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Time and space-dependent development of dislocation densities during uniaxial deformation

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THE EVOLUTION of internal variables, mobile and immobile dislocation densities during plastic deformation is studied within the framework of continuum mechanics utilizing the assumption of minimization of potential of the two types of dislocations. A pair of partial differential equations are derived and various simplified forms are solved for dislocation densities in space with increasing time. These equations yield a periodic spatial oscillation of the dislocation densities in agreement with the dislocation patterning in uniaxial deformation.

W ramach mechaniki ośrodka ciągłego przeanalizowano problem ewolucji zmiennych wewnętrznych, ruchomych i nieruchomych dyslokacji, posługując się założeniem o minimalizacji potencjałów obu typów dyslokacji. Zagadnienie sprowadzono do układu dwóch równań różniczkowych cząstkowych, które rozwiązać można w pewnych szczególnych przypadkach rozwoju dyslokacji w czasie i przestrzeni. Równania te prowadzą do okresowych drgań przestrzennych gęstości dyslokacji, zgodnych z ich rozkładem obserwowanym podczas deformacji jedno-osiowej.

В рамках механики сплошных сред проанализирована проблема эволюции внутренних переменных, подвижных и неподвижных дислокаций, послуживаясь предположением о минимизации потенциалов обоих типов дислокаций. Задача сведена к системе двух дифференциальных уравнений в частных производных, которые можно решить в некоторых частных случаях развития дислокаций во времени и в пространстве. Эти уравнения приводят к периодическим пространственным колебаниям плотности дислокаций, согласно с их распределением, наблюдаемым во время одноосевых деформаций.

1. Introduction

THE PLASTIC DEFORMATION of most metals for a range of strains, strain rates and temperatures is accompanied by the formation of heterogeneous dislocation structures such as coils, tangles, forests, persistent slip bands, and dislocation cells. This means that the plastic deformation due to the independent motion of dislocations uniformly distributed throughout the volume is an unstable process [1, 2]. During plastic deformation, the dislocations interact with each other and such interactions follow the general principle of minimization of free energy to yield mechanically stable equilibrium configurations [2,3]. Recently, a number of attempts have been made to study the evolution of dislocation patterns during plastic deformation by writing the usual rate equations for internal microsturctural variables [4–7]. WALGRAEF and EIFANTIS [8], EIFANTIS [8, 9] and a number of other workers [10–13] have studied the evolution and stability of dislocation patterns through balance laws of continuum mechanics containing rate and flux terms.

In this paper the continuum mechanical approach to dislocation dynamics is extended to include the assumption of minimization of potential of two types of dislocations by diffusive rearrangement of dislocations during uniaxial deformation. Various closed form solutions of the system of PDE's of mobile and immobile dislocation densities are obtained under various simplifying assumptions.

2. Basic equations

2.1. Internal variable formulation

Let us consider, for simplicity, a purely reactive type of transformation of mobile and immobile dislocations under applied stress. The general set of equations in terms of internal variables can be expressed as:

$$\varrho^m = (\varrho_1^m, \varrho_2^m, \dots \varrho_{S^m}^m), \tag{2.1}$$

$$\varrho^i = (\varrho_1^i, \varrho_2^i, \dots \varrho_{S^i}^i). \tag{2.2}$$

Equations (2.1) and (2.2) represent mobile and immobile dislocation densities in slip systems S^m and S^i, respectively. The superscripts m and i correspond to mobile and immobile dislocations and subscripts $(1, \dots, S^m)$ and $(1, \dots, S^i)$ correspond to the individual slip systems. ϱ_i^m and ϱ_j^i represent the mobile and immobile dislocation density in the i-th and j-th slip system, respectively.

