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On a constitutive relation for heterogeneous thermoelastic media

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OUR AIM is to work out a general formulation of a constitutive relation for heterogeneous thermoelastic media. In the first part, we consider a material with a given homogeneous temperature field: we are interested in the calculation of the local thermal stresses and the global behaviour of the homogeneous equivalent medium. In the second part, the transient temperature field is nonhomogeneous. Particular attention is devoted to thermoelastic damping. The global behaviour of the body appears to be viscoelastic due to thermoelastic coupling. In both cases, the material consists of grains with uniform mechanical properties in each grain. The field equations of the problem are transformed into integral equations which are solved through a discretization scheme.

Celem pracy jest ogólne sformułowanie związku konstytutywnego dla niejednorodnych ośrodków termosprężystych. Na wstępie rozpatruje się materiał w znanym, jednorodnym polu temperatury: chodzi tu o wyznaczenie lokalnych naprężeń termicznych oraz globalnego zachowania się równoważnego ośrodka jednorodnego. W dalszej części pracy nieustalone pole temperatury jest niejednorodne. Szczególną uwagę zwrócono na tłumienie termosprężyste. Globalne zachowanie się ciała ma charakter lepkosprężysty w wyniku sprzężenia termosprężystego. W obu przypadkach materiał składa się z ziaren o stałych własnościach mechanicznych w obrębie poszczególnych ziaren. Równania polowe zagadnienia przekształcono w równania całkowe rozwiązywane metodą dyskretyzacji.

Целью работы является общая формулировка определяющего состношения для неоднородных термоупругих сред. Вначале рассматривается материал в известном, однородном поле температур: здесь имеется в виду определение локальных термических напряжений и глобального поведения эквивалентной однородной среды. В дальнейшей части работы неустановившееся поле температур является неоднородным. Особенное внимание обращено на термоупругие затухания. Глобальное поведение тела имеет вязкоупругий характер в результате термоупругого сопряжения. В обоих случаях материал состоит из зерен о постоянных механических свойствах в области отдельных зерен. Полевые уравнения задачи преобразованы в интегральные уравнения, решаемые методом дискретизации.

1. Introduction

WE CONSIDER a heterogeneous polycrystalline medium consisting of disordered grains. Each grain has a linear thermoelastic behaviour. We are mostly interested in the mechanical behaviour of the aggregate.

Thermoelastic damping has been discussed by ZENER [26] who pointed out that a homogeneous thermoelastic solid (e.g., a vibrating beam) dissipates mechanical energy via thermomechanical coupling and irreversibility due to heat conduction between the heated compressed and cooled extended region of the beam. The global behaviour appears to be a linear viscoelastic response. Zener also discussed the viscoelastic response of a polycrystal with disoriented thermoelastic grains having identical mechanical properties. In both cases, the viscoelastic macroscopic behaviour is due to heat fluxes associated with

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a nonhomogeneous temperature field:

in the case of a vibrating beam, the nonhomogeneity of the temperature generated by a boundary condition is at a macroscopic scale;

on the other hand, in the polycrystalline material, nonhomogeneity of the temperature field appears also at the microscopic grain scale.

Our objective is a general formulation of a constitutive relation for a granular material with thermomechanical properties varying from grain to grain. For the assumed homogeneous strain and temperature boundary conditions, we derive, under certain simplifying assumptions, relations which estimate the local strain and temperature. Effective properties (particularly viscoelastic effects when the temperature field is nonuniform) result from these relations.

Recent work has been devoted to the homogenization of thermoelastic heterogeneous media (FRANCFORT [10]; SUQUET [23]), where a multiple scale technique is used and is found to be specially useful for a material with space periodicity in the mechanical properties. Another general theory, proposed by BIOT [4] and investigated by BRUN [5], uses a variational principle to derive quite general results. Our approach relies upon an integral formulation of the problem. Such a method has been used in other works dealing with homogenization (see WILLIS [25] and MURA [18]):

1. Research of the effective elasticity tensor of polycrystals by ZELLER and DEDERICHS [27]; determination of the equivalent homogeneous medium of an elastoplastic material (KRÖNER [14], HILL [12]). For perfectly disordered polycrystals, BERVEILLER and ZAOUI [3] use a self-consistent scheme. Results are obtained when the Eshelby's solution is used for an inclusion embedded in a homogeneous matrix or when using integral formulations. In this latter case, BERVEILLER [2] requires the knowledge of the Green's tensor of elasticity for a homogeneous isotropic infinitely extended medium. In a formal way, our calculations will require the use of Green's tensor of elasticity and of the elementary solution of the heat equation for a homogeneous, anisotropic, infinitely extended medium.

2. In the case of a composite with periodic microstructures, effective properties may be deduced from characteristic field quantities which are periodic functions of space. This fact is made of by T. IWAKUMA and S. NEMAT-NASSER [19] to obtain an accurate estimation of overall mechanical properties of the material. Assuming the displacement field in the form of Fourier series, their quantitative results are similar to the results obtained with the use of Eshelby's tensor.

After presentation of the general equations of the problem in Sect. 2, we successively consider the case of a material with a given homogeneous temperature field (Sect. 3) and with a nonhomogeneous transient temperature field (Sect. 4).

The case of homogeneous temperature field has been considered by HASHIN and ROSEN [11], LAWS [15], SEIICHI NOMURA and TSU-WEI CHOU [20]. A viscoelastic effect does not appear in this case since temperature gradients are absent. Under the assumption of homogeneity of deformation in each grain, we use an integral formulation very similar to an elastoplastic approach (thermal strains are similar to given plastic strains) by which we determinate the localization tensors.

In Sect. 4 the temperature field may be nonhomogeneous and transient. Then irreversibility due to heat gradients leads to viscoelastic macroscopic effects. We obtain a for-

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mulation of these effects via an integral formulation of the heat equation. Under specific simplifying assumptions, we derive localization tensors.

The methods employed here may be used in principle in other problems involving diffusion processes and coupling.

2. Governing equations

Consider a thermoelastic medium whose constitutive equations are given by:

1. The Duhamel–Neumann local law

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta, \tag{2.1}$$

where C_{ijkl} are the anisotropic elastic isothermal coefficients, β_{ij} are the components of the thermal expansion tensor, σ_{ij} is the Cauchy stress tensor, ε_{ij} is the infinitesimal strain tensor and θ measures temperature variation from some reference absolute uniform value T_0.

Equivalent forms of Eq. (2.1) are

$$\sigma_{ij} = C_{ijkl}(-\alpha_{kl}\theta + \varepsilon_{kl}),$$

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} + \alpha_{ij}\theta,$$

where S_{ijkl} are the elastic compliances and α_{ij} the components of the free thermal expansion tensor.

Identification of these equations requires

$$\beta_{ij} = C_{ijkl}\alpha_{kl},$$

2. The heat equation which governs the evolution of the temperature θ

$$C_e\dot{\theta} = (k_{ij}\theta_{,i})_{,j} - T_0\beta_{ij}\dot{\varepsilon}_{ij}, \tag{2.2}$$

where j represents the partial derivative $\dfrac{\partial}{\partial x_j}$, C_e is the heat capacity at constant deformation and k_{ij} is the heat conductivity tensor.

For a heterogeneous medium, the coefficients $C_{ijkl}, \beta_{ij}, C_e, k_{ij}$ are space-dependent.

We will only consider in this paper quasi-static problems. Then the equilibrium equations in the absence of body forces

$$\sigma_{ij,j} = 0 \tag{2.3}$$

complete the set of equations for the unknowns $\sigma_{ij}, \varepsilon_{ij}, \theta$.

In the following, we will examine two types of problems:

thermoelasticity with a known stationary time-independent temperature field θ,

thermoelasticity with a transient temperature field governed by the heat equation.

In both cases, we transform the governing equations and the boundary conditions to equivalent integral equations from which it is possible to derive, through and adequate approximate scheme, concentration tensors and effective properties at the macroscopic scale.

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3. Thermoelasticity with stationary temperature field

We assume in this section that the field θ of temperature is known, homogeneous and time-independent. This situation may occur in some stationary heat conduction problems. One may be interested in the calculation of local thermal stresses (which play an important role, for example, in crack initiation) as well as in the macroscopic behaviour of the material.

3.1. Integral equation

The Duhamel law (2.1) may be written in the following form:

$$\sigma_{ij} = C_{ijkl}\,\varepsilon^e_{kl}, \tag{3.1}$$

where the elastic strain ε^e_{kl} is the difference between the total strain ε_{kl} and thermal strain ε^a_{kl},

$$\varepsilon^e_{kl} = \varepsilon_{kl} - \varepsilon^a_{kl}, \tag{3.2}$$

$$\varepsilon^a_{kj} = \alpha_{kj}\theta. \tag{3.3}$$

An analogy appears between Eqs. (3.1), (3.2) and (2.3) of the stationary problem and the elastoplastic problem (KRÖNER [14]).

The thermal strain $\boldsymbol{\epsilon}^a$ plays a role similar to plastic strain in an elastoplastic problem. This analogy does not work of course when we consider the evolution law of the thermal deformation (heat equation for thermoelasticity, but flow law for plasticity). Nevertheless, for the stationary thermoelastic problem considered in this section, the evolution law of thermal strain $\varepsilon^a_{ij} = \alpha_{ij}\theta$ plays no role and the analogy is complete. Then we can apply methods used in the context of elastoplasticity (KRONER [14], BERVEILLER and ZAOUI [3]), to derive some results specific to stationary thermoelasticity.

