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On a constitutive theory for materials undergoing microstructural changes

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A CONSTITUTIVE theory is discussed for materials which undergo microstructural changes, and thus have different micromechanisms for the generation of stress in different regimes of response. Of particular interest is a two-network theory of polymer response in which, at some state of deformation, molecular cross-links are broken and then reformed in a new reference state. The mechanical response then depends on the deformation of both the remaining portion of the original material and newly formed one. A particular constitutive equation is introduced in order to develop the methodology for performing calculations, and to study material behavior. The original and newly formed material are both treated as incompressible isotropic nonlinear neo-Hookean elastic materials, but with different reference configurations. Several homogeneous deformations are analyzed, and permanent set on release of load is calculated. Nonhomogeneous deformations are studied by means of the problem of the combined extension and torsion of a circular cylinder. Unloading and loading response is determined, as well as permanent set on release of load.

Przedyskutowano teorię równań konstytutywnych dla materiałów podlegających zmianom mikrostrukturalnym, a zatem charakteryzujących się różnymi mechanizmami powstawania naprężeń w różnych zakresach pracy. Szczególnie interesująca jest dwusieciowa teoria zachowania się polimerów, zgodnie z którą w pewnych stanach deformacji usieciowanie międzycząsteczkowe ulega załamaniu, prowadząc do nowego stanu odniesienia. Deformacja takiego ciała zależy wtedy od odkształcenia obu jego składników w stanie oryginalnym i po przemianie. Wprowadzono szczególne równanie konstytutywne pozwalające opracować metodologię obliczeń oraz badać zachowanie się materiału, który w obu stanach oryginalnym i przekształconym traktowany jest jako nieściśliwy, izotropowy i nieliniowy materiał sprężysty (tzw. neo-Hookean) charakteryzujący się jednak innymi konfiguracjami odniesienia. Przeanalizowano szereg przypadków deformacji jednorodnych oraz niejednorodnych wywołanych rozciąganiem i skręcaniem walca kołowego, ze zwróceniem uwagi na procesy obciążania i odciążania.

Обсуждается теория определяющих уравнений для материалов подвергающихся микроструктурным изменениям, а затем характеризующиеся разными механизмами образования напряжений в различных рабочих диапазонах. Особый интерес представляет двусетевая теория поведения полимеров, согласно которой при некоторых состояниях деформации в межмолекулярном структурировании наступает перелом, ведущий к новому состоянию отнесения. Деформация такого тела зависит тогда от деформации обеих составляющих (в подлинном состоянии и после преобразования). Было введено частное определяющее уравнение, позволяющее разработать методологию расчетов, а также исследовать поведение материала, который в обоих состояниях (подлинном и преобразованном) рассматривается как несжимаемый, изотропный и нелинейный упругий материал (так называемый neo-Hookean), характеризующийся, однако, другими конфигурациями отнесения. Был проанализирован ряд случаев однородных и неоднородных деформаций, вызванных растяжением и кручением кругового цилиндра с учетом процесса нагружания и разгружания.

1. Introduction

MANY OF THE CONSTITUTIVE equations which have been developed in continuum mechanics are based on assumptions which imply that the generation of stress is due to the response of a single material micromechanism which does not change as the body is being deformed.

In nonlinear elasticity, the current value of the Cauchy stress depends only on the gradient of the current configuration with respect to the reference configuration, and this dependence is expressed in terms of a Helmholtz free energy function. Such a constitutive assumption can be motivated by a single molecular mechanism if one considers the kinetic theory of rubber elasticity, where the form of the Helmholtz free energy is related to configurational changes in macromolecules.

However, in many applications, the mechanical response of most materials, whether polymer or metal, requires consideration of more than one micromechanism. The theory of plasticity for metals is an example of this kind of behavior. Within a certain range of response, the mechanical response is due to the elastic distortion of the underlying crystal structure and is modeled by linear or non-linear elasticity theory. However, at some stage the mechanical response also becomes affected by the action of dislocations at grain boundaries between crystals. When this second mechanism is induced, and the stress is released, there is permanent deformation. The amount of permanent deformation depends on the amount of deformation prior to the removal of the stress. The subsequent mechanical properties also depend on the amount of deformation prior to the release of stress. For example, an isotropic cylinder subjected to a sufficiently large uniaxial extension will remain cylindrical upon removal of stress, but will have new dimensions. Relative to the new state, the material will be anisotropic.

In this paper, we consider a large class of materials which undergo microstructural changes when deformed. That is, a second micromechanism arises in the mechanical response of these materials which leads to permanent set on release of load and induces anisotropy. One such micromechanism in polymers is provided by the two network theory of TOBOLSKY and co-workers [1, 2]. In this theory, a certain number of cross-links are present in the initial stress-free state, and additional cross-links are introduced in a later state. The initial system of cross-links produces one network or micromechanism. The second micromechanism is produced by the appearance of the new cross-links. TOBOLSKY, PRETTYMAN and DILLON [3] postulated another example of materials undergoing microstructural changes in which molecular cross-links are broken and reformed in a new reference state. A two-network theory was used by LODGE [4] to discuss permanent set in rubbers caused by uniaxial and biaxial extensional deformation. FONG and ZAPAS [5] used both two network theory and the molecular model of Tobolsky, Prettyman and Dillon to discuss chemical stress relaxation and permanent set in rubber. They also discussed aspects of the development of anisotropy associated with permanent set. A third example of material responses in which a second micromechanism occurs might be that of strain-induced crystallization. PETERLIN [6] suggested that this mechanism develops when a number of macromolecular chain segments associate together to form a bundle-like cluster with fairly good orientation. The remaining segments may be in an amorphous region. Thus, a new constituent or micromechanism is formed which contributes to the mechanical response. A discussion of various aspects of anisotropic mechanical behavior due to such microstructural changes is presented in the book by WARD [7].

Thus, in these materials, there are different constitutive expressions for the stress tensor in different regimes of response of the material. We shall focus our attention on such a class in our discussions. We hasten to add that we discuss a specific example in order to elucidate our point. However, the ideas proposed herein are applicable whenever there is more than one micromechanism responsible for the development of stresses in a continuum. In an earlier paper [8], we have discussed a possible method for modeling changes from one manner of response to another, based on the ideas of bifurcation and the selection of the appropriate branch.

2. Theoretical preliminaries

Let $\varkappa_0$ denote the reference configuration of a solid body in its unstrained, unstressed state. Let $\mathbf{X}_0$ denote the position of a particle in $\varkappa_0$, i.e., positions of particles in $\varkappa_0$ will have the suffix 0. Let $\varkappa(t)$ denote the current configuration of the body and $\mathbf{x}_0(t)$ denote the current coordinates of the particle which was initially at $\mathbf{X}_0$. Let the deformation gradient with regard to the reference configuration be denoted as $\mathbf{F}_{\varkappa_0}(t) = \partial\mathbf{x}(t)/\partial\mathbf{X}_0$.