The average dislocation densities, ϱ_i^m and ϱ_j^i, in the present case, are only time-dependent and thus the dynamic change is given by the following equations:

$$\dot{\varrho}_i^m = f_i^m(\varrho^m, \varrho^i), \tag{2.3}$$

$$\dot{\varrho}_j^i = f_j^i(\varrho^m, \varrho^i). \tag{2.4}$$

2.2. Specific forms for dislocation interactions

Various specific forms for Eqs. (2.3) and (2.4) based on different types of physically observed dislocation interactions have been suggested in the past [4–7]. For simplicity, we consider only the multiplication and annihilation of mobile and immobile dislocations. The dislocation densities of mobile and immobile dislocations are governed by the rates at which such multiplication and annihilation events take place. Here we propose that these processes be continuous and have the following simplest forms:

$$\dot{\varrho}^m = (\varepsilon_m - \mu_{mm}\varrho^m - \mu_{mi}\varrho^i)\varrho^m, \tag{2.5}$$

$$\dot{\varrho}^i = (\varepsilon_i - \mu_{im}\varrho^m - \mu_{ii}\varrho^i)\varrho^i, \tag{2.6}$$

where $\varepsilon_m \varrho^m$ and $\varepsilon_i \varrho^i$ denote the rate of multiplication of mobile dislocations and immobile dislocation groups, respectively. Annihilation events take place when two mobile, immobile or a mobile and immobile dislocation of opposite signs come in contact. Thus the rate of annihilation of mobile and immobile dislocations are $(-\mu_{mm}\varrho^{m^2} - \mu_{mi}\varrho^i\varrho^m)$ and $(-\mu_{ii}\varrho^{i^2} - \mu_{im}\varrho^m\varrho^i)$, respectively. Other remobilization and immobilization terms are neglected. Clearly, if $\mu_{im} = \mu_{mi} = 0$, there is no interaction between the mobile and immobile dislocations and Eqs. (2.5) and (2.6) degenerate into two logistic equations.

2.3. Diffusion of dislocations

Let us now consider the case of pure diffusion of dislocations under stress. A one-dimensional balance law for mobile dislocations, assuming no multiplication or annihilation along the glide plane [8–9], can be written as

$$\dot{\varrho}+\frac{\partial J}{\partial x} = 0, \tag{2.7}$$

where J is the dislocation flux. The dot $(\cdot)$ above ϱ denotes the time derivative of dislocation density. If we assume the dislocation flux J to be given by the Fick's first law,

$$J = -D\left(\frac{\partial \varrho}{\partial x}\right) \tag{2.8}$$

then, from Eqs. (2.7) and (2.8) one obtains

$$\dot{\varrho} = D\left(\frac{\partial^2 \varrho}{\partial x^2}\right), \tag{2.9}$$

where $\varrho = \varrho(x, t)$ and D is the diffusivity or the diffusion coefficient and is assumed to be constant. Equation (2.9) is also referred to as Fick's second law. If the diffusivity is not assumed to be independent of position x, then Eq. (2.9) can be written as

$$\dot{\varrho} = \frac{\partial^2 (D\varrho)}{\partial x^2}. \tag{2.10}$$

2.4. Diffusive flux for dislocation and rearrangement during plastic deformation

In this section we introduce a quantity $-U(x)$ as the potential of the dislocations at the position x. $U(x)$ can be thought of as a measure of the "favorableness" for the residency of dislocations since each dislocation tends to move toward lower potential area where the conditions are more favorable. Thus it may be plausible to assume that the mean velocity of the movement caused by the favorableness of the system is proportional to the force produced by the potential function $U(x)$, that is $\nabla_x U(x)$.

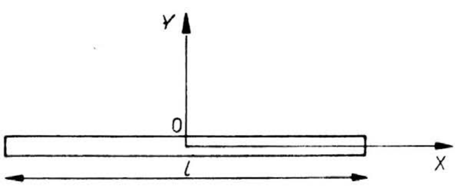

FIG. 1. Uniaxial tensile deformation of a crystal with a slip plane oriented in the stress direction.