Combining the Duhamel law (2.1.) and the equilibrium equations (2.3), we obtain

$$[C_{ijkl}(r)(u_{l,k}(r) - \varepsilon^a_{kl}(r))]_{,i} = 0, \tag{3.4}$$

where r is the position vector with the components x_1, x_2, x_3, and $u_{k,l}$ is the gradient of the total displacement.

Now we proceed to the transformation of Eqs. (3.4) to equivalent integral equations in a similar way as done by ZELLER and DEDERICHS [27] in the context of heterogeneous elasticity. We emphasize, as boundary conditions, that the total deformation

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{3.5}$$

on the boundary ∂v is equal to a given uniform strain E^0_{kl}.

Let us decompose the elastic tensor to a sum of an arbitrary uniform part and a space-dependent part:

$$C_{ijkl}(r) = C^0_{ijkl} + \tilde{C}_{ijkl}(r). \tag{3.6}$$

In the same way, from boundary conditions we decompose the gradient of total displacement as follows:

$$u_{i,j}(r) = U^0_{i,j} + \tilde{u}_{i,j}(r), \tag{3.7}$$

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where $\mathbf{U}^0(r)$ defined modulo a rigid body displacement is a solution of

$$E^0_{kl} = \frac{1}{2}(U^0_{k,l}+U^0_{l,k}).$$

Assuming the uniformity of E^0_{kl} in the body, it appears that

$$\tilde{u}_i(r) = 0 \quad \text{on} \quad \partial v.$$

Substitution of Eqs. (3.6) and (3.7) in Eq. (3.4) leads to the following equation:

$$C^0_{ijkl}\tilde{u}_{l,kl}(r)+f_j(r) = 0, \tag{3.8}$$

where f_j may be regarded as the j-th component of a fictitious body force given by

$$f_j = [\tilde{C}_{ijkl}u_{l,k}-C_{ijkl}\varepsilon^a_{kl}]_{,i}. \tag{3.9}$$

The Navier equation of elasticity (3.8) can be solved by the Green function method. We define, for the homogeneous elastic reference medium (with uniform elastic constant C^0_{ijkl} in the volume v), Green functions G_{km} solutions of the following equations:

$$C^0_{ijkl}G_{km,lj}(r,r')+\delta_{im}\delta(r-r') = 0 \tag{3.10}$$

with the boundary condition

$$G_{km} = 0 \text{ on } \partial v. \tag{3.11}$$

Let us observe that our presentation of integral equation relies upon a pure mathematical treatment. In the first place, we assume for a finite anisotropic body the existence of the Green tensor (which is generally not known; WILLIS [25)]. In the second place, we will give some simplifying arguments leading us to the use of the Green tensor for an infinitely extended anisotropic medium (whose computation is possible by using, for example, the Fourier transform; FAIVRE [9]).

In the formula (3.10), δ_{im} is the Kronecker symbol and $\delta(r-r')$ is the Dirac function. For a fixed m, $\delta_{im}.\delta(r-r')$ represents the i-th component of a unit concentrated force at point r' and parallel to the m axis. It is clear from Eq. (3.10) that G_{km} is the k-th displacement component at point r due to this force and that the solution $\tilde{u}$ of the Navier equation (3.8) is in the form

$$\tilde{u}_n(r) = \int_v G_{nj}(r,r')f_j(r')dr'. \tag{3.12}$$

Let us adopt a practical symbolism with the following notation:

$$\tilde{u}_n(r) = (G_{nj}*f_j)(r).$$

Using the expression (3.9) for fictitious forces in Eq. (3.12), then substituting Eq. (3.12) in Eq. (3.7) we require, after differentiation,

$$u_{n,m}(r) = U^0_{n,m}+[G_{nj,im}*(\tilde{C}_{ijkl}u_{l,k}-C_{ijkl}\varepsilon^a_{kl})](r). \tag{3.13}$$

Taking the symmetric part of $G_{nj,im}$ with respect to the couple (m,n), we define the tensor Γ:

$$\Gamma_{mnij} = \frac{1}{2}[G_{mj,in}+G_{nj,im}]. \tag{3.14}$$

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We get by symmetrization

$$\varepsilon_{mn}(r) = E^0_{mn} + [\Gamma_{mnij} * (\tilde{C}_{ijkl}\,\varepsilon_{kl} - C_{ijkl}\,\varepsilon^a_{kl})](r). \tag{3.15}$$

In these integral equations, the total strains are the only unknowns since the thermal strains are given as well as the uniform strain E^0_{mn} at the boundary.

Recalling that the thermal strain $\boldsymbol{\epsilon}^a$ is written

$$\varepsilon^a_{kl} = \alpha_{kl}\theta$$

we point out that the integral equations (3.15) constitute a functional relation of the local strain $\boldsymbol{\epsilon}$ and the uniform strain $\mathbf{E}^0$ for a given homogeneous temperature field θ. Using a self-consistent scheme for a granular material, we establish more precisely in the following a similar relation which expresses local strains in terms of averaged strains.

3.2. Concentration tensor and effective properties

Let us recall briefly the notion of concentration tensors $\mathbf{A}$ and $\mathbf{a}$ which link the local total strain $\boldsymbol{\epsilon}$ to the macroscopic total strain $\mathbf{E}$ and temperature θ by (see LAWS [15]):

$$\varepsilon_{ij}(r) = A_{ijkl}(r)E_{kl} - a_{ij}(r)\theta, \tag{3.16}$$

where r is the position vector. We can also get the concentration tensors $\mathbf{B}$ and $\mathbf{b}$ for the local stress $\boldsymbol{\sigma}$:

$$\sigma_{ij}(r) = B_{ijkl}(r)\Sigma_{kl} + b_{ij}(r)\theta, \tag{3.17}$$

where $\boldsymbol{\Sigma}$ is the macroscopic stress.

We assume that our heterogeneous medium statisfies Hill's hypothesis (no body forces, material with macrohomogeneous properties undergoes small strains, grain size is small compared to the size of the volume) under which the macroscopic stress $\boldsymbol{\Sigma}$ and strain $\mathbf{E}$ are the volume mean values of the local stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\epsilon}$,

$$\begin{aligned} \langle\boldsymbol{\sigma}\rangle &= \boldsymbol{\Sigma}, \\ \langle\boldsymbol{\epsilon}\rangle &= \mathbf{E}. \end{aligned} \tag{3.18}$$

We also have, under Hill's hypothesis,

$$\langle\sigma_{ij}\rangle\langle\varepsilon_{ij}\rangle = \langle\sigma_{ij}\,\varepsilon_{ij}\rangle. \tag{3.19}$$

By taking the mean value of Eqs. (3.16) and (3.17), we require

$$\langle\mathbf{A}\rangle = \mathbf{1} \quad \langle\mathbf{B}\rangle = \mathbf{1}, \quad \langle\mathbf{a}\rangle = \mathbf{0}, \quad \langle\mathbf{b}\rangle = \mathbf{0}, \tag{3.20}$$

where $\mathbf{1}$ is unit tensor.

When the concentration tensor $\mathbf{A}$ is known, it is an easy task to obtain the effective coefficients in the macroscopic Duhamel law,

$$\Sigma_{ij} = C^{\text{eff}}_{ijkl}E_{kl} - \beta^{\text{eff}}_{ij}\theta, \tag{3.21}$$

where $\mathbf{C}^{\text{eff}}$ and $\boldsymbol{\beta}^{\text{eff}}$ are the effective elastic and thermal expansion tensors. Using the relation (3.16) and the local constitutive law (2.1), LAWS [15] has obtained the following result:

$$\begin{aligned} \mathbf{C}^{\text{eff}} &= \langle\mathbf{CA}\rangle, \\ \boldsymbol{\beta}^{\text{eff}} &= \langle\boldsymbol{\beta}\mathbf{A}\rangle, \end{aligned} \tag{3.22}$$

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where the tensor **CA** represents the contracted product of **C** and **A**. As we assume θ is known, the constitutive equation (3.21) is sufficient to describe the behaviour of the material.

In a similar manner, we can write

$$E_{ij} = S^{\text{eff}}_{ijkl}\Sigma_{kl} + \alpha^{ij}_{\text{eff}}\theta \tag{3.23}$$

with

$$\begin{aligned} \mathbf{S}^{\text{eff}} &= \langle \mathbf{SB} \rangle, \\ \boldsymbol{\alpha}^{\text{eff}} &= \langle \boldsymbol{\alpha}\mathbf{B} \rangle. \end{aligned} \tag{3.24}$$

Let us note the following relations (see NOMURA *et al.* [20]):

$$\boldsymbol{\beta}^{\text{eff}} = \mathbf{C}^{\text{eff}}\boldsymbol{\alpha}^{\text{eff}} = \langle \boldsymbol{\beta}\mathbf{A} \rangle = \langle (\mathbf{C}\boldsymbol{\alpha})\mathbf{A} \rangle.$$

The preceding functional relation (3.16) of the total local strain $\boldsymbol{\epsilon}$ and the macroscopic strain **E** for a given homogeneous temperature field θ is often of greater value (see S. NEMAT-NASSER *et al.* [19]) than the functionnal relation of $\boldsymbol{\epsilon}$ and the boundary uniform strain $\mathbf{E}^0$ given by the integral equations (3.15). Such a relation may be formally derived from the integral equations in that way:

we define the tensors $\mathbf{A}^0$ and $\mathbf{a}^0$ such that Eqs. (3.15) may be formally written as

$$\varepsilon_{mn}(r) = A^0_{mnij}(r)E^0_{ij} - a^0_{mn}(r)\theta.$$

After averaging and using Eqs. (3.18), we deduce the relation between the total strain $\boldsymbol{\epsilon}$, the uniform temperature θ and the macroscopic deformation **E**

$$\boldsymbol{\epsilon} = \mathbf{A}^0\langle \mathbf{A}^0 \rangle^{-1}\mathbf{E} - [\mathbf{a}^0 - \mathbf{A}^0\langle \mathbf{A}^0 \rangle^{-1}\langle \mathbf{a}^0 \rangle]\theta$$

which is of the conventional form

$$\boldsymbol{\epsilon} = \mathbf{AE} - \mathbf{a}\theta$$

with

$$\mathbf{A} = \mathbf{A}^0\langle \mathbf{A}^0 \rangle^{-1}$$

and

$$\mathbf{a} = -\mathbf{A}^0\langle \mathbf{A}^0 \rangle^{-1}\langle \mathbf{a}^0 \rangle + \mathbf{a}^0.$$

This scheme has been indicated by Beradai and Berveiller (to be published in Cahiers du Groupe Français de Rhéologie) in the context of elastoplasticity.