The constitutive equation for the Cauchy stress for elastic materials (TRUESDELL [9]) has the form

$$\mathbf{T}(t) = \mathscr{F}_{\varkappa_0}[\mathbf{F}_{\varkappa_0}(t)]. \tag{2.1}$$

Now, suppose that at some later state of deformation there is a change in the microstructure of the material. The precise change depends on the situation under consideration. There may be a conversion of a portion or all of the material to a new microstructure, while the rest of the material retains the original microstructure. The changes could occur gradually with changes in deformation or time, or they could take place very quickly. After the change has occurred, it is assumed that a point in space is occupied simultaneously by two particles, or material elements. One represents the remaining portion of the material with the original microstructure, and the other represents the portion of the material with the new microstructure. Such an assumption that two particles occupy the same point in space is not entirely new to continuum mechanics. The theory of interacting continua (cf. TRUESDELL [10], BOWEN [11], ATKIN and CRAINE [12]) is founded on the basis of such an assumption.

We shall regard $\varkappa_1$ to be the reference configuration for particles which represent the portion of the material comprising the new microstructure. Henceforth, we shall refer to such particles as the "newly formed material". As we shall see in a later section, it is possible that the spatial formation of the new material due to nonhomogeneous deformations requires the consideration of an infinite sequence of new reference configurations.

Let $\mathbf{X}_1$ denote the position of a particle of the new material in configuration $\varkappa_1$. Also, let $\mathbf{x}(t)$ denote the position of the same particle in the current configuration which is achieved by deforming $\varkappa_1$ further. Denote the deformation gradient with respect to the new material configuration $\varkappa_1$ as $\mathbf{F}_{\varkappa_1}(t) = \partial\mathbf{x}(t)/\partial\mathbf{X}_1$. We shall assume that the constitutive equation for the stress depends on the deformation of both the remaining portion of the original microstructure and the newly formed microstructure, and is given by

$$\mathbf{T}(t) = \mathscr{F}[\mathbf{F}_{\varkappa_0}(t); \mathbf{F}_{\varkappa_1}(t)]. \tag{2.2}$$

If a second microstructure is to affect the generation of stress in the material, then it is necessary to introduce a criterion which determines when exactly the change in the material microstructure occurs. In general, this criterion, which we shall refer to as an 'activation' criterion, depends on the history of the deformation of the original material. For instance, it could be denoted by a functional of the history of the deformation gradient:

$$A[\mathbf{F}_{\varkappa_0}(s)|_{s=0}^{t}] = 0. \tag{2.3}$$

Thus, when the functional reaches a certain value, the new microstructure emerges.

The constitutive equations must be subjected to the restrictions of material symmetry and frame indifference. We discuss these briefly here. A detailed discussion of the consequences of material symmetry will be discussed in a separate work.

Let G_0 denote the set of material symmetry transformations (cf. Peer group, TRUESDELL [9]) for the original material whose reference configuration is $\varkappa_0$. Let G_1 denote the material symmetry transformation for the newly formed material whose reference configuration is $\varkappa_1$. Then, the material symmetry restriction on the constitutive equation (2.2) is

$$\mathscr{F}[\mathbf{F}_{\varkappa_0}(t); \mathbf{F}_{\varkappa_1}(t)] = \mathscr{F}[\mathbf{F}_{\varkappa_0}(t)\mathbf{H}; \mathbf{F}_{\varkappa_1}(t)\mathbf{M}], \qquad \mathbf{H} \in G_0, \quad \mathbf{M} \in G_1. \tag{2.4}$$

Note that the newly formed material need not have the same material symmetry properties as the original material. An example of such a situation might arise during the process of strain-induced crystallization in polymers. Due to the orientation of macromolecular segments into lamellae, the crystallized portion would be transversely isotropic, while the remaining portion is randomly coiled, and hence considered isotropic.

In order to consider the restrictions of material frame indifference, consider a motion of the material, $\mathbf{x}(t) = \mathbf{x}(\mathbf{X}, t)$, $t \geqslant 0$, where $\mathbf{X} \in \varkappa_0$. Then $\mathbf{F}_{\varkappa_0}(t) = \partial\mathbf{x}(t)/\partial\mathbf{X}$. Suppose that the activation $\varkappa_1$ criterion is satisfied at time t_1. The configuration $\varkappa_1$ is defined by the mapping

$$\mathbf{X}_1 = x(\mathbf{X}_0, t_1). \tag{2.5}$$

For times $t \geqslant t_1$:

$$\mathbf{F}_{\varkappa_1}(t) = (\partial\mathbf{x}(t)/\partial\mathbf{X}_1) = \mathbf{F}_{\varkappa_0}(t)(\partial\mathbf{X}_0/\partial\mathbf{X}_1) = \mathbf{F}_{\varkappa_0}(t)[\mathbf{F}_{\varkappa_0}(t_1)]^{-1}. \tag{2.6}$$

Now consider a second motion of the form

$$\hat{\mathbf{x}}(t) = \mathbf{Q}(t)\,\mathbf{x}(\mathbf{X}, t) + \mathbf{d}(t), \qquad t \geqslant 0, \tag{2.7}$$

where $\mathbf{x}(\mathbf{X}, t)$ is the first motion, $\mathbf{d}(t)$ is independent of $\mathbf{X}$ and $\mathbf{Q}(t)$ is an orthogonal transformation, i.e.,

$$\mathbf{Q}(t)\,\mathbf{Q}(t)^T = \mathbf{Q}(t)^T\mathbf{Q}(t) = \mathbf{I}, \qquad t \geqslant 0. \tag{2.8}$$

In this second motion, the deformation gradient for the material with the original microstructure is

$$\hat{\mathbf{F}}_{\varkappa_0}(t) = \mathbf{Q}(t)\mathbf{F}_{\varkappa_0}(t). \tag{2.9}$$

The activation criterion should also be satisfied at time t_1 for the second motion. The reference configuration for the material with the newly formed microstructure $\hat{\varkappa}_1$, is defined as in (2.5) by the mapping

$$\hat{\mathbf{X}}_1 = \hat{\mathbf{x}}(\mathbf{X}_0, t_1) = \mathbf{Q}(t_1)\,\mathbf{x}(\mathbf{X}_0, t_1) + \mathbf{d}(t_1) = \mathbf{Q}(t_1)\,\mathbf{X}_1 + \mathbf{d}(t_1). \tag{2.10}$$

For times $t \geqslant t_1$, the deformation gradient is given by:

$$\hat{\mathbf{F}}_{\hat{\varkappa}_1}(t) = \frac{\partial \hat{\mathbf{x}}(t)}{\partial \hat{\mathbf{X}}_1} = \mathbf{Q}(t)\mathbf{F}_{\varkappa_1}(t)\mathbf{Q}(t_1)^T. \tag{2.11}$$

In virtue of Eqs. (2.2), (2.9) and (2.11), the requirement of frame indifference implies that:

$$\mathscr{F}[\mathbf{Q}(t)\mathbf{F}_{\varkappa_0}(t); \mathbf{Q}(t)\mathbf{F}_{\varkappa_1}(t)\mathbf{Q}(t_1)^T] = \mathbf{Q}(t)\mathscr{F}[\mathbf{F}_{\varkappa_0}(t); \mathbf{F}_{\varkappa_1}(t)]\mathbf{Q}(t)^T. \tag{2.12}$$