Let a crystal of length, l with a slip plane along its length and slip direction parallel to x direction be deformed at a constant strain rate under tensile stress in the x direction. Let us consider the flux at the origin O (Fig. 1). The number of mobile dislocation to the left of the origin O at time τ is denoted by ϱ_A^m and the potential by U_A^m. Similarly, the number of mobile dislocations to the right of the origin at time t is denoted by ϱ_B^m and the potential by U_B^m. The number of dislocations moving from the left of the origin to the right of the origin during the time interval τ and those moving in the opposite direction are given by the expressions

$$\left\{\alpha_m+\beta_m\varrho_A^m+\frac{\gamma_m}{l}[U_A^m-U_B^m]\right\}\varrho_A^m,$$

(2.11)

$$\left\{\alpha_m+\beta_m\varrho_B^m+\frac{\gamma_m}{l}[U_B^m-U_A^m]\right\}\varrho_B^m,$$

where the coefficients $\alpha_m, \beta_m, \gamma_m$ are dependent on l and τ. Therefore the net flux at the origin O is given by

$$J_m = \frac{1}{\tau}[(\alpha_m+\beta_m\varrho_A^m)\varrho_A^m-(\alpha_m+\beta_m\varrho_B^m)\varrho_B^m]+\frac{\gamma_m}{\tau l}(U_A^m-U_B^m)(\varrho_A^m+\varrho_B^m). \tag{2.12}$$

Here if both l and τ tend to zero, we obtain the following expression for the flux in the continuously changing system as the limit

$$J_m(x,t) = -\frac{\partial}{\partial x}[(\alpha_m+\beta_m\varrho^m(x,t))\varrho^m(x,t)]-\gamma_m\varrho^m(x,t)\frac{dU^m}{dx}, \tag{2.13}$$

where we put

$$\lim_{l,\tau\to 0}\alpha_m\frac{l^2}{\tau} = \alpha_m,$$

$$\lim_{l,\tau\to 0}\beta_m\frac{l^3}{\tau} = \beta_m,$$

$$\lim_{l,\tau\to 0}\frac{\varrho^m}{l} = \varrho^m,$$

$$\lim_{l,\tau\to 0}\frac{\gamma_m l}{\tau}\left[\frac{U_B^m-U_A^m}{l}\right] = \frac{dU^m}{dx}.$$

The first and second term on the right side of Eq. (2.13) represent the diffusivity flux and dislocation or potential flux, respectively. The above analysis can easily be extended to the case of immobile dislocations. For immobile dislocations one obtains,

$$J_i(x,t) = -\frac{\partial}{\partial x}[(\alpha_i+\beta_i\varrho^m(x,t))]\varrho^i(x,t)]-\gamma_i\varrho^i(x,t)\frac{dU^i}{dx}. \tag{2.14}$$

3. Dislocation modelling

So far we have considered the independent cases of dislocation transformation, diffusive transport, and rearrangement of dislocations to minimize the strain energy of the system. This section presents equations of evolution of dislocation densities involving all the above processes.

3.1. Evolution equations of dislocation densities

Equations (2.5)–(2.7), (2.13) and (2.14) lead to the following system of nonlinear partial differential equations governing the evolution of mobile and immobile dislocation densities:

$$\frac{\partial \varrho^m(x,t)}{\partial t} = -\frac{\partial J_m(x,t)}{\partial x} + (\varepsilon_m - \mu_{mm}\varrho^m - \mu_{mi}\varrho^i)\varrho^m, \tag{3.1}$$

$$\frac{\partial \varrho^i(x,t)}{\partial t} = -\frac{\partial J_i(x,t)}{\partial x} + [\varepsilon_i - \mu_{im}\varrho^m - \mu_{ii}\varrho^i)\varrho^i, \tag{3.2}$$

where

$$J_m(x,t) = -\frac{\partial}{\partial x}[D_m\varrho^m(x,t)] - \gamma_m\varrho^m(x,t)\frac{dU^m}{dx} \tag{3.3}$$

and

$$J_i(x,t) = -\frac{\partial}{\partial x}[D_i\varrho^i(x,t)] - \gamma_i\varrho^i(x,t)\frac{dU^i}{dx}. \tag{3.4}$$