For a macrohomogeneous granular material, we will present an equivalent result where some simplifications are first introduced in the integral equations.

3.3. Equivalent form of integral equation

Recalling from Hill's hypothesis that the volume of the body is large enough to be regarded as an infinitely extended medium, Green's tensor **G** becomes the Green-tensor for an infinitely extended homogeneous body with elastic coefficients C^0_{ijkl}:

$$G_{jn}(r, r') = G_{jn}(r - r') \tag{3.25}$$

with $G_{jn} = 0$ and $G_{jn,m} = 0$ at infinity.

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We will describe the field of deformations in terms of the elastic- and the thermal-components, respectively $\boldsymbol{\epsilon}^e$ and $\boldsymbol{\epsilon}^a$, of the total strain $\boldsymbol{\epsilon}$

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^a_{ij}.$$

The same decomposition is formally taken for the uniform strain $\mathbf{E}^0$, *this will be justified by using the self-consistent scheme*

$$E^0_{ij} = E^{0e}_{ij} + E^{0a}_{ij}. \tag{3.26}$$

Taking account of Eq. (3.25), the integral equations appear as convolutions. For example, we note Eq. (3.12) in the form

$$\tilde{u}_n(r) = \int_v G_{nj}(r-r') f_j(r') dr' = (G_{nj} * f_j)(r). \tag{3.27}$$

Substituting the modified Green tensor Γ_{mnij} from Eq. (3.14) into Eq. (3.15), the integral equations may be written as

$$\varepsilon_{mn}(r) = E^0_{mn} + \frac{1}{2}[G_{nj,m} + G_{mj,n}] * [\tilde{C}_{ijlk}\varepsilon_{kl} - C_{ijkl}\varepsilon^a_{kl}]_{,i}(r). \tag{3.28}$$

By uniformity of $\mathbf{C}^0$ and $\mathbf{E}^a$, the term

$$[C^0_{ijkl} E^{0a}_{kl}]_{,i}$$

is zero.

Then, by introducing it in the last term of the preceding equation, we obtain

$$\varepsilon_{mn} = E^0_{mn} + \Gamma_{mnij} * [\tilde{C}_{ijkl}\varepsilon_{kl} - C_{ijkl}\varepsilon^a_{kl} + C^0_{ijkl}E^{0a}_{kl}]. \tag{3.29}$$

Consequently, using the decompositions of $\boldsymbol{\epsilon}$ (3.2) and $\mathbf{E}^0$ (3.26), we get an equivalent form of Eq. (3.29) in terms of the pure local elastic strain $\boldsymbol{\epsilon}^e$, the macroscopic one $\mathbf{E}^{0e}$ and the thermal deviation $\boldsymbol{\epsilon}^a - \mathbf{E}^{0a}$:

$$\varepsilon^e_{mn} = E^{0e}_{mn} - (\varepsilon^a_{mn} - E^{0a}_{mn}) + \Gamma_{mnij} * [\tilde{C}_{ijkl}\varepsilon^e_{kl} - C^0_{ijkl}(\varepsilon^a_{kl} - E^{0e}_{kl})]. \tag{3.30}$$

Introducing tensor $\boldsymbol{\alpha}^0$ such that

$$\mathbf{E}^{0a} = \boldsymbol{\alpha}^0\theta,$$

we obtain an equivalent concentration relation in terms of macroscopic elastic strain and the uniform temperature:

$$\varepsilon^e_{mn} = E^{0e}_{mn} - (\alpha_{mn} - \alpha^0_{mn})\theta + \Gamma_{mnij} * [\tilde{C}_{ijkl}\varepsilon^e_{kl} - C^0_{ijkl}(\alpha_{kl} - \alpha^0_{kl})\theta].$$

An approximate solution of these linear integral equations can be sought by looking at the piecewise constant solution $\boldsymbol{\epsilon}^e$ on a subdivision of the infinitely extended volume. We propose in the following a discretization based on a finite number of elements by using a self-consistent scheme; the subdomains are grains constituting a representative finite volume of the body.

3.4. Granular material, self-consistent scheme

We define the characteristic function of grain α:

$\Delta^\alpha(r) = 1$ for r located in grain α of volume V_α,

$\Delta^\alpha(r) = 0$ for r outside of grain α.

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The discretization is based on a simple approximation which is to replace for each grain all characteristic properties by their volume averages over the grain. Such piecewise uniformity has been previously used and given accurate results for low and moderate concentration (see S. NEMAT-NASSER *et al.* [19]; BERVEILLER [2]).

Taking account of this approximation, it appears that for an infinitely extended medium, any parameter should be practically given in terms of a sum over an infinite number of grains. Defining the average on the grain α of any space-dependent variable (say $Q(r)$) by

$$Q^{\alpha} = \frac{1}{V_{\alpha}} \int_{V_{\alpha}} Q(r)\,dr,$$

the deviation of elastic properties from the reference homogeneous medium

$$\tilde{\mathbf{C}}(r) = \mathbf{C}(r) - \mathbf{C}^{0}$$

can be written as

$$\tilde{C}_{ijkl}(r) = \sum_{\alpha=1}^{\infty} [C^{\alpha}_{ijkl} - C^{0}_{ijkl}]\Delta^{\alpha}(r).$$

However, using a N-sites self-consistent scheme, we propose an acceptable approximation yielding a finite summation: recalling that the grain-sizes are small compared to the size of the volume, we assimilate the body to an infinitely extended medium constituted of a finite aggregate of N grains embedded in a matrix whose characteristics are those of the homogeneous equivalent medium. This means that we neglect any contributions of grains located at the boundaries and that we focus our attention on the N grains constituting the major volume part *representative* of the body.

The N-sites self-consistent scheme. In this case the uniform strain $\mathbf{E}^{0}$ is equal to the macroscopic deformation

$$\mathbf{E}^{0} = \mathbf{E}$$

and we justify the notion adopted for the elastic and thermal components

$$\mathbf{E}^{0e} = \mathbf{E}^{e},$$
$$\mathbf{E}^{0a} = \mathbf{E}^{a}.$$

Looking at integral equations, it is possible to proceed to a discretization over a finite number of grains. Assuming a uniform temperature θ, the deviation of thermal strain $\boldsymbol{\epsilon}^{a} - \mathbf{E}^{a}$ is constant in the grain α and may be first written in terms of an infinite serie:

$$\boldsymbol{\epsilon}^{a}(r) - \mathbf{E}^{a} = \sum_{\alpha=1}^{N} (\boldsymbol{\epsilon}^{a\alpha} - \mathbf{E}^{a})\Delta^{\alpha}(r) + \sum_{\alpha=N+1}^{\infty} (\boldsymbol{\epsilon}^{a\alpha} - \mathbf{E}^{a})\Delta^{\alpha}(r)$$

but the last summation is equal to zero due to the homogeneity of the surrounding matrix (for $\alpha > N$, grain corresponds to the matrix). Consequently, the discretization of Eqs. (3.30) will be expressed in terms of finite summations by using the following equalities:

Deviation of thermal strain

$$\boldsymbol{\epsilon}^{a}(r) - \mathbf{E}^{a} = \sum_{\alpha=1}^{N} (\boldsymbol{\epsilon}^{a\alpha} - \mathbf{E}^{a})\Delta^{\alpha}(r). \tag{3.31}$$

Deviation of elastic properties

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The coefficients C^0_{ijkl} and $\boldsymbol{\alpha}^0$ of the surrounding matrix are those of the homogeneous equivalent medium. We note:

$$C^{\text{eff}}_{ijkl} = C^0_{ijkl} \quad \text{and} \quad \boldsymbol{\alpha}^{\text{eff}} = \boldsymbol{\alpha}^0$$

and the discretization is

$$\tilde{\mathbf{C}}(r) = \mathbf{C}(r) - \mathbf{C}^0 = \sum_{\alpha=1}^{N} (\mathbf{C}^\alpha - \mathbf{C}^{\text{eff}}) \Delta^\alpha(r),$$

(3.32)

$$\tilde{\boldsymbol{\alpha}}(r) = \boldsymbol{\alpha}(r) - \boldsymbol{\alpha}^0 = \sum_{\alpha=1}^{N} (\boldsymbol{\alpha}^\alpha - \boldsymbol{\alpha}^{\text{eff}}) \Delta^\alpha(r).$$