The relevant forms of the polar decompostions of $\mathbf{F}_{\varkappa_0}$ and $\mathbf{F}_{\varkappa_1}$ are denoted by

$$\mathbf{F}_{\varkappa_0}(t) = \mathbf{R}_0(t)\mathbf{U}_0(t), \quad \mathbf{F}_{\varkappa_1}(t) = \mathbf{V}_1(t)\mathbf{R}_1(t). \tag{2.13}$$

Let $\mathbf{Q}(s)$, $0 \leqslant s \leqslant t$ be defined so that $\mathbf{Q}(t) = \mathbf{I}$ and $\mathbf{Q}(t_1) = \mathbf{R}_1(t)$. It follows from Eqs. (2.12) and (2.13) that

$$\mathscr{F}[\mathbf{F}_{\varkappa_0}(t); \mathbf{F}_{\varkappa_1}(t)] = \mathscr{F}[\mathbf{F}_{\varkappa_0}(t); \mathbf{V}_1(t)]. \tag{2.14}$$

It follows from the polar decomposition of $\hat{\mathbf{F}}_{\hat{\varkappa}_1}(t)$, (2.11) and (2.13), that

$$\mathbf{Q}(t)\mathbf{F}_{\varkappa_1}(t)\,\mathbf{Q}(t_1)^T = \mathbf{Q}(t)\mathbf{V}_1(t)\mathbf{Q}(t)^T\mathbf{Q}(t)\mathbf{R}_1(t)\mathbf{Q}(t_1)^T. \tag{2.15}$$

The statement of frame indifference in Eq. (2.12) reduces to

$$\mathscr{F}[\mathbf{Q}(t)\mathbf{F}_{\varkappa_0}(t); \mathbf{Q}(t)\mathbf{V}_1(t)\mathbf{Q}(t)^T] = \mathbf{Q}(t)\mathscr{F}[\mathbf{F}_{\varkappa_0}(t); \mathbf{V}_1(t)]\mathbf{Q}(t)^T. \tag{2.16}$$

Now let $\mathbf{Q}(s)$, $0 \leqslant s \leqslant t$ be such that $\mathbf{Q}(t) = \mathbf{R}_0(t)^T$. In view of Eq. (2.13), we find that

$$\mathscr{F}[\mathbf{F}_{\varkappa_0}(t); \mathbf{V}_1(t)] = \mathbf{R}_0(t)\mathscr{F}[\mathbf{U}_0(t); \mathbf{R}_0^T(t)\mathbf{V}_1(t)\mathbf{R}_0(t)]\mathbf{R}_0^T(t). \tag{2.17}$$

By Eqs. (2.2), (2.14) and (2.17), and without loss in generality, the constitutive equation can finally be written as:

$$\mathbf{T} = \mathbf{R}_0(t)\mathscr{S}[\mathbf{C}_{\varkappa_0}(t); \mathbf{R}_0^T(t)\mathbf{B}_{\varkappa_1}(t)\mathbf{R}_0(t)]\mathbf{R}_0^T(t), \tag{2.18}$$

where

$$\begin{aligned} \mathbf{C}_{\varkappa_0}(t) &= \mathbf{F}_{\varkappa_0}(t)^T\mathbf{F}_{\varkappa_0}(t) = \mathbf{U}_0(t)^2, \\ \mathbf{B}_{\varkappa_1}(t) &= \mathbf{F}_{\varkappa_1}(t)\mathbf{F}_{\varkappa_1}(t)^T = \mathbf{V}_1(t)^2. \end{aligned} \tag{2.19}$$

3. A specific microstructural change — two-network theory

In order to illustrate the application of our theory, we introduce the following specific model. Both the original and newly formed material are assumed to be incompressible, isotropic and neo-Hookean nonlinear elastic solids. The form of the constitutive equation in (2.1) for the original material, before new material comes into play, is:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{B}_{\varkappa_0}, \quad \mathbf{B}_{\varkappa_0} = \mathbf{F}_{\varkappa_0}\mathbf{F}_{\varkappa_0}^T, \tag{3.1}$$

where μ is a constant. The activation criterion (2.3), which determines when the new material forms, is given by:

$$A(I_1, I_2) = 0, \tag{3.2}$$

where I_1 and I_2 are the invariants of $\mathbf{B}_{\varkappa_0}$. The constitutive equation (2.2), which applies once Eq. (3.2) is satisfied, has the form

$$\mathbf{T} = -p\mathbf{I} + \zeta_1 \mu \mathbf{B}_{\varkappa_0} + (1-\zeta_1)\hat{\mu}\mathbf{B}_{\varkappa_1}, \tag{3.3}$$

where $\mathbf{B}_{\varkappa_1} = \mathbf{F}_{\varkappa_1}\mathbf{F}_{\varkappa_1}^T$. We have also assumed that the shear modulus of the newly formed material is different from that for the original network. The parmeter ζ_1 denotes the mass fraction of the original network which remains and $1-\zeta_1$ is the mass fraction of the newly formed material.

Equation (3.3) represents the special case of (2.18) in which $\mathcal{S}$ is given by:

$$\mathcal{S} = \zeta_1 \mu \mathbf{C}_{\varkappa_0}(t) + (1-\zeta_1)\hat{\mu}\mathbf{R}_0^T\mathbf{B}_{\varkappa_1}(t)\,\mathbf{R}_0. \tag{3.4}$$

Notice that when the activation criterion is satisfied, there is an instantaneous change from a continuum consisting of the original material to one consisting of the remainder of the original material and the newly formed material. It is assumed that no further change occurs.

Although this model is very simple, it incorporates the essential features of the ideas being considered here. It is also very convenient for presenting examples. As will be seen in later sections, analytical solutions to several important problems can be obtained within the context of such a theory. The model also provides valuable insight for situations in which features such as compressibility and stress relaxation are incorporated.

4. Homogeneous deformation — triaxial extension

Let the components of $\mathbf{X}_0$ with respect to a Cartesian coordinate system be given by $X_i^{(0)}$. Consider a cube of material which undergoes the following triaxial deformation from reference configuration $\varkappa_0$:

$$x_i(t) = \lambda_i(t)\,X_i^{(0)}, \quad i = 1, 2, 3, \quad \text{no sum over } i \tag{4.1}$$

with

$$\lambda_1(t)\,\lambda_2(t)\,\lambda_3(t) = 1.$$

Then

$$\mathbf{F}_{\varkappa_0} = \begin{bmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{bmatrix}. \tag{4.2}$$

Suppose the activation criterion (3.2) is satisfied at time t_1. Since the triaxial deformation (4.1) is homogeneous, the activation criterion will be satisfied simultaneously at all particles of the cube. Then the coordinates of a newly formed material particle, denoted by $X_i^{(1)}$, are related to coordinates $X_i^{(0)}$ by:

$$X_i^{(1)} = \lambda_i(t_1)X_i^{(0)}, \quad i = 1, 2, 3. \tag{4.3}$$

The mapping in Eq. (4.3) defines the reference configuration $\varkappa_1$ of the newly formed material.