Equations (3.3) and (3.4) can be written as

$$J_m(x,t) = -\frac{\partial}{\partial x}[(D_m\varrho^m(x,t)] - \frac{\gamma_m}{g_m}\varrho^m(x,t)\nu^m \tag{3.5}$$

and

$$J_i(x,t) = -\frac{\partial}{\partial x}[(D_i\varrho^i(x,t)] - \frac{\gamma_i}{g_i}\varrho^i(x,t)\nu^i \tag{3.6}$$

where the mean velocities are given by

$$\nu^m = g_m\left(\frac{\partial U^m}{\partial x}\right), \tag{3.7}$$

$$\nu^i = g_i\left(\frac{\partial U^i}{\partial x}\right) \tag{3.8}$$

as discussed earlier in Subsect 2.4. Equations (3.1) and (3.2) can be rearranged employing Eqs. (3.5) and (3.6) as

$$\frac{\partial \varrho^m(x,t)}{\partial t} + h_m\left(\frac{\partial j^m}{\partial x}\right) - \frac{\partial^2(D_m\varrho^m)}{\partial x^2} = (\varepsilon_m - \mu_{mm}\varrho^m - \mu_{mi}\varrho^i)\varrho^m, \tag{3.9}$$

$$\frac{\partial \varrho^i(x,t)}{\partial t} + h_i\left(\frac{\partial j^i}{\partial x}\right) - \frac{\partial^2(D_i\varrho^i)}{\partial x^2} = (\varepsilon_i - \mu_{im}\varrho^m - \mu_{ii}\varrho^i)\varrho^i, \tag{3.10}$$

where $j^m = \varrho^m\nu^m$ and $j^i = \varrho^i\nu^i$, respectively. Also, in Eqs. (3.9) and (3.10), $h_m = (\gamma_m/g_m)$ and $h_i = (\gamma_i/g_i)$ are constants for mobile and immobile dislocations, respectively.

Also the plastic strain rate $\dot{\varepsilon}^p$ can be written as

$$\dot{\varepsilon}^p = \phi b(\varrho^m\nu^m + \varrho^i\nu^i), \tag{3.11}$$

where b is the Burgers vector and ϕ is an orientation factor relating single crystal to polycrystal deformation.

3.2. Stationary behavior of dislocations during plastic deformation

If the mobile dislocation diffusivity is assumed to be density-dependent in the following manner

$$D_m = \alpha_m + \beta_m \varrho^m, \tag{3.12}$$

then the flow of dislocations is given from Eq. (3.3) as, (assuming $\gamma_m = 1$),

$$J_m(x, t) = -\frac{\partial}{\partial x}\{(\alpha_m + \beta_m \varrho^m)\varrho^m(x, t)\} - \varrho^m(x, t)\left(\frac{dU^m}{dx}\right). \tag{3.13}$$

The change of dislocation density is given by

$$\dot{\varrho}^m(x, t) = -\Delta J_m(x, t). \tag{3.14}$$

Let us now consider a one-dimensional space with the boundaries at which $J_m = 0$. A stationary distribution $\varrho^{m*}(x)$ can be obtained as a solution of the equation $J_m = 0$, i.e.,

$$(\alpha_m + 2\beta_m \varrho^m)\frac{d\varrho^{m*}}{dx} + \frac{dU^m}{dx}\varrho^{m*} = 0. \tag{3.15}$$

The solution of Eq. (3.15) is given by

$$2\beta_m\{\varrho^{m*}(x) - \varrho^m(0)\} + \alpha_m \ln\left\{\varrho^{m*}\frac{(x)}{\varrho^m}(0)\right\} = -\{U^m(x) - U^m(0)\}, \tag{3.16}$$

where $\varrho^m(0)$ and $U^m(0)$ are arbitrary chosen points in the system where we can set the origin of the coordinates. Here we can show that the stationary solution given by Eq. (3.16) which satisfies Eq. (3.15) is really a globally stable stationary solution of Eq. (3.13) and if the solution $\varrho^m(x, t)$ of Eq. (3.13) starting from an initial condition $\varrho^m(x, 0) > 0$ (for all x) is a smooth function of x and t, it approaches this stationary solution $\varrho^{m*}(x)$. Let us consider a function defined by