Elastic strain

We assume that the elastic strain is uniform in each grain. ESHELBY [8] has proved this to be exact in the context of an elastoplastic inclusion problem when the plastic strain is supposed to be uniform in an ellipsoidal inclusion even if the elastic coefficient of the matrix and the inclusion are different. Our assumption of uniform elastic deformation in each grain, although not rigorously satisfied, may be considered as a reasonable approximation for ellipsoidal grains.

$$\boldsymbol{\epsilon}^e(r) = \sum_{\alpha=1}^{N} \boldsymbol{\epsilon}^{e\alpha} \Delta^\alpha(r). \tag{3.33}$$

After substitution of the strains $\boldsymbol{\epsilon}^e$, $\boldsymbol{\epsilon}^a$ given by Eqs. (3.33) and (3.32) in the integral equation (3.30), we get

$$\begin{aligned} \varepsilon^e_{mn}(r) = E^e_{mn} - \sum_{\alpha=1}^{N} (\varepsilon^{a\alpha} - E^a_{mn}) \Delta^\alpha(r) \\ + \sum_{\alpha=1}^{N} [\tilde{C}^\alpha_{ijkl} \varepsilon^{e\alpha}_{kl} - C^{\text{eff}}_{ijkl}(\varepsilon^{a\alpha}_{kl} - E^a_{kl})] \int_V \Gamma_{mnij}(r-r') \Delta^\alpha(r') dr'. \end{aligned} \tag{3.34}$$

It turns out that the elastic deformation ε^e_{mn} given by Eq. (3.34) may be nonuniform over a grain α. For consistency with our hypothesis of uniformity (3.33), we define the volume mean value on the grain β of the elastic deformation:

$$\varepsilon^{e\beta}_{mn} = \frac{1}{V_\beta} \int_{V_\beta} \varepsilon^e_{mn} dV^\beta. \tag{3.35}$$

Consequently, the discretized equation becomes for $\beta = 1, \ldots, N$

$$\varepsilon^{e\beta}_{mn} = E^e_{mn} - (\varepsilon^{a\beta}_{mn} - E^a_{mn}) + \sum_{\alpha=1}^{N} \Gamma^{\beta\alpha}_{mnij} [\tilde{C}^\alpha_{ijkl} \varepsilon^{e\alpha}_{kl} - C^{\text{eff}}_{ijkl}(\varepsilon^{a\alpha}_{kl} - E^a_{kl})], \tag{3.36}$$

where

$$\Gamma^{\beta\alpha}_{mnij} = \frac{1}{V_\beta} \int_{V_\alpha} \int_{V_\beta} \Gamma_{mnij}(r^{(\beta)} - r^{(\alpha)}) dv^{(\beta)} dv^{(\alpha)}. \tag{3.37}$$

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Substituting the thermal strains, we obtain in terms of θ

$$\varepsilon_{mn}^{e\beta} = E_{mn}^{e} - \tilde{\alpha}_{mn}^{\beta}\theta + \sum_{\alpha=1}^{N} \Gamma_{mnij}^{\beta\alpha}[\tilde{C}_{ijkl}^{\alpha}\varepsilon_{kl}^{e\alpha} - C_{ijkl}^{\mathrm{eff}}\tilde{\alpha}_{kl}^{\alpha}\theta]. \tag{3.38}$$

Recalling that the thermal strains ($\boldsymbol{\epsilon}^{a\alpha} = \boldsymbol{\alpha}^{\alpha}\theta$) are known since θ is given, the N tensorial equations (3.36) constitute a linear system describing the field of elastic deformation as a function of the macroscopic elastic strain $\mathbf{E}^e$ and the temperature field θ.

The $N+1$ tensorial unknowns are namely the N elastic deformations $\boldsymbol{\epsilon}^{e\beta}$ (for $\beta = 1$ to N) and the effective elastic tensor $\mathbf{C}^{\mathrm{eff}}$. It appears that the elastic deformation in grain β is the superposition of the macroscopic elastic strain $\mathbf{E}^e$ and complicated interactions between grains characterized by the coupling terms $\boldsymbol{\Gamma}^{\beta\alpha}$. The supplementary equation is given by the self-consistency condition

$$\mathbf{E} = \sum_{\gamma=1}^{N} f^{\gamma}\boldsymbol{\epsilon}^{\gamma} \tag{3.39}$$

which means (cf. Hill' results (3.18)) that the macroscopic strain $\mathbf{E}$ is the volumic mean value of microscopic strain $\boldsymbol{\epsilon}^{\gamma}$ and where f^{γ} is the voluminal fraction of grain γ:

$$f^{\gamma} = \frac{V^{\gamma}}{V}$$

(3.40) and

$$\boldsymbol{\epsilon}^{\beta} = \boldsymbol{\epsilon}^{e\beta} + \boldsymbol{\epsilon}^{a\beta}.$$

The scheme of calculation is the so-called N-site self-consistent scheme. We propose in Appendix 3 a presentation of the linear system in a matrix form which may be resolved with numerical computation. Such a resolution may be rather complicated for a great number of grains: however, it enables us to obtain a concentration relation which gives the interactions of a grain with its nearest neighbours.

The one-site self-consistent scheme. It is well known that accurate results are obtained when using a one-site self-consistent scheme in case of a matrix having perfectly disordered grains. Looking at Eqs. (3.36) and assuming that only one grain (β) is embedded in the matrix, we obtain

$$\varepsilon_{mn}^{e\beta} = E_{mn}^{e} - (\varepsilon_{mn}^{a\beta} - E_{mn}^{a}) + \Gamma_{mnij}^{\beta\beta}[\tilde{C}_{ijkl}^{\beta}\varepsilon_{kl}^{e\beta} - C_{ijkl}^{\mathrm{eff}}(\varepsilon_{kl}^{a\beta} - E_{kl}^{a})].$$

Introducing the following tensor $l_{ijkl}^{\beta\alpha}$

$$\mathbf{l}^{\alpha\beta} = \delta_{\alpha\beta}\mathbf{1}, \tag{3.41}$$

where $\delta_{\alpha\beta}$ is the Krönecker symbol and $\mathbf{1}$ the fourth order unit tensor

$$1_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

we get the approximate strain localization law (no summation for repeated Greek indices):

$$\begin{aligned}\varepsilon_{kl}^{e\beta} = {} & [l_{mnkl}^{\beta\beta} - \Gamma_{mnij}^{\beta\beta}\tilde{C}_{ijkl}^{\beta}]^{-1}E_{mn}^{l} \\ & - [l_{mnkl}^{\beta\beta} - \Gamma_{mnij}^{\beta\beta}\tilde{C}_{ijkl}^{\beta}]^{-1}[1_{mnrs}^{\beta\beta} + \Gamma_{mnpq}^{\beta\beta}C_{pqrs}^{\mathrm{eff}}](\varepsilon_{rs}^{a\beta} - E_{rs}^{a})\end{aligned} \tag{3.42}$$

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which may be written for each grain $\beta(\beta = 1, \ldots, N)$ representative of the material. We add to these N tensorial equations with $N+1$ unknowns (namely, $\boldsymbol{\epsilon}^{e\beta}$ for $\beta = 1$ to N and $\mathbf{C}^{\mathrm{eff}}$) the self-consistency condition (3.39)

$$\mathbf{E} = \sum_{\beta=1}^{N} f^{\beta}\boldsymbol{\epsilon}^{\beta}.$$

Then one possible computation involves the substitution of $\boldsymbol{\epsilon}^{e\beta}$ from Eqs. (3.42) and (3.40) into the self-consistency condition. The resulting tensorial equation contains the tensorial unknown $\mathbf{C}^{\mathrm{eff}}$ which appears implicit due to the expression of the tensor $\boldsymbol{\Gamma}^{\beta\beta}$ ((3.37), (3.14), (3.10) with $C^{\mathrm{eff}} = \mathbf{C}^0$). This method is also applicable but more complex for the N-site self-consistent scheme (see BERVEILLER, ZAOUI [3] in the context of elasto-plasticity).

We proceed in the following to the treatment of thermoelasticity with a transient temperature field. The formulae indicated above dealing with thermostatics (LAWS [15]), although formal, are rather classical. For consideration, quantitative results are given by NOMURA *et al.*, [20] for a binary glass-epoxy composite system; LAWS [16] in the case of a transversely isotropic composite (by using a self-consistent method) and BUDIANSKY [6] for an isotropic material.

4. Thermoelasticity with transient temperature field

We consider now problems with nonhomogeneous and transient temperature fields. To the Duhamel–Neumann law (2.1) and equilibrium equations (2.3) we must add the heat equation (2.2): we will have temperature gradients and viscoelastic effects.

In the first part, we derive an integral formulation for the heat equation. Furthermore, in the same manner as in Sect. 3 we obtain an integral equation in which the local and macroscopic strains appear. Under certain simplifications we get, for a granular material, concentration relations which give the local strain as a linear function of all the past history of the macrostrain. Then the macroscopic constitutive law appears to be the same as that for a linear viscoelastic medium.

4.1. Integral formulation of heat equation

The transformation of the heat equation (2.2)

$$C_e(r)\dot{\theta}(r, t) = [k_{ij}(r)\theta_{,j}(r, t)]_{,i} + H(r, t) \tag{4.1}$$

with the source term

$$H(r, t) = -T_0\beta_{ij}(r)\dot{\varepsilon}_{ij}(r, t) \tag{4.2}$$

into an integral equation starts with the following first step.

