time covered by the characteristics (2.5) passing through Ω.

Let us also notice that along the characteristic curve the gradient of the solution $D = \frac{\partial}{\partial x^j} v^i$ increases linearly with time. We have

$$\frac{\partial v}{\partial x} = \frac{\partial v(0, x)}{\partial x_0} \frac{\partial x_0}{\partial x} = D_0(I - D_0 t + D_0^2 t^2) = D_0 - D_0^2 t,$$

which shows that the gradient of v increases linearly with time in the case of double waves ($D_0^2 \neq 0$) or it stays bounded in the case of simple waves ($D_0^2 = 0$). Thus the isobaric flows can not provide us with the counterexample of global existence. They can be still interesting, however, from the point of view of applications (geostrophic flows), and they provide us with the wide class of solutions of the Euler equations.

References

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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