$$H = \int\left\{\alpha_m \varrho^m \ln\left[\frac{\varrho^m}{\varrho^{m*}}\right] - \alpha_m(\varrho^m - \varrho^{m*}) + \beta_m(\varrho^m - \varrho^{m*})^2\right\}dx \geqslant 0, \tag{3.17}$$

where the equality holds only when $\varrho^m(x, t) = \varrho^{m*}(x)$ for all x. The time derivative of this function H is calculated using Eq. (3.14) as

$$\frac{dH}{dt} = \int\left\{\alpha_m \ln\left(\frac{\varrho^m}{\varrho^{m*}}\right) + 2\beta_m(\varrho^m - \varrho^{m*})\right\}\frac{\partial \varrho^m}{\partial t}dx$$
$$= \int \frac{\partial J_m}{\partial x}\left\{\alpha_m \ln\left(\frac{\varrho^m}{\varrho^{m*}}\right) + 2\beta_m(\varrho^m - \varrho^{m*})\right\}dx \tag{3.18}$$

and by partial integration with the boundary contition $J_m = 0$, we have

$$\frac{dH}{dt} = \int J_m\left\{\frac{\alpha_m}{\varrho^m}\frac{\partial \varrho^m}{\partial x} - \frac{\alpha_m}{\varrho^{m*}}\frac{\partial \varrho^{m*}}{\partial x} + 2\beta_m\left(\frac{\partial \varrho^m}{\partial x} - \frac{\partial \varrho^{m*}}{\partial x}\right)\right\}dx = -\int \frac{J_m^2}{\varrho^m}dx \leqslant 0, \tag{3.19}$$

where Eqs. (3.13) and (3.15) have been used. The equality of Eq. (3.19) holds again only when $J_m = 0$, that is $\varrho^m = \varrho^{m*}$. Therefore we can say that the solution of Eq. (3.14), always approaches the stationary solution $\varrho^{m*}(x)$ given by Eq. (3.16). The stationary solution of Eq. (3.16) has a simple form,

$$\varrho^{m*} = \varrho^m(0)\exp\left[-\frac{(U^m(x)-U^m(0))}{\alpha_m}\right], \tag{3.20}$$

if $\beta_m = 0$. Equation (3.20) implies, in the case of a density-independent diffusion coefficient, an exponential dependence between steady state mobile dislocation density and the corresponding potential difference between the two mobile states. The effect of density-dependent diffusion is to make the dislocation distribution flatter as the total number of dislocation increases. Steady state solution for the case of density-dependent diffusivity of immobile dislocations ($D_i = \alpha_i + \beta_i\varrho^i$) can similarly be written as

$$\varrho^{i*}(x) = \varrho^i(0)\exp\left[-\frac{(U^i(x)-U^i(0))}{\alpha_i}\right]. \tag{3.21}$$

3.2.1. Solution at the boundaries in stationary state. Equations (3.3) and (3.4) at the boundaries where $J_m = J_i = 0$, as well as $j^m = j^i = 0$, yield

$$\frac{d\varrho^{m*}}{dx} = 0 \tag{3.22}$$

and

$$\frac{d\varrho^{i*}}{dx} = 0. \tag{3.23}$$

This implies that ϱ^{m*} and ϱ^{i*} are constant. The same conclusion can be drawn from Eqs. (3.20) and (3.21). Also from Eqs. (3.9) and (3.10) for the steady state,

$$\begin{aligned} \varepsilon_m &= \mu_{mm}\varrho^m + \mu_{mi}\varrho^i, \\ \varepsilon_i &= \mu_{im}\varrho^m + \mu_{ii}\varrho^i. \end{aligned} \tag{3.24}$$

The solution of equations (3.24) for $\mu_{im} = \mu_{mi} = \mu$ yields

$$\varrho^m(x)_{1,2} = \varepsilon_m \pm \frac{\sqrt{\{\varepsilon_m^2 + 4\mu_{mm}(\mu_{ii}\varrho_i^2 - \varepsilon_{ii}\varrho^i)\}}}{2\mu_{mm}}. \tag{3.25}$$