Let us consider a decomposition of heat capacity and conductivity tensor into an arbitrary uniform part and a fluctuating part:

$$\begin{aligned} C_e(r) &= C_e^0 + \tilde{C}_e(r), \\ k_{ij}(r) &= k_{il}^0 + \tilde{k}_{ij}(r). \end{aligned} \tag{4.3}$$

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By $\psi(r, t/r', t')$ we denote the elementary solution of the heat equation for a homogeneous material C_e^0, k_{ij}^0 in a domain V initially at uniform temperature and subjected at time $t = 0$ to an instantaneous heat deposit located at point r':

$$C_e^0 \frac{\partial \Psi}{\partial t}(r, t/r', t') = k_{ij}^0 \frac{\partial^2 \Psi}{\partial x_i \partial x_j}(r, t/r', t') + \delta(r-r')\delta(t-t') \tag{4.4}$$

with the boundary condition

$$\psi(a, t/r', t') = 0 \tag{4.5}$$

for (a) located on the boundary ∂V.

In Appendix 2, we derive the integral equation governing the evolution of the solution θ of the equation (4.1):

$$\begin{aligned}\theta(r, t) = {} & \int_0^t \int_v \Psi(r, t/r', t') H(r', t') dr' dt' + C_e^0 \int_v \Psi(r, t/r', 0)\theta(r', 0) dr' \\ & - \int_0^t \int_{\partial V} [\Psi(r, t/r', t') q_\theta(r', t') - \theta(r', t') \cdot q_\psi(r, t/r', t')] dS dt' \\ & - \int_0^t \int_v \left(\tilde{C}_e(r') \Psi(r, t/r', t') \theta(r', t') - \theta(r', t') \frac{\partial}{\partial x_i'} \left[\tilde{k}_{ij}(r') \frac{\partial}{\partial x_j'} \Psi(r, t/r', t') \right] \right) dr dt'.\end{aligned} \tag{4.6}$$

The generalized heat flux q_θ is defined by

$$q_\theta(r, t) = -\mathbf{n}(r)[\mathbf{k}\,\mathrm{grad}\,\boldsymbol{\theta}](r, t) \tag{4.7}$$

(same form for q_Ψ with θ replaced by Ψ), where $\mathbf{n}$ is the external normal of the boundary S of volume V.

The solution θ appears in an implicit form in Eq. (4.6). The first three terms in the second member of Eq. (4.6) take, respectively, account of heat deposit by the source term H, initial conditions θ (r, θ) and thermal boundary conditions involving the temperature θ and the gradient q_θ on boundary S.

The last term where θ appears as an implicit unknown, involves the heterogeneous parts $\tilde{C}_e$ and $\tilde{\mathbf{k}}$ of heat capacity and conductivity tensor.

4.2. Recapitulation of the governing equations

Equation (3.15),

$$\varepsilon_{mn}(r) = E_{mn}^0 + \Gamma_{mnij} * [\tilde{C}_{ijkl}\varepsilon_{kl} - C_{ijkl}\varepsilon_{kl}^a](r) \tag{4.8}$$

or its equivalent form (see p. 3.3)

$$\varepsilon_{mn}^e = E_{mn}^{0e} - (\varepsilon_{mn}^a - E_{mn}^{0a}) + \Gamma_{mnij} * [\tilde{C}_{ijkl}\varepsilon_{kl}^e - C_{ijkl}^0(\varepsilon_{kl}^a - E_{kl}^{0a})]$$

which were derived from the equilibrium equations (2.3), in the case of an infinitely extended

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medium, with the Duhamel–Neumann law (2.1) still remains valid at each instant. Let us recall that the thermal strain is related to the temperature field θ by

$$\varepsilon^a_{kl}(r, t) = \alpha_{kl}(r)\theta(r, t). \tag{4.9}$$

The tensor Γ_{mnij} is related through the formula (3.14) to the Green tensor $\mathbf{G}$ of the boundary problem for an infinitely extended homogeneous elastic medium whose elastic coefficients C^0_{ijkl} differ from $\tilde{C}_{ijkl}$ by C_{ijkl}.

Our aim is to show that the system of equations (4.8), (4.9), (4.6) (representative of the constitutive law of the material) can be reduced to a form which exhibits the viscoelastic behaviour. We simplify the integral equation (4.6) governing the evolution of temperature by neglecting the three following terms:

a) We assume that the body consists of N grains embedded in an infinitely extended matrix with uniform properties $\mathbf{C}^0$, $\mathbf{k}^0, \boldsymbol{\alpha}^0$, C^0_e (see justification in paragraph (3.4) which involves the use of a N-sites self-consistent scheme).

In the sequel the uniform quantities $\mathbf{C}^0$, $\mathbf{k}^0, \boldsymbol{\alpha}^0$, C^0_e, $\mathbf{E}$ correspond to the homogeneous equivalent medium. In such a case it can be shown (Appendix 1) that the elementary solution ψ has the following form:

$$\Psi(r, t/r', t') = \Psi(r-r', t-t')$$

(4.10) *and*

$$\Psi(r, t) = [|\det(C^0_e k^{ij}_0)|]^{\frac{1}{2}} (4\Pi t)^{-3/2} \exp\left[-\frac{C^0_e k^{m_1}_0 x_m x_l}{4t}\right].$$

Let us note the boundary properties:

$$\Psi(r, t) = 0 \quad \textit{and} \quad q_\Psi(r, t) = 0 \textit{ for } r \textit{ located at infinity.}$$

Still supposing that the matrix is infinitely extended, we neglect the contribution of the heat boundary-fluxes q_θ and q_ψ at the location of the cluster. It appears that the third term in Eq. (4.6) disappears.

b) The evolution of initial temperature is described by the term

$$\int_V \Psi(r-r', t)\theta(r' 0)dr'.$$

This term becomes almost stationary for large time and does not create thermomechanical irreversibility. So we can neglect it for our purposes.

c) The last term in Eq. (4.6) also disappears for a material whose thermal properties C_e and $\mathbf{k}$ are assumed to be homogeneous:

$$\tilde{C}_e = 0,$$
$$\tilde{\mathbf{k}} = 0.$$

We only assume that the following properties are nonuniform:

$$\tilde{C} \neq 0, \quad \tilde{\boldsymbol{\alpha}} \neq 0 \quad \text{and} \quad \tilde{\boldsymbol{\beta}} \neq 0.$$

Let us note that the general problem, without these simplifications, can be solved for the granular material, but the algebra is more complicated.

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From our simplifications Eq. (4.6) reads with H given by Eq. (4.2)

$$\theta(r, t) = -\int_0 \int_v \Psi(r-r', t-t') T_0 \beta_{ij}(r') \dot{\varepsilon}_{ij}(r', t') dr' dt' \tag{4.11}$$

and it follows after substitution in the Duhamel law (2.1) that

$$\sigma_{ij}(r, t) = C_{ijkl}(r) \varepsilon_{kl}(r, t) + \beta_{ij}(r) \int_0^t \int_v \Psi(r-r', t-t') T_0 \beta_{kl}(r') \dot{\varepsilon}_{kl}(r', t') dr' dt' \tag{4.12}$$

which is a nonlocal (1) hereditary constitutive law.

4.3. Granular material

We adopt the same discretization scheme as in Sect. 3.4 for the cluster constituted of N grains embedded in the homogeneous equivalent medium. In each grain, say α, the total strain $\boldsymbol{\epsilon}(r, t)$ and temperature $\theta(r, t)$ may be approximated by their mean values:

$$\boldsymbol{\epsilon}^{\alpha}(t) = \frac{1}{V_\alpha} \int_{V_\alpha} \boldsymbol{\epsilon}(r, t) dV_\alpha, \tag{4.13}$$

$$\theta^{\alpha}(t) = \frac{1}{V_\alpha} \int_{V_\alpha} \theta(r, t) dV_\alpha. \tag{4.14}$$

The thermal strain $\boldsymbol{\epsilon}^{a\alpha}$ and the elastic strain $\boldsymbol{\epsilon}^{e\alpha}$ are also assumed uniform in grain α:

$$\begin{aligned} \boldsymbol{\epsilon}^{a\alpha}(t) &= \boldsymbol{\alpha}^{\alpha} \theta^{\alpha}(t), \\ \boldsymbol{\epsilon}^{\alpha}(t) &= \boldsymbol{\epsilon}^{a\alpha}(t) + \boldsymbol{\epsilon}^{e\alpha}(t). \end{aligned} \tag{4.15}$$

Under these assumptions, the discretized form of Eq. (3.38) in terms of $\boldsymbol{\epsilon}^{e\alpha}$ and $(\boldsymbol{\epsilon}^{a} - \mathbf{E}^{a})^{\alpha}$ still remains valid at each instant

$$\varepsilon^{e\beta}_{mn}(t) = E^{e}_{mn}(t) - (\varepsilon^{a\beta}_{mn} - E^{a}_{mn})(t) + \sum_{\alpha=1}^{N} \Gamma^{\beta\alpha}_{mnij} [\tilde{C}^{\alpha}_{ijkl} \varepsilon^{e\alpha}_{kl} - C^{0}_{ijkl} (\varepsilon^{a\alpha}_{kl} - E^{a}_{kl})](t).$$

Recalling that the integral equation (3.15) (or (4.8)) is equivalent to Eq. (3.30), we would rather work with the following discretized form of Eq. (4.8) which appears in terms of the piecewise uniform functions $\boldsymbol{\epsilon}^{\alpha}$ and $\boldsymbol{\epsilon}^{a\alpha}$:

$$\varepsilon^{\beta}_{mn} = E_{mn} + \sum_{\alpha=1}^{N} \Gamma^{\beta\alpha}_{mnij} (\tilde{C}^{\alpha}_{ijkl} \varepsilon^{\alpha}_{kl} - C^{\alpha}_{ijkl} \varepsilon^{a\alpha}_{kl}) \tag{4.16}$$

for $(\beta = 1, \ldots, N)$ (2).