For times $t > t_1$, the mapping of the original material from reference configuration $\varkappa_0$ to current configuration $\varkappa(t)$ is still given by Eq. (4.1). For the newly formed material, the mapping from reference configuration $\varkappa_1$ to the current configuration $\varkappa(t)$ is found from Eqs. (4.1) and (4.3) to be:

$$x_i(t) = [\lambda_i(t)/\lambda_i(t_1)]X_i^{(1)}, \quad i = 1, 2, 3, \quad \text{no sum over } i. \tag{4.4}$$

Then

$$F_{\varkappa_1} = \begin{bmatrix} [\lambda_1(t)/\lambda_1(t_1)] & 0 & 0 \\ 0 & [\lambda_2(t)/\lambda_2(t_1)] & 0 \\ 0 & 0 & [\lambda_3(t)/\lambda_3(t_1)] \end{bmatrix}. \tag{4.5}$$

The stress, by (3.3), has the following form for all further triaxial deformations:

$$T_{ii} = -p+\zeta_1\mu\lambda_i(t)^2+(1-\zeta_1)\hat{\mu}[\lambda_i(t)/\lambda_i(t_1)]^2, \quad i = 1,2,3, \text{ no sum over } i. \tag{4.6}$$

Suppose that the stress reduces to zero at time $t = \bar{t}$. If $\zeta_1 \neq 1$ or $\zeta_1 \neq 0$, the stress-free configuration will coincide with neither $\varkappa_0$ nor $\varkappa_1$. The stress-free configuration is found by solving Eq. (4.6) for strech ratios $\lambda_i(\bar{t}) = \bar{\lambda}_i$ and then using these in Eq. (4.1).

In order to solve for these stretch ratios, let (4.6) be rewritten as:

$$-p+\zeta_1\mu\phi_i[\bar{\lambda}_i/\lambda_i(t_1)]^2 = 0, \quad i = 1, 2, 3, \quad \text{no sum over } i, \qquad \phi_i = (\lambda_i(t_1))^2+[\hat{\mu}(1-\zeta_1)/\mu\zeta_1]. \tag{4.7}$$

Equation (4.7) implies that

$$\phi_1[\bar{\lambda}_1/\lambda_1(t_1)]^2 = \phi_2[\bar{\lambda}_2/\lambda_2(t_1)]^2 = \phi_3[\bar{\lambda}_3/\lambda_3(t_1)]^2. \tag{4.8}$$

This, together with the condition that the deformation be isochoric, leads to the following relations,

$$\begin{aligned} \bar{\lambda}_1 &= \{\phi_2\phi_3/\phi_1^2\}^{1/6}\lambda_1(t_1), \\ \bar{\lambda}_2 &= \{\phi_1\phi_3/\phi_2^2\}^{1/6}\lambda_2(t_1), \\ \bar{\lambda}_3 &= \{\phi_1\phi_2/\phi_3^2\}^{1/6}\lambda_3(t_1). \end{aligned} \tag{4.9}$$

Thus, each of the stretch ratios which define the stress-free configuration depends on the ratio of the moduli, $\hat{\mu}/\mu$, the ratio of the mass fraction of newly formed material to the mass fraction of the original network which remains, and the corresponding stretch ratio at time t_1 when the activation condition is satisfied.

5. Simple shear deformation

Suppose that a cube of material in reference configuration $\varkappa_0$ undergoes the simple shearing motion:

$$\begin{aligned} x_1(t) &= X_1^{(0)}+K(t)X_2^{(0)}, \\ x_2(t) &= X_2^{(0)}, \\ x_3(t) &= X_3^{(0)}. \end{aligned} \tag{5.1}$$

As is well known for simple shear, $I_1(t) = I_2(t) = 3+K(t)^2$. At some time t_1, the amount of shear $K(t_1)$ is such that

$$A(3+K(t_1)^2, 3+K(t_1)^2) = 0, \tag{5.2}$$

and the activation criterion (3.2) is satisfied. Letting $K \equiv K(t_1)$, the coordinates of the newly formed material particles, $X_i^{(1)}$ are related to coordinates $X_i^{(0)}$ by

$$\begin{aligned} X_1^{(1)} &= X_1^{(0)}+KX_2^{(0)},\\ X_2^{(1)} &= X_2^{(0)},\\ X_3^{(1)} &= X_3^{(0)}. \end{aligned} \tag{5.3}$$

For times $t > t_1$, the mapping of the original material particles from reference configuration $\varkappa_0$ to current configuration $\varkappa(t)$ is still given by Eq. (5.1). The mapping from reference configuration $\varkappa_1$ of the newly formed material particles to the current configuration $\varkappa(t)$ is found from Eqs. (5.1) and (5.3) to be

$$\begin{aligned} x_1(t) &= X_1^{(1)}+[K(t)-K]X_2^{(1)},\\ x_2(t) &= X_2^{(1)},\\ x_3(t) &= X_3^{(1)}. \end{aligned} \tag{5.4}$$

The stress, by Eq. (3.3), has the following form for further simple shear deformation

$$\mathbf{T} = -p\mathbf{I}+\zeta_1\mu\begin{bmatrix} 1+(K(t))^2 & K(t) & 0\\ K(t) & 1 & 0\\ 0 & 0 & 0\end{bmatrix}+(1-\zeta_1)\hat{\mu}\begin{bmatrix} 1+(K(t)-K)^2 & K(t)-K & 0\\ K(t)-K & 1 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{5.5}$$

According to this model, the shear stress-shear strain relation is initially

$$T_{12} = \mu K(t). \tag{5.6}$$

After the newly formed material appears, (5.5) gives

$$T_{12} = \zeta_1\mu K(t)+(1-\zeta_1)\hat{\mu}(K(t)-K) = \{\zeta_1[\mu-\hat{\mu}]+\hat{\mu}\}K(t)-(1-\zeta_1)\hat{\mu}K. \tag{5.6'}$$

For $K(t) < K$, the shear stress-strain graph is linear with slope μ. At K the graph has a jump discontinuity downward by amount $\mu(1-\zeta_1)K$. For $K(t) > K$, the graph is again linear with slope $\zeta_1(\mu-\hat{\mu})+\hat{\mu}$. The jump discontinuity downward occurs because the newly formed material is assumed to be in a stress-free state.

Now suppose that we wish to determine the new traction-free configuration. Becuse of the presence of distinct normal stresses in Eq. (5.5), the stress-free configuration cannot be related to $\varkappa_0$ or $\varkappa_1$ by a simple shear deformation. In [13], RAJAGOPAL and WINEMAN considered the shearing of a cube of a nonlinear elastic isotropic material in the absence of normal tractions. They showed that the corresponding deformation consisted of shear superposed on unequal triaxial extension, and determined the extensions.

Although the total traction on the material is to vanish, the terms in Eq. (5.5) associated with $\mathbf{B}_{\varkappa_0}$ and $\mathbf{B}_{\varkappa_1}$ will not equal zero. The newly formed material can be thought of as exerting normal and shear tractions on the remaining material. As in [13], these will not be what is necessary to maintain simple shear deformations from $\varkappa_0$ and $\varkappa_1$ to the traction-free configuration. Guided by the work in [13], we now let the deformation from $\varkappa_0$ to the traction-free configuration be given by shear superposed on unequal triaxial extensions.