4. Analytical solution

Let us consider the solution of partial differential equations given by (3.9) and (3.10) in a one-dimensional finite interval [0, 1]. The nonlinear system (3.9)–(3.10) is difficult to solve analytically and we shall consider here a simplification of the above system by assuming $\mu_{mm} = \mu_{ii} = 0$. Also the mobile and immobile dislocation fluxes are assumed to be constant i.e., $j^m = j^i = j_0$, where j_0 is a constant. Equations (3.9)–(3.10) thus become

$$\begin{aligned} \frac{\partial\varrho^m(x,t)}{\partial t} &= D_m\frac{\partial^2\varrho^m}{\partial x^2} + (\varepsilon_m\varrho^m - \mu_{mi}\varrho^i\varrho^m), \\ \frac{\partial\varrho^i(x,t)}{\partial t} &= D_i\frac{\partial^2\varrho^i}{\partial x^2} + (\varepsilon_i\varrho^i - \mu_{im}\varrho^i\varrho^m), \end{aligned} \tag{4.1}$$

where $0 < x < l$, $t > 0$.

The point $(\varepsilon_i/\mu_{im}, \varepsilon_m/\mu_{mi})$ is an equilibrium point of the system (4.1). The nonlinearity is still quite awkward and we shall limit ourselves to small deviations from the temporal and special equilibrium state. Accordingly, if we set

$$\varrho^m(x,t) = \frac{\varepsilon_i}{\mu_{im}} + u^m(x,t), \qquad \varrho^i(x,t) = \frac{\varepsilon_m}{\mu_{mi}} + u^i(x,t), \tag{4.2}$$

then Eq. (4.1) leads to

$$\frac{\partial u^m(x,t)}{\partial t} = D_m \frac{\partial^2 u^m}{\partial x^2} - \left[\mu_{mi}\frac{\varepsilon_i}{\mu_{im}}\right] u^i(x,t) - \mu_{mi} u^m u^i, \qquad \frac{\partial u^i(x,t)}{\partial t} = D_i \frac{\partial^2 u^i}{\partial x^2} - \left[\mu_{im}\frac{\varepsilon_m}{\mu_{mi}}\right] u^m(x,t) - \mu_{im} u^m u^i, \tag{4.3}$$

where $l > x > 0$, $t > 0$. The linear system corresponding to Eq. (4.3) is

$$\frac{\partial u^m(x,t)}{\partial t} = D_m \frac{\partial^2 u^m}{\partial x^2} - au^i \tag{4.4}$$

and

$$\frac{\partial u^i(x,t)}{\partial t} = D_i\, \partial^2 \frac{u^i}{\partial x^2} - bu^m, \tag{4.5}$$

where a and b are given by $(\mu_{mi}\varepsilon_i/\mu_{im})$ and $(\mu_{im}\varepsilon_m/\mu_{mi})$, respectively. Let the initial and boundary conditions for the system (4.4)–(4.5) and hence for Eqs. (4.1)–(4.2), as required by physical considerations, be

$$\begin{aligned} u^m(x,0) &= 0, \quad u^i(x,0) = 0, \quad 0 < x < l,\\ u^m(0,t) &= u^m(l,t) = K^m(t), \quad 0 < x < l, \quad t > 0,\\ u^i(0,t) &= u^i(l,t) = K^i(t), \quad 0 < x < l, \quad t > 0. \end{aligned} \tag{4.6}$$