(1) Stress at point r depends on values of strains in the entire volume V.

(2) Let us note that the equivalent integral equation (3.38) in terms of the deviation of the pure local thermal and elastic strains was first necessary to justify finite summations over the N representative grains embedded in the infinitely extended equivalent medium.

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Substituting the expression of $\boldsymbol{\epsilon}$ from

$$\boldsymbol{\epsilon}(r) = \sum_{\gamma=1}^{N} \boldsymbol{\epsilon}^{\gamma} \Delta^{\gamma}(r)$$

in the heat integral equation (4.11) gives

$$\theta(r, t) = -\int_0^t T_0 \sum_{\gamma=1}^{N} \beta_{ij}^{\gamma} \dot{\varepsilon}_{ij}^{\gamma}(t') \int_v \Psi(r-r', t-t') \Delta^{\gamma}(r') dr' dt'. \tag{4.17}$$

From thi swe deduce the mean value θ^{α} of temperature on a grain α:

$$\theta^{\alpha}(t) = \frac{1}{V_{\alpha}} \int_{V_{\alpha}} \theta(r, t) \Delta^{\alpha}(r) dr = \sum_{\gamma=1}^{N} \left[-T_0 \beta_{ij}^{\gamma} \int_0^t \dot{\varepsilon}_{ij}^{\gamma}(\tau) \Psi^{\alpha\gamma}(t-\tau) d\tau \right], \tag{4.18}$$

where $\Psi^{\alpha\gamma}$ is a coupling term function characterizing the influence of grain γ on grain α:

$$\Psi^{\alpha\gamma} = (t, \tau) \frac{1}{V_{\alpha}} \int_{V_{\gamma}} \int_{V_{\alpha}} \Psi(r-r', t-\tau) dr' dr. \tag{4.19}$$

Then Eq. (4.16) takes the form

$$\varepsilon_{mn}^{\beta}(t) = E_{mn}(t) + \sum_{\alpha=1}^{N} \Gamma_{mnij}^{\beta\alpha} \left[\tilde{C}_{ijkl}^{\alpha} \varepsilon_{kl}^{\alpha} + C_{ijkl}^{\alpha} \alpha_{kl}^{\alpha} \sum_{\gamma=1}^{N} T_0 \beta_{pq}^{\gamma} \int_0^t \dot{\varepsilon}_{pq}^{\gamma}(\tau) \Psi^{\alpha\gamma}(t-\tau) d\tau \right]. \tag{4.20}$$

This is an integro-differential system where the $\boldsymbol{\epsilon}^{\beta}$ are the N tensorial unknowns and where the macroscopic deformation $\mathbf{E}(t)$ is known by assumption. As $\boldsymbol{\epsilon}^{\beta}$ depends linearly upon $\mathbf{E}$, it appears that

$$\varepsilon_{ij}^{\beta}(t) = \int_0^t J_{ijkl}^{\beta}(t-\tau) E_{kl}(\tau) d\tau, \tag{4.21}$$

where $J_{ijkl}^{\beta}(t)$ are creep localization tensors.

Let us consider the following special case:

$$E_{mn}(t) = \delta(t-t') \delta_{\alpha m} \delta_{\eta n}, \tag{4.22}$$

we denote by $\delta(t-\tau)$ the Dirac function at τ. Then, putting Eq. (4.22) in the law (4.21) gives

$$\varepsilon_{ij}^{\beta}(t) = J_{ij\alpha\eta}^{\beta}(t-t').$$

It follows that $J_{ij\alpha\eta}^{\beta}(t-t')$ is the total strain in grain β at time t, resulting from a macroscopic Dirac input applied at time t' having the form (4.22).

The localization law (4.21) has the same form as the elastic localization law:

$$\varepsilon_{ij}^{\beta} = A_{ijkl}^{\beta} E_{kl} \tag{4.23}$$

but in Eq. (4.21) we have an integral taking account of the past history of the applied macro deformation $\mathbf{E}$. It is easy to derive the stress in grain α as a function of $\mathbf{E}$ using (4.18), (4.21) and the discretized form of the Duhamel–Neumann law (2.1):

$$\sigma_{ij}^{\alpha} = C_{ijkl}^{\alpha} \varepsilon_{kl}^{\alpha} - \beta_{ij}^{\alpha} \theta^{\alpha}.$$

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Substituting ε_{kl} from Eq. (4.21) and θ from Eq. (4.18), we obtain

$$\sigma_{ij}^{\alpha}(t) = C_{ijkl}^{\alpha}\Big[\int_0^t J_{klmn}^{\alpha}(t-\tau)E_{mn}(\tau)d\tau\Big]+\beta_{ij}^{\alpha}\Big[\sum_{\gamma=1}^{N} T_0\beta_{pq}^{\gamma}\int_0^t \dot{\varepsilon}_{pq}^{\gamma}(\tau)\Psi^{\alpha\gamma}(t-\tau)d\tau\Big]. \tag{4.24}$$

In the sense of the time derivation of distributions (SCHWARTZ [21]), the last term of this expression reads (with the boundary condition (4.10))

$$\beta_{ij}^{\alpha}\sum_{\gamma=1}^{N} T_0\beta_{pq}^{\gamma}\int_0^t \varepsilon_{pq}^{\gamma}(\tau)\dot{\Psi}^{\alpha\gamma}(t-\tau)d\tau$$

which becomes, by substituting again $\varepsilon_{pq}^{\gamma}(\tau)$ from Eq. (4.21),

$$\beta_{ij}^{\alpha}\sum_{\gamma=1}^{N} T_0\beta_{pq}^{\gamma}\int_0^t \dot{\Psi}^{\alpha\gamma}(t-\tau)\Big[\int_0^{\tau} J_{pqmn}^{\gamma}(\tau-\tau')E_{mn}(\tau')d\tau'\Big]d\tau.$$

Then, Eq. (4.24) becomes

$$\sigma_{ij}^{\alpha}(t) = \int_0^t C_{ijkl}^{\alpha}J_{klmn}^{\alpha}(t-\tau)E_{mn}(\tau)d\tau$$
$$+\int_0^t\int_0^{\tau}\sum_{\alpha=1}^{N}[T_0\beta_{ij}^{\alpha}\beta_{pq}^{\gamma}\dot{\psi}^{\alpha\gamma}(t-\tau)J_{pqmn}^{\gamma}(\tau-\tau')E_{mn}(\tau')]d\tau'd\tau.$$

Adopting the following notation:

$$B_{ijmn}^{\alpha}(t,t') = C_{ijkl}^{\alpha}J_{klmn}^{\alpha}(t)\,\partial(t')+\sum_{\gamma=1}^{N} T_0\beta_{ij}^{\alpha}\beta_{pq}^{\gamma}\dot{\psi}^{\alpha\gamma}(t)J_{pqmn}^{\gamma}(t')$$

the preceding complicated result may be written as

$$\sigma_{ij}^{\alpha}(t) = \int_0^t\int_0^{\tau} B_{ijmn}^{\alpha}(t-\tau,\tau-\tau')E_{mn}(\tau')d\tau'd\tau. \tag{4.25}$$

This expression appears to be a linear functional with argument E_{mn}.

We thus obtain a localization law through the formula (4.25) by which we have access to the macroscopic law

$$\Sigma_{ij}(t) = \int_0^t\int_0^{\tau} B_{ijmn}(t-\tau,\tau-\tau')E_{mn}(\tau')d\tau'd\tau.$$

This relation has the form of a macroscopic viscoelastic law where Σ_{ij} is given by

$$\Sigma_{ij} = \langle\sigma_{ij}\rangle = \sum_{\alpha=1}^{N}\frac{V_\alpha}{V}\sigma_{ij}^{\alpha}$$

and

$$B_{ijmn}(t-\tau,\tau-\tau') = \sum_{\alpha=1}^{N}\frac{V_\alpha}{V}B_{ijmn}^{\alpha}(t-\tau,\tau-\tau').$$

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Noting that we have obtained, for an infinitely extended medium, integral equations which involve convolution products in space and time, we point out that a more complex mathematical treatment of these products (see SCHWARTZ [21], where a formalism for finite domains is also investigated) would establish also this analogy with a macroscopic viscoelastic behaviour and without assuming the restriction (4.2c) on the fluctuations of thermomechanical properties (e.g. uniformity of specific heat and of conductivity-tensor but only space-dependence of the elasticity coefficients).

In the same spirit, the hypothesis (4.2.b) (e.g., initial temperature is zero) is not of crucial importance because the initial thermal gradients have, after a short time, a negligible contribution to thermoelastic coupling.

To conclude, let us comment on the two parts of restriction (4.2a):

1. The hypothesis of a great size of the matrix is not in fact necessary to prove macroscopic viscoelasticity: it allows us to obtain the classical integral equations in terms of the Green tensor of elasticity and of the elementary solution of heat equation for an infinitely extended homogeneous medium; it also leads us to a natural resolution scheme based on a self-consistent method with respect to the granular microstructure.

2. The second part (e.g., thermal insulated boundaries) which appears here as a consequence of the assumed great size of the matrix is in fact essential and constitutes the key argument of this work. The justification relies upon the classification of thermal gradients given by ZENER [26]:

the first type of thermal fluctuation, at the scale of the body, is related to boundary conditions: this is a macroscropic thermal difference;

the second one, at the grain scale, results from the material heterogeneity.