Let the deformation from configuration $\varkappa_1$ to the traction-free configuration be given by

$$(5.7)\qquad \begin{aligned} x_1 &= \bar{\lambda}_1 X_1^{(1)} + \bar{K}\bar{\lambda}_2 X_2^{(1)},\\ x_2 &= \bar{\lambda}_2 X_2^{(1)},\\ x_3 &= \bar{\lambda}_3 X_3^{(1)}. \end{aligned}$$

In view of Eq. (5.3), the deformation from configuration $\varkappa_0$ to the traction-free configuration is given by

$$(5.8)\qquad \begin{aligned} x_1 &= \bar{\lambda}_1 X_1^{(0)} + (\bar{\lambda}_1 K + \bar{K}\bar{\lambda}_2) X_2^{(0)},\\ x_2 &= \bar{\lambda}_2 X_2^{(0)},\\ x_3 &= \bar{\lambda}_3 X_3^{(0)}. \end{aligned}$$

If the tractions of the cube are to vanish, then $\mathbf{T} = \mathbf{0}$. By Eq. (3.3), $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$, $\bar{K}$, and p must be chosen to satisfy

$$(5.9)\qquad -p\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \zeta_1\mu\begin{bmatrix}\bar{\lambda}_1^2+(K^*)^2 & K^*\bar{\lambda}_2 & 0\\ K^*\bar{\lambda}_2 & \bar{\lambda}_2^2 & 0\\ 0 & 0 & \bar{\lambda}_3^2\end{bmatrix} + (1-\zeta_1)\hat{\mu}\begin{bmatrix}\bar{\lambda}_1^2+(\bar{K}\bar{\lambda}_2)^2 & \bar{K}\bar{\lambda}_2^2 & 0\\ \bar{K}\bar{\lambda}_2^2 & \bar{\lambda}_2^2 & 0\\ 0 & 0 & \bar{\lambda}_3^2\end{bmatrix} = \mathbf{0}$$

where $K^* = \bar{\lambda}_1 K + \bar{K}\bar{\lambda}_2$. According to Eqs. (5.8), K^* represents the residual shear displacement of the surface $X_2 =$ constant. The condition that $T_{12} = 0$ implies

$$(5.10)\qquad \begin{aligned} \bar{K}\bar{\lambda}_2 &= [-K\bar{\lambda}_1/(1+D)],\\ D &= [(1-\zeta_1)\hat{\mu}/\zeta_1\mu],\\ K^* &= [D/(1+D)]K\bar{\lambda}_1. \end{aligned}$$

Moreover, the conditions $T_{22} = 0$ and $T_{33} = 0$ imply that

$$(5.11)\qquad \bar{\lambda}_2^2 = \bar{\lambda}_3^2, \quad p = [\zeta_1\mu + (1-\zeta_1)\hat{\mu}]\bar{\lambda}_2^2.$$

On setting $T_{11} = 0$ and using Eqs. (5.10) and (5.11), we find that

$$(5.12)\qquad \bar{\lambda}_2^2 = \{1 + [D/(1+D)^2]K^2\}\bar{\lambda}_1^2.$$

The condition that there is no volume change, $\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = 1$, together with Eqs. (5.11) and (5.12), leads to the result that

$$(5.13)\qquad \begin{aligned} \bar{\lambda}_1 &= \{1 + [D/(1+D)^2]K^2\}^{-1/3} < 1,\\ \bar{\lambda}_2 = \bar{\lambda}_3 &= \{1 + [D/(1+D)^2]K^2\}^{1/6} > 1. \end{aligned}$$

Finally, the residual shear displacement is

$$(5.14)\qquad K^* = [D/(1+D)]K/\{1 + [D/(1+D)^2]K^2\}^{1/3} < K,$$

since $D > 0$.

Equations (5.13) and (5.14) show how the traction-free deformed state of the cube depends on the amount of new material and the shear deformation when it forms.

6. Shear superposed on triaxial extension

Suppose that a cube of material in reference configuration $\varkappa_0$ is subjected to a motion consisting of shear superposed on triaxial extension

$$\begin{aligned} x_1(t) &= \lambda_1(t)X_1^{(0)}+K(t)\lambda_2(t)X_2^{(0)},\\ x_2(t) &= \lambda_2(t)X_2^{(0)},\\ x_3(t) &= \lambda_3(t)X_3^{(0)}, \end{aligned} \tag{6.1}$$

where $\lambda_1(t)\,\lambda_2(t)\,\lambda_3(t) = 1$. Let $\lambda_1, \lambda_2, \lambda_3, K$ denote the values of the stretch ratio and shear when activation criterion (3.2) is satisfied. The coordinates of the newly formed material particles in reference configuration $\varkappa_1$ are given by

$$\begin{aligned} X_1^{(1)} &= \lambda_1 X_1^{(0)}+K\lambda_2 X_2^{(0)},\\ X_2^{(1)} &= \lambda_2 X_2^{(0)},\\ X_3^{(1)} &= \lambda_3 X_3^{(0)}. \end{aligned} \tag{6.2}$$

For any subsequent deformation, the mapping of the original material particles from reference configuration $\varkappa_0$ to current configuration $\varkappa(t)$ is still given by Eqs. (6.1). The mapping of the newly formed material from reference configuration $\varkappa_1$ to current configuration $\varkappa(t)$ is found from Eqs. (6.1) and (6.2) to be

$$\begin{aligned} x_1(t) &= [\lambda_1(t)/\lambda_1]X_1^{(1)}+\{K(t)[\lambda_2(t)/\lambda_2]-K[\lambda_1(t)/\lambda_1]\}X_2^{(1)},\\ x_2(t) &= [\lambda_2(t)/\lambda_2]X_2^{(1)},\\ x_3(t) &= [\lambda_3(t)/\lambda_3]X_3^{(1)}. \end{aligned} \tag{6.3}$$

The stress, by Eq. (3.3), is given by

$$\begin{aligned} \mathbf{T} = -p\mathbf{I}+\zeta_1\mu &\begin{bmatrix} \lambda_1(t)^2+(K(t)\lambda_2(t))^2 & K(t)\,\lambda_2(t)^2 & 0\\ K(t)\,\lambda_2(t)^2 & \lambda_2(t)^2 & 0\\ 0 & 0 & \lambda_3(t)^2 \end{bmatrix}\\ &+(1-\zeta_1)\hat{\mu}\begin{bmatrix} (\lambda_1(t)/\lambda_1)^2+\hat{K}^2 & \hat{K}(\lambda_2(t)/\lambda_2) & 0\\ \hat{K}(\lambda_2(t)/\lambda_2) & (\lambda_2(t)/\lambda_2)^2 & 0\\ 0 & 0 & (\lambda_3(t)/\lambda_3)^2 \end{bmatrix}, \end{aligned} \tag{6.4}$$

where

$$\hat{K} = K_2(t)\,[\lambda_2(t)/\lambda_2]-[K\lambda_1(t)/\lambda_1].$$

Consider now the problem of determining the traction-free residual shape of the original cube. Recalling the discussion in Sect. 5, we can expect it to be related to $\varkappa_0$ by a deformation of form Eqs. (6.1). Let the value of the parameters for the traction-free state be $\lambda_1(t) = \tilde{\lambda}_1$, $\lambda_2(t) = \tilde{\lambda}_2$, $\lambda_3(t) = \tilde{\lambda}_3$, $K(t) = \tilde{K}$. Expressions for $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{K}$ in terms of $\zeta_1, \lambda_1, \lambda_2, \lambda_3, K$ can be obtained by solving Eq. (6.4) with $\mathbf{T} = \mathbf{0}$, i.e.,

$$\mathbf{0} = -p\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}+\zeta_1\mu\begin{bmatrix} \tilde{\lambda}_1^2+(\tilde{K}\tilde{\lambda}_2)^2 & \tilde{K}\tilde{\lambda}_2^2 & 0\\ \tilde{K}\tilde{\lambda}_2^2 & \tilde{\lambda}_2^2 & 0\\ 0 & 0 & \tilde{\lambda}_3^2 \end{bmatrix} \tag{6.5}$$