The conditions $u^m(l,t)$, $u^i(l,t)$, $u^m(0,t)$, and $u^i(0,t)$ represent the fact that mobile and immobile dislocation densities at the surface increase with time and reach a saturated value indicated by K^m and K^i, respectively. If we use the nonsingular linear transformation

$$\begin{aligned} \begin{bmatrix} u^m(x,t) \\ u^i(x,t) \end{bmatrix} &= \begin{bmatrix} a/b^{1/2}\cosh(ab)^{1/2}t & \sinh(ab)^{1/2}t \\ -\sinh(ab)^{1/2}t & -b/a^{1/2}\cosh(ab)^{1/2}t \end{bmatrix} \begin{bmatrix} y^m(x,t) \\ y^i(x,t) \end{bmatrix},\\ \begin{bmatrix} y^m(x,t) \\ y^i(x,t) \end{bmatrix} &= \begin{bmatrix} b/a^{1/2}\cosh(ab)^{1/2}t & \sinh(ab)^{1/2}t \\ -\sinh(ab)^{1/2}t & -a/b^{1/2}\cosh(ab)^{1/2}t \end{bmatrix} \begin{bmatrix} u^m(x,t) \\ u^i(x,t) \end{bmatrix}, \end{aligned} \tag{4.7}$$

then the system (4.4)–(4.6) becomes

$$\frac{\partial y^m(x,t)}{\partial t} = D_m \frac{\partial^2 y^m}{\partial x^2}, \qquad \frac{\partial y^i(x,t)}{\partial t} = D_i \frac{\partial^2 y^i}{\partial x^2}, \tag{4.8}$$

$$l > x > 0, \quad t > 0;$$

$$y^m(x, 0) = 0, \quad y^i(x, 0) = 0, \quad 0 < x < l,$$

$$\text{(4.9)} \quad y^m(0, t) = y^m(l, t) = K^m \left(\frac{b}{a}\right)^{1/2} \cosh(ab)^{1/2}t + K^i \sinh(ab)^{1/2}t, \quad x > \infty, \quad t > 0,$$

$$y^i(0, t) = y^i(l, t) = -K^m \sinh(ab)^{1/2}t - \frac{a}{b} K^i \cosh(ab)^{1/2}t, \quad x > \infty, \quad t > 0,$$

where $l > x > 0,\ t > 0$.

The unique solution of (4.7)–(4.9) [14] is given by

$$\text{(4.10)} \quad y^m(x, t) = 2\pi D \sum_{n=1,\infty} n\left\{\int_0^t y^m(0, \tau)\exp[-n^2\pi^2 D_m(t-\tau)]\,d\tau\right\} \sin n\pi x;$$

and a similar equation holds for $y^i(x, t)$. Equations (4.7)–(4.9) lead to

$$\text{(4.11)} \quad u^m(x, t) = 2\pi D_m \sum_{n=1,\infty} n \int_0^t \left\{\left\{K^m \cos(ab)^{1/2}(t-\tau) + \left(\frac{a}{b}\right)^{1/2} K^i \sinh(ab)^{1/2}(t-\tau)\right.\right.$$
$$\exp[-n^2\pi^2 D_m(t-\tau)]\,d\tau \sin n\pi x, \quad 0 < x \leqslant 1.$$

It is easy to see from Eq. (4.11) that as $t \to \infty$,

$$\text{(4.12)} \quad \lim_{l\to\infty} u^m(x, t) = 2\pi D_m \sum_{n=1,\infty} n\left[-\frac{\varepsilon_i}{\mu_{im}} \frac{n^2\pi^2 D_m}{n^4\pi^4 D_m^2 - ab} - \frac{\varepsilon_m}{\mu_{mi}} \frac{a}{n^4\pi^4 D^2 - ab}\right] \sin \pi x$$

which is nothing but the Fourier sine expansion of the stationary solution of the system (4.4)–(4.6). If ϕ and ψ are such stationary solution, then

$$\text{(4.13)} \quad D_m \frac{d^2\phi}{dx^2} = a\psi, \quad D_i \frac{d^2\psi}{dx^2} = b\phi,$$

and the boundary conditions become

$$\text{(4.14)} \quad \phi(0) = \phi(l) = K^m, \quad \psi(0) = \psi(l) = K^i$$

for which a unique solution pair exists. Thus for the case of a finite interval, stationary solutions of the system (4.4)–(4.6) exist and are unique; furthermore the nonstationary solutions of (4.4)–(4.6) asymptotically approach the stationary solutions as $t > \infty$. This implies that both mobile and immobile dislocation densities vary periodically in space and time to reach saturation values.

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