We have thus eliminated thermal gradients due to macroscopic causes in order to focus our attention on thermal currents which result only from the material heterogeneity. In that way, and using a discretization of integral equations adjusted to ellipsoidal grains, we have shown that thermoelastic coupling combined with thermal conduction acting from grain to grain confers to a heterogeneous granular material a macroscopic viscoelastic behaviour.

Other techniques may also be investigated. For consideration, MOLINARI and ORTIZ [17] have used Fourier transformation of the set of linearized coupled partial differential equations of thermoelasticity for the general heterogeneous material. The macroscopic viscoelastic behaviour has been proved, keeping in mind the hypothesis of thermal insulated boundaries.

Appendix 1. Form of the elementary solution $\theta_{\text{anis}}(z, t)$ of the heat equation for an infinitely extended homogeneous anisotropic body

We denote by z_1, z_2, z_3 the components of the position vector $\mathbf{z}$ and we assume that the elementary solution θ_{iso} of heat equation for an infinitely extended homogeneous isotropic body is known. In fact, SCHWARTZ [21] and SOBOLEV [22] have shown that

$$\theta_{\text{iso}}(\mathbf{z}, t) = \frac{1}{(4\Pi t)^{3/2}} e^{-z^2/4t}, \tag{1}$$

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where

$$\frac{\partial\theta_{\text{iso}}}{\partial t}(\mathbf{z}, t) = \left[\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\frac{\partial^2}{\partial z_3^2}\right]\theta_{\text{iso}}(\mathbf{z}, t). \tag{2}$$

We can find $\theta_{\text{anis}}(\mathbf{z}, t)$ which verifies

$$\frac{\partial\theta_{\text{anis}}}{\partial t}(\mathbf{z}, t) = k_{ij}^0\frac{\partial^2}{\partial z_i\,\partial z_j}(\theta_{\text{anis}}(\mathbf{z}, t)) \tag{3}$$

by using Fourier's transformation F. Let us note $\hat{\theta}_{\text{anis}}$ the transform of θ_{anis} by F. We have

$$\theta_{\text{anis}}(\mathbf{z}, t) \xrightarrow{F} \theta_{\text{anis}}(\xi, t).$$

Equation (3) becomes in the transformed space

$$\frac{\partial\hat{\theta}_{\text{anis}}}{\partial t}(\xi, t)+4\Pi^2\xi_i\xi_j k_{ij}^0\hat{\theta}_{\text{anis}}(\xi, t) = 0. \tag{4}$$

In the ξ space we define the vector $\mathbf{X}$ by the transformation

$$\xi = A\mathbf{X} \text{ (equivalent to: } \xi_i = A_{im}X_m), \tag{5}$$

where A is the matrix which verify

$$A_{il}k_{ij}^0A_{jm} = \partial_{lm} \tag{6}$$

or in matrix notation,

$$A\tilde{k}A = 1. \tag{7}$$

In most cases the conductivity matrix is a positive symmetric matrix and consequently admits an inverse matrix k^{-1} of components k_0^{ij}.

$$k^{-1} = A\tilde{A}. \tag{8}$$

Then, by substituting ξ from Eq. (5) into Eq. (4), we obtain

$$\frac{\partial}{\partial t}\hat{\theta}_{\text{anis}}(\xi, t)+4\Pi^2\partial_{lm}X_lX_m\hat{\theta}_{\text{anis}}(\xi, t) = 0. \tag{9}$$

Let us note (this will be later justified)

$$\hat{\theta}_{\text{iso}}(\mathbf{X}, t) = \hat{\theta}_{\text{anis}}(\xi, t) \tag{10}$$

so that Eq. (9) becomes

$$\frac{\partial}{\partial t}\hat{\theta}_{\text{iso}}(X, t)+4\Pi^2(X_1^2+X_2^2+\hat{X}_3^2)\theta_{\text{iso}}(\mathbf{X}, t) = 0. \tag{11}$$

By using the inverse Fourier transformation F^{-1}

$$\theta_{\text{iso}}(\mathbf{X}, t) \xrightarrow{F^{-1}} \theta_{\text{iso}}(\mathbf{z}, t),$$

Eq. (11) becomes

$$\frac{\partial\theta_{\text{iso}}}{\partial t}-\left(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\frac{\partial^2}{\partial z_3^2}\right)\theta_{\text{iso}}(\mathbf{z}, t) = 0.$$

This is the heat equation (2) in an isotropic body. It justifies our previous notation (10) which is equivalent to the composition of functions,

$$\hat{\theta}_{\text{iso}} \circ A^{-1} = \hat{\theta}_{\text{anis}}. \tag{12}$$

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Let us recall the relation between the composition symbol $\circ$ and Fourier's transformation (VO KHAC KOAN [24]:

$$\hat{\theta}_{\text{iso}} \circ A^{-1} = |\det A| \widehat{\theta_{\text{iso}} \circ \tilde{A}} \tag{13}$$

so that by substituting Eq. (12) and by linearity of F

$$\hat{\theta}_{\text{anis}} = \widehat{|\det A| \theta_{\text{iso}} \circ \tilde{A}}. \tag{14}$$

By inverting Eq. (14)

$$\theta_{\text{anis}} = |\det A| \theta_{\text{iso}} \circ \tilde{A}.$$

The functionnal equality leads, after substitution of θ_{iso} from Eq. (1), to

$$\theta_{\text{anis}}(\mathbf{y}, t) = |\det A| (4\Pi t)^{-3/2} e^{-z^2/4t},$$

where the relation between the vectors $\mathbf{z}$ and $\mathbf{y}$ is

$$\mathbf{z} = \tilde{A}\mathbf{y}. \tag{15}$$

Then the scalar product z^2 of Eq. (15) becomes

$$\mathbf{z}^2 = \mathbf{z} \cdot \mathbf{z} = (\tilde{A}y)(\tilde{A}y) = A_{lm}\tilde{A}_{ml} y_l y_l. \tag{16}$$

By using Eq. (8) in Eq. (16) we substitute the new expression of z^2 in Eq. (15) and we observe as a consequence of Eq. (8) that

$$\sqrt{|\det k^{-1}|} = |\det A|$$

so that Eq. (15) becomes

$$\theta_{\text{anis}}(\mathbf{y}, t) = \sqrt{|\det k_0^{ij}|}\, (4\Pi t)^{-3/2} e^{[-k_0^{ml} y_l y_m/4t]}. \tag{17}$$

Then, by simple translation of the vector $\mathbf{x}$ and linearity with the coefficient C_ε^0 in Eq. (3), we can deduce that the expression ψ as

$$\Psi(\mathbf{x}-\mathbf{x}', t-t') = [|\det(C_\varepsilon^0 k_0^{ij})|]^{1/2} (4\Pi t)^{-3/2} \exp\left[\frac{-C_\varepsilon^0 k_0^{ml}}{4(t-t')} (x_m - x_m')(x_l - x_l')\right]$$

is the solution of the equation

$$C_\varepsilon^0 \frac{\partial \Psi}{\partial t}(\mathbf{x}-\mathbf{x}', t-t') = k_{ij}^0 \frac{\partial^2}{\partial x_i \partial x_j} \Psi(\mathbf{x}-\mathbf{x}', t-t') \tag{18}$$

which is equivalent in a generalized sense to Eq. (4.4).

Appendix 2. Integral equation governing the evolution of the solution of heat equation

Let us recall the two following equations from which we derive the required integral equation. The heat equation with the presence of a source term H is given by

$$C_\varepsilon(r) \frac{\partial \theta}{\partial t}(r, t) = \frac{\partial}{\partial x_i}\left[k_{ij}(r) \frac{\partial \theta}{\partial x_j}(r, t)\right] + H(r, t). \tag{1}$$

The elementary solution Ψ of this equation is defined by

$$C_\varepsilon^0 \frac{\partial \Psi}{\partial t}(r, t/r', t') = k_{ij}^0 \frac{\partial^2 \Psi}{\partial x_i \partial x_j}(r, t/r', t') + \delta(r-r')\delta(t-t'), \tag{2}$$

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where Ψ verifies the principle of causality:

$$\Psi(r, t/r', t') = 0 \quad \text{for} \quad t \leqslant t'. \tag{3}$$

It means that the time at which we observe the temperature produced by the r'-located heat source is always after the date t' at which that source was instantaneously created.