(6.5) [cont.]
$$+(1-\zeta_1)\hat{\mu}\begin{bmatrix}(\tilde{\lambda}_1/\lambda_1)^2+\hat{K}^2 & \hat{K}(\tilde{\lambda}_2/\lambda_2) & 0\\ \hat{K}(\tilde{\lambda}_2/\lambda_2) & (\tilde{\lambda}_2/\lambda_2)^2 & 0\\ 0 & 0 & (\tilde{\lambda}_3/\lambda_3)^2\end{bmatrix},$$

where $\hat{K} = \tilde{K}[\tilde{\lambda}_2/\lambda_2]-K[\tilde{\lambda}_1/\lambda_1]$. This system of equations can be solved by a procedure similar to that in Sect. 5. The details are quite tedious and are omitted for purposes of brevity. The results are

$$\begin{aligned}\tilde{\lambda}_1 &= (A_2^{1/2}A_3^{1/6}/A^{1/3})\,\lambda_1,\\ \tilde{\lambda}_2 &= (A^{1/6}A_3^{1/6}/A_2^{1/2})\,\lambda_2,\\ \tilde{\lambda}_3 &= (A^{1/6}/A_3^{1/3})\,\lambda_3,\\ \tilde{K}\tilde{\lambda}_2 &= (A_3^{1/6}A_2^{1/2}/A^{1/3})[\lambda_2 D/(\lambda_2^2+D)]\,K,\end{aligned}\tag{6.6}$$

where

$$\begin{aligned}A_i &= \lambda_i^2+D, \qquad i = 1, 2, 3,\\ A &= A_1 A_2+\lambda_2^2 D K^2,\\ D &= (1-\zeta_1)\hat{\mu}/\zeta_1\mu.\end{aligned}\tag{6.7}$$

7. Nonhomogeneous deformations

We consider nonhomogeneous deformations of an incompressible homogeneous solid circular cylinder. Each material element of the cylinder undergoes a different local homogeneous deformation. The activation criterion will be satisfied at different elements at different stages of the global deformation. In this and the next section, we provide examples which illustrate how such problems can be treated and the physical consequences of two-network material response.

The examples to be considered involve uniaxial extension and torsion of the cylinder. In the present section, we consider the following deformation history. The cylinder is first subjected to a uniaxial extension which is sufficiently large that the activation criterion is satisfied at all material elements simultaneously. The cylinder is then stretched further and twisted. This causes local extension and shear of material elements of both the remaining original and the newly formed material.

The initial radius and length of the cylinder are R_0 and L_0. Let the Z-axis of a cylindrical coordinate system coincide with the centerline of the cylinder. Denote the coordinates of a material particle in reference configuration $\varkappa_0$ by (R, Θ, Z) and let (r, θ, z) denote its coordinates in current configuration $\varkappa(t)$. The extension of the cylinder in $\varkappa_0$ to a cylinder in $\varkappa(t)$ is described by the mapping

$$\begin{aligned}r &= r(R),\\ \theta &= \Theta,\\ z &= \lambda Z.\end{aligned}\tag{7.1}$$

The physical components with respect to cylindrical coordinates of deformation gradient $\mathbf{F}_{\varkappa_0}$ are given by

$$\mathbf{F}_{\varkappa_0} = \begin{bmatrix} dr/dR & 0 & 0 \\ 0 & r/R & 0 \\ 0 & 0 & \lambda \end{bmatrix}. \tag{7.2}$$

According to the incompressibility condition, det $|\mathbf{F}_{\varkappa_0}| = 1$,

$$r = (1/\sqrt{\lambda})R. \tag{7.3}$$

Let the value of λ, when activation criterion (3.2) is satisfied, by denoted by $\bar{\lambda}$. Let the coordinates of a newly formed material particle in $\varkappa_1$ be denoted by $(\bar{r}, \bar{\theta}, \bar{z})$. By Eqs. (7.1) and (7.3) these are related to the coordinates in $\varkappa_0$ by

$$\begin{aligned} \bar{r} &= (1/\sqrt{\bar{\lambda}})R, \\ \bar{\theta} &= \Theta, \\ \bar{z} &= \bar{\lambda}Z. \end{aligned} \tag{7.4}$$

In the second phase of deformation, the cylinder is subjected to additional extension and torsion. The deformation from $\varkappa_1$ to the current configuration $\varkappa(t)$ is given by

$$\begin{aligned} r &= r(\bar{r}), \\ \theta &= \bar{\theta}+\hat{\psi}\hat{\lambda}\bar{z}, \\ z &= \hat{\lambda}\bar{z}, \end{aligned} \tag{7.5}$$

where $\hat{\lambda}$ is the stretch ratio of the centerline in configuration $\varkappa(t)$ with respect to that in configuration $\varkappa_1$, and $\hat{\psi}$ is the angle of twist per unit length of the cylinder in configuration $\varkappa(t)$.

The deformation from configuration $\varkappa_0$ to configuration $\varkappa(t)$ is obtained as the composition of the mappings in Eqs. (7.4) and (7.5),

$$\begin{aligned} r &= r(\bar{r}(R)), \\ \theta &= \Theta+\hat{\psi}\hat{\lambda}\bar{\lambda}Z, \\ z &= \hat{\lambda}\bar{\lambda}Z. \end{aligned} \tag{7.6}$$

The physical components of the deformation gradients with respect to cylindrical coordinates are given by

$$\begin{aligned} \mathbf{F}_{\varkappa_0} &= \begin{bmatrix} dr/dR & 0 & 0 \\ 0 & r/R & r(d\theta/dZ) \\ 0 & 0 & \hat{\lambda}\bar{\lambda} \end{bmatrix}, \\ \mathbf{F}_{\varkappa_1} &= \begin{bmatrix} dr/d\bar{r} & 0 & 0 \\ 0 & r/\bar{r} & r(d\theta/d\bar{z}) \\ 0 & 0 & \hat{\lambda} \end{bmatrix}. \end{aligned} \tag{7.7}$$

Incompressibility conditions det $|\mathbf{F}_{\varkappa_0}| = 1$, det $|\mathbf{F}_{\varkappa_1}| = 1$ imply that

$$r = \left(1/\sqrt{\bar{\lambda}}\right)\bar{r}, \qquad r = \left(1/\sqrt{\hat{\lambda}\bar{\lambda}}\right)R, \tag{7.8}$$

and thus, by Eqs. (7.7),

$$\mathbf{F}_{\varkappa_0} = \begin{bmatrix} \left(1/\sqrt{\hat{\lambda}\bar{\lambda}}\right) & 0 & 0 \\ 0 & \left(1/\sqrt{\hat{\lambda}\bar{\lambda}}\right) & r\hat{\psi}\hat{\lambda}\bar{\lambda} \\ 0 & 0 & \hat{\lambda}\bar{\lambda} \end{bmatrix}, \qquad \mathbf{F}_{\varkappa_1} = \begin{bmatrix} \left(1/\sqrt{\hat{\lambda}}\right) & 0 & 0 \\ 0 & \left(1/\sqrt{\hat{\lambda}}\right) & r\hat{\psi}\hat{\lambda} \\ 0 & 0 & \hat{\lambda} \end{bmatrix}. \tag{7.9}$$

The stress, by Eq. (3.3), has the form

$$\mathbf{T} = -p\mathbf{I} + \zeta_1\mu \begin{bmatrix} (1/\hat{\lambda}\bar{\lambda}) & 0 & 0 \\ 0 & (1/\hat{\lambda}\bar{\lambda}) + (r\hat{\psi})^2(\hat{\lambda}\bar{\lambda})^2 & r\hat{\psi}(\hat{\lambda}\bar{\lambda})^2 \\ 0 & r\hat{\psi}(\hat{\lambda}\bar{\lambda})^2 & (\hat{\lambda}\bar{\lambda})^2 \end{bmatrix} + (1-\zeta_1)\hat{\mu} \begin{bmatrix} (1/\hat{\lambda}) & 0 & 0 \\ 0 & (1/\hat{\lambda}) + (r\hat{\psi})^2\hat{\lambda}^2 & r\hat{\psi}\hat{\lambda}^2 \\ 0 & r\hat{\psi}\hat{\lambda}^2 & \hat{\lambda}^2 \end{bmatrix}. \tag{7.10}$$

The individual components of Eq. (7.10) have the form

$$T_{z\theta} = r\hat{\psi}[\mu\zeta_1(\hat{\lambda}\bar{\lambda})^2 + (1-\zeta_1)\hat{\mu}\hat{\lambda}^2], \qquad T_{ii} = -p + \tilde{\sigma}_{ii}(\hat{\lambda}, \bar{\lambda}, r\hat{\psi}), \quad ii = rr, \theta\theta, zz. \tag{7.11}$$

In view of Eqs. (7.11), the equilibrium equations reduce to

$$dT_{rr}/dr + (T_{rr} - T_{\theta\theta})/r = 0, \qquad p = p(r). \tag{7.12}$$

It is easily shown that

$$p(r) = \tilde{\sigma}_{rr} + \int_{r_0}^{r} [(\tilde{\sigma}_{rr} - \tilde{\sigma}_{\theta\theta})/r']dr', \tag{7.13}$$

where the outer surface of the cylinder in configuration $\varkappa(t)$, with radius r_0, is assumed to be free of traction, i.e. $T_{rr}(r_0) = 0$.

In the remainder of this section, we confine attention to small angles of twist, i.e., $|\hat{\psi}| \ll 1$. Then, by Eqs. (7.10), (7.11) and (7.13), we find

$$
(7.14)\qquad
\begin{aligned}
\tilde{\sigma}_{rr} &= \zeta_1 \mu(1/\hat{\lambda}\bar{\lambda}) + (1-\zeta_1)\hat{\mu}(1/\hat{\lambda}),\\
\tilde{\sigma}_{\theta\theta} &= \sigma_{rr} + O(\hat{\psi}^2),\\
\tilde{\sigma}_{zz} &= \zeta_1 \mu(\hat{\lambda}\bar{\lambda})^2 + (1-\zeta_1)\hat{\mu}\hat{\lambda}^2,\\
p(r) &= \tilde{\sigma}_{rr} + O(\hat{\psi}^2),\\
T_{zz} &= \zeta_1 \mu[(\hat{\lambda}\bar{\lambda})^2 - (1/\hat{\lambda}\bar{\lambda})] + (1-\zeta_1)\hat{\mu}[1/\hat{\lambda})] + O(\hat{\psi}^2).
\end{aligned}
$$

The axial force N and twisting moment M required to maintain the deformation are given by

$$
(7.15)\qquad
\begin{aligned}
N &= 2\pi \int_0^{r_0} r T_{zz} dr,\\
M &= 2\pi \int_0^{r_0} r^2 T_{z\theta} dr.
\end{aligned}
$$

Upon substituting $T_{z\theta}$ in Eqs. (7.11), T_{zz} in Eqs. (7.14), and making use of Eqs. (7.8), it is found that, to within terms of $O(\hat{\psi}^2)$,

$$
(7.16)\qquad
\begin{aligned}
N &= (\pi R_0^2/\hat{\lambda}\bar{\lambda})\ \{\zeta_1 \mu[(\hat{\lambda}\bar{\lambda})^2 - (1/\hat{\lambda}\bar{\lambda})] + (1-\zeta_1)\hat{\mu}[\hat{\mu}[\hat{\lambda}^2 - (1/\hat{\lambda})]\}\hat{\lambda}^2 - (1/\hat{\lambda})]\},\\
M/\hat{\psi} &= (\pi R_0^4/2)\ [1/(\hat{\lambda}\bar{\lambda})^2]\{\zeta_1 \mu(\hat{\lambda}\bar{\lambda})^2 + (1-\zeta_1)\hat{\mu}\hat{\lambda}^2\}.
\end{aligned}
$$

The ratio of the torsional stiffness for small angles of twist, $M/\hat{\psi}$, to the axial force can be written as

$$
(7.17)\qquad (M/\hat{\psi})/N = (R_0^2/2)\,[1 + D/\bar{\lambda}^2]/\{[(\hat{\lambda}\bar{\lambda}) - 1/(\hat{\lambda}\bar{\lambda})^2] + (D/\hat{\lambda}\bar{\lambda})\,[\hat{\lambda}^2 - 1/\hat{\lambda}]\},
$$

where

$$
D = (1-\zeta_1)\hat{\mu}/(\zeta_1 \mu).
$$

If $\zeta_1 = 1$, so that no new material forms, Eq. (7.17) reduces to Rivlin's global universal relation for the torsional stiffness in terms of axial force (cf. [14]). If $\zeta_1 = 0$, so that there is total conversion of the original material, Eq. (7.17) again reduces to a global universal relation. Otherwise, if $0 < \zeta_1 < 1$, the ratio $(M/\hat{\psi})/N$ depends on ζ_1 and no universal relation is possible.

In concluding this section, we wish to point out that this analysis has been carried out using a more general constitutive equation. The original and the newly formed material have different constitutive expressions for incompressible isotropic nonlinear elastic response. In the present section both materials are taken to be neo-Hookean for purposes of simplicity.

8. Extension and torsion of a cylinder — II

In this problem, the cylinder is first subjected to a uniaxial extension of an amount less than is needed to cause the activation criterion to be satisfied. The bar is then twisted. At some angle of twist, the additional strain, and its radial dependence, causes the activation criterion to be satisfied initially at the outermost material elements of the cylinder.

As twisting increases, the interior material elements become further deformed. At any angle of twist, there is an interior radius at which the activation criterion is satisfied. Inside this radius, the material elements consist only of the original material. Outside this radius, there are material elements of both the original and newly formed material.

Let (R, Θ, Z) denote the coordinates of a particle in $\varkappa_0$. Let $\varkappa'$ denote the configuration of the cylinder at the end of the uniaxial extension portion of the deformation. Let $\bar{\lambda}$ denote the corresponding axial stretch ratio. If the particle coordinates in $\varkappa'$ are denoted by (r', θ', z'), then the mapping from $\varkappa_0$ to $\varkappa'$ can be written as

$$
\begin{aligned}
r' &= (1/\sqrt{\bar{\lambda}})R,\\
\theta' &= \Theta,\\
z' &= \bar{\lambda}Z,
\end{aligned}
\tag{8.1}
$$

where we have incorporated the restriction of incompressibility which led to Eq. (7.3).

Now let the cylinder be twisted to configuration $\varkappa(t)$, in which particle coordinates are denoted by (r, θ, z). The deformation from $\varkappa'$ to $\varkappa(t)$ is given by

$$
\begin{aligned}
r &= r',\\
\theta &= \theta'+\psi z',\\
z &= z',
\end{aligned}
\tag{8.2}
$$

and from $\varkappa_0$ to $\varkappa(t)$ is given by

$$
\begin{aligned}
r &= (1/\sqrt{\bar{\lambda}})R,\\
\theta &= \Theta+\psi\bar{\lambda}Z,\\
z &= \bar{\lambda}Z.
\end{aligned}
\tag{8.3}
$$

The physical components of the deformation gradient $\mathbf{F}_{\varkappa_0}$ with respect to cylindrical coordinates are given by

$$
\mathbf{F}_{\varkappa_0} = \begin{bmatrix} dr/dR & 0 & 0 \\ 0 & r/R & r(d\theta/dZ) \\ 0 & 0 & \bar{\lambda} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{\bar{\lambda}} & 0 & 0 \\ 0 & 1/\sqrt{\bar{\lambda}} & r\psi\bar{\lambda} \\ 0 & 0 & \bar{\lambda} \end{bmatrix}.
\tag{8.4}
$$

Clearly, $\det|\mathbf{F}_{\varkappa_0}| = 1$. The strain tensor $\mathbf{B}_{\varkappa_0} = \mathbf{F}_{\varkappa_0}\mathbf{F}^T_{\varkappa_0}$ is given by

$$
\mathbf{B}_{\varkappa_0} = \begin{bmatrix} (1/\bar{\lambda}) & 0 & 0 \\ 0 & (1/\bar{\lambda})+(r\psi\bar{\lambda})^2 & r\psi\bar{\lambda}^2 \\ 0 & r\psi\bar{\lambda}^2 & \bar{\lambda}^2 \end{bmatrix},
\tag{8.5}
$$

and its invariants are given by

$$
\begin{aligned}
I_1 &= 2/\bar{\lambda}+\bar{\lambda}^2+(r\psi)^2\bar{\lambda}^2,\\
I_2 &= 2\bar{\lambda}+1/\bar{\lambda}^2+(r\psi)^2\bar{\lambda}.
\end{aligned}
\tag{8.6}
$$

When $\psi = 0$, I_1 and I_2 have the same value at all material elements of the cylinder, that for uniaxial extension. Consider the material element at a typical radius r. As $r\psi$ increases, both I_1 and I_2 increase. At some value of $r\psi$, say K^*, the activation criterion will be satisfied at that material element. By Eqs. (8.6), the dependence of I_1 and I_2 on $r\psi$ is the same for all material elements of the cylinder. Since all material elements have the same activation criterion, this criterion will be satisfied whenever

$$r\psi = K^*. \tag{8.7}$$

Let ψ^* be the angle of twist when Eq. (3.2) is satisfied at the outermost element. Then, by Eqs. (8.3) and (8.7),

$$r_0\psi^* = \left(R_0/\sqrt{\bar{\lambda}}\right)\psi^* = K^*. \tag{8.8}$$

Now consider values of $\psi > \psi^*$. By Eqs. (8.7) and (8.8), the relation between the radius of a material element r_a and the angle of twist ψ when the activation criterion is met at that element is given by

$$r_a\psi = r_0\psi^*, \tag{8.9}$$

or

$$R_a\psi = R_0\psi^*.$$

Accordingly, given the angle of twist $\psi > \psi^*$, the material elements with $r < (\psi^*/\psi)\, r_0$ consist solely of the original material, while for $r > (\psi^*/\psi)\, r_0$ there are elements of both the original and the newly formed material. We will refer to $r_a = (\psi^*/\psi)\, r_0$ as the activation radius.

Now consider a typical radius r between the current activation radius r_a and the outer radius. The deformation gradient for the original material is given by Eq. (8.4). The deformation gradient of the newly formed material element relates its current configuration in $\varkappa(t)$ to its configuration when it was formed. However, the latter configuration varies with the material element because the activation criterion is satisfied at different radii at different angles of twist. Let $\varkappa_1(r)$ denote the local configuration of material element when it is formed, and let $\mathbf{F}_{\varkappa_1(r)}$ denote the deformation gradient relating $\varkappa_1(r)$ to the current configuration in $\varkappa(t)$.

Denote the deformation gradient for the local homogeneous deformation of a material element from its state in $\varkappa_0$ to $\varkappa_1(r)$ by $\mathbf{F}_{\varkappa_0,\varkappa_1(r)}$. This is given by Eq. (8.4) with $r\psi = K^*$, by (8.7). Making use of Eq. (8.8), we see that

$$\mathbf{F}_{\varkappa_0,\varkappa_1(r)} = \begin{bmatrix} 1/\sqrt{\bar{\lambda}} & 0 & 0 \\ 0 & 1/\sqrt{\bar{\lambda}} & r_0\psi^*\bar{\lambda} \\ 0 & 0 & \bar{\lambda}^{1} \end{bmatrix}. \tag{8.10}$$

Recalling Eq. (2.6), the deformation gradient $\mathbf{F}_{\varkappa_1(r)}$ is given by

$$\mathbf{F}_{\varkappa_1(r)} = \mathbf{F}_{\varkappa_0}(\mathbf{F}_{\varkappa_0,\varkappa_1(r)})^{-1}. \tag{8.11}$$

Since

$$(8.12)\qquad (\mathbf{F}_{\varkappa_0,\varkappa_1(r)})^{-1} = \begin{bmatrix} \sqrt{\bar\lambda} & 0 & 0 \\ 0 & \sqrt{\bar\lambda} & -r_0\psi^*\sqrt{\bar\lambda} \\ 0 & 0 & 1/\bar\lambda \end{bmatrix},$$

it follows from Eqs. (8.4), (8.11) and (8.12) that

$$(8.13)\qquad \mathbf{F}_{\varkappa_1(r)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r\psi - r_0\psi^* \\ 0 & 0 & 1 \end{bmatrix}.$$

The form of the matrix in Eq. (8.13) represents the fact that there has only been shear and no extension since the new material element was formed, and the amount of shear is $r\psi - r_0\psi^*$. The local homogeneous deformation represented by $F_{\varkappa_1(r)}$ is analogous to that in Eqs. (5.4).

The stress distribution for $0 \leqslant r \leqslant r_a$ is given by

$$(8.14)\qquad \mathbf{T} = -p\mathbf{I} + \mu\mathbf{B}_{\varkappa_0} = -p\mathbf{I} + \mu\begin{bmatrix} 1/\bar\lambda & 0 & 0 \\ 0 & (1/\bar\lambda) + (r\psi\bar\lambda)