Our process starts with searching two expressions of the following integral:

$$I(r, t) = \int_0^t \int_V \left\{ \begin{array}{l} \Psi(r, t/r', t') \dfrac{\partial}{\partial x_i'} \left[k_{ij}(r') \dfrac{\partial \theta}{\partial x_j'} (r', t') \right] \\ -\theta(r', t') \dfrac{\partial}{\partial x_i'} k_{ij}(r') \dfrac{\partial}{\partial x_j'} \Psi(r, t/r', t') \Big] \end{array} \right\} dV' dt'. \tag{4}$$

In the first step we use Eq. (1) and the decomposition (4.3) of C_ε and k:

$$\Psi(r, t/r', t') \left[\frac{\partial}{\partial x_i'} \left(k_{ij}(r') \frac{\partial}{\partial x_j'} \theta(r', t') \right) \right] = \Psi(r, t/r', t') [\tilde{C}_\varepsilon(r') \dot{\theta}(r', t') + C_\varepsilon^0 \dot{\theta}(r', t') - H(r', t')]$$

and by using the equivalent expression of Eq. (2) SOBOLEV [22], (CHANG [7])

$$-C_\varepsilon^0 \frac{\partial \Psi}{\partial t'} (r, t/r', t') = \frac{\partial}{\partial x_i'} \left[k_{ij}^0 \frac{\partial}{\partial x_j'} \Psi(r, t/r', t') \right] + \delta(r-r') \delta(t-t').$$

We obtain from Eq. (2)

$$\theta(r', t') \left[\frac{\partial}{\partial x_i'} \left[k_{ij}(r') \frac{\partial \Psi}{\partial x_j'} (r, t/r', t') \right] \right. = \theta(r', t') \left[\frac{\partial}{\partial x_i'} \left[\tilde{k}(r') \frac{\partial}{\partial x_j'} \Psi(r, t/r', t') \right] \right.$$
$$+ \theta(r', t') \left[-\delta(r-r') \delta(t-t') - C_\varepsilon^0 \frac{\partial \Psi}{\partial t'} (r, t/r', t') \right]$$

so that Eq. (4) reads

$$\begin{aligned} I(r, t) = & \int_0^t \int_V \theta(r', t') \delta(r-r') \delta(t-t') dr' dt' \\ & + C_\varepsilon^0 \int_0^t \int_V \left[\frac{\partial \Psi}{\partial t'} (r, t/r', t') \theta(r', t') + \frac{\partial \theta}{\partial t'} (r', t') \Psi(r, t/r', t') dr' dt' \right] \\ & + \int_0^t \int_V \left\{ \begin{array}{l} \tilde{C}_\varepsilon(r') \left[\Psi(r, t/r', t') \dfrac{\partial \theta}{\partial t'} (r', t') \right] - [\Psi(r, t/r', t') H(r', t')] \\ -\left[\dfrac{\partial}{\partial x_i'} (\tilde{k}_{ij}(r') \dfrac{\partial}{\partial x_j'} \Psi(r, t/r', t') \right] \theta(r', t') \end{array} \right\} dr' dt. \end{aligned} \tag{5}$$

The first term of Eq. (5) may be written by definition of the Dirac function:

$$\int_0^t \int_V \theta(r', t') \delta(r-r') \delta(t-t') dr' dt' = \theta(r, t).$$

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By time integration of the second term of Eq. (5), we obtain

$$C_\varepsilon^0 \int_0^t \int_V \left[\frac{\partial \Psi}{\partial t'}(r, t/r', t')\theta(r', t') + \frac{\partial \theta}{\partial t'}(r', t')\Psi(r, t/r', t') dr' dt' \right.$$

$$= C_\varepsilon^0 \int_0^t \int_V \frac{\partial}{\partial t'} [\Psi(r, t/r', t')\theta(r', t')] dr' dt'$$

$$= C_\varepsilon^0 \int_V [\Psi(r, t/r', t)\theta(r', t) - \Psi(r, t/r', 0)\theta(r', 0)] dr'.$$

Then, due to the principle of causality (3), we write for the second term

$$-C_\varepsilon^0 \int_V \Psi(r, t/r', 0)\theta(r', 0) dr'.$$

The conclusion of this first step is that we can use the following expression of $I(r, t)$:

$$(6) \qquad I(r, t) = \theta(r, t) - C_\varepsilon^0 \int_V \Psi(r, t/r', 0)\theta(r', 0) dr'$$

$$+ \int_0^t \int_V \left\{ \begin{array}{l} \tilde{C}_\varepsilon(r') \left[\Psi(r, t/r', t') \frac{\partial \theta}{\partial t'}(r', t') \right] - [\Psi(r, t/r', t') H(r', t')] \\ - \frac{\partial}{\partial x_i'} \left(\tilde{k}_{ij}(r') \frac{\partial \theta}{\partial x_j'} \Psi(r, t/r', t') \right) \theta(r', t') \end{array} \right\} dr' dt.$$

In the second step we use the second Green's formula together with Eq. (4) to obtain

$$I(r, t) = \int_0^t \int_S \left[\Psi(r, t/r', t') n_i' k_{ij}(r') \frac{\partial}{\partial x_j'}(r', t') \right.$$

$$\left. - \theta(r', t') n_i' k_{ij} \frac{\partial}{\partial x_j'} \Psi(r, t/r', t') \right] dS' dt'.$$

We denote by q_θ and q_Ψ the following expressions of generalized heat flux:

$$q_\theta(a, t) = -n_i k_{ij} \frac{\partial}{\partial x_j} \theta(a, t),$$

$$q_\Psi'(r, t/a', t') = -n_i' k_{ij} \frac{\partial}{\partial x_j'} \Psi(r, t/a', t'),$$

where $n_i(n_i')$ is the component of the external normal at point $a(a')$ of the boundary S. With this notation we have

$$(7) \qquad I(r, t) = \int_0^t \int_S [\theta(a', t') q_\Psi'(r, t/a', t') - \Psi(r, t/a', t') q_\theta'(a', t')] da' dt'.$$

By comparing the two expressions of $I(r, t)$, Eq. (6) and (7), we obtain the required integral equation (4.6):

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$$\theta(r,t) = C_\varepsilon^0 \int_V \Psi(r,t/r',0)\theta(r',0)dr' + \int_0^t \int_V \Psi(r,t/r',t')H(r',t')dr'dt'$$

$$+ \int_0^t \int_V \left[\frac{\partial}{\partial x_i'}\left(\tilde{k}_{ij}(r')\frac{\partial}{\partial x_j'}\Psi(r,t/r',t')\right)\theta(r',t')\right.$$

$$\left. - \tilde{C}_\varepsilon(r')\Psi(r,t/r',t')\frac{\partial\theta}{\partial t'}(r',t')\right]dv'dt'$$

$$+ \int_0^t \int_S [\theta(a',t')q_\Psi'(r,t/a',t') - \Psi(r,t/a',t')q_\theta'(a',t')]da'dt'.$$

Appendix 3. Presentation of the linear system (3.36) in a matrix form

The N equations of the linear system (3.36) take on the following form for $\beta = 1$ to N:

$$\varepsilon_{mn}^{e\beta} = E_{mn}^e - (\varepsilon_{mn}^{a\beta} - E_{mn}^a) + \sum_{\alpha=1}^{N} \Gamma_{mnij}^{\beta\alpha}[\tilde{C}_{ijkl}^{\alpha}\varepsilon_{kl}^{e\alpha} - C_{ijkl}^0(\varepsilon_{kl}^{a\alpha} - E_{kl}^a)], \tag{1}$$

where the N unknowns are the elastic strains $\boldsymbol{\epsilon}^{e\beta}$ for $\beta = 1$ to N (we consider here that the tensor $\mathbf{C}^0$ is given for example from the computation of the implicit equation deduced from the self-consistency condition).

Recalling the formula (3.41) of tensor $\mathbf{I}^{\alpha\beta}$

$$\mathbf{I}^{\alpha\beta} = \delta_{\alpha\beta}\mathbf{1}, \tag{2}$$

where $\delta_{\alpha\beta}$ is the Krönecker symbol and $\mathbf{1}$ the fourth order unit tensor

$$1_{ijke} = \frac{1}{2}(\delta_{ik}\delta_{jl} + {}_{il}\delta_{jk}),$$

we get from Eq. (1) and (2) the approximate strain localization law:

$$\sum_{\alpha=1}^{N}[\mathbf{I}^{\alpha\beta} - \boldsymbol{\Gamma}^{\beta\alpha}\tilde{C}^{\alpha}]\boldsymbol{\epsilon}^{e\alpha} = \mathbf{E}^e - \sum_{\alpha=1}^{N}[\mathbf{I}^{\beta\alpha} + \boldsymbol{\Gamma}^{\beta\alpha}\mathbf{C}^0](\boldsymbol{\epsilon}^{a\alpha} - \mathbf{E}^a). \tag{3}$$

If we denote by $[\varepsilon^e]^t$ the transpose of the matrix

$$[(\varepsilon_{ij}^{e1}), (\varepsilon_{ij}^{e2}), \ldots, (\varepsilon_{ij}^{e\alpha}), \ldots, (\varepsilon_{ij}^{eN})],$$

Eq. (3) becomes

$$[I - \Gamma\tilde{C}][\varepsilon^e]^t = [E^e]^t - [I + \Gamma C^0][\varepsilon^a - E^a]^t, \tag{4}$$

where the matrices

$$[I - \Gamma\tilde{C}],\ [E^e],\ [I + \Gamma C^0],\ [\varepsilon^a - E^a]$$

are formed with blocks

$$(I_{ijkl}^{\beta\alpha} - \Gamma_{ijmn}^{\beta\alpha}\tilde{C}_{mnkl}^{\alpha}),\ (E_{ij}^e),\ (I_{ijkl}^{\beta\alpha} + \Gamma_{ijmn}^{\beta\alpha}C_{mnkl}^{0\alpha}),\ (\varepsilon_{kl}^a - E_{kl}^a).$$

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The solution of the system is written

$$[\varepsilon^e]^t = [I-\Gamma\tilde{C}]^{-1}[E^e]^t - [I-\Gamma\tilde{C}]^{-1}[I+\Gamma C^0][\varepsilon^a - E^a]^t \tag{5}$$

which describes the field of elastic deformation $\boldsymbol{\epsilon}^e$ as a function of the elastic and thermal components of the macroscopic strain and of the local thermal strains field. Total deformation is given by

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^a$$

and local stresses

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}^e.$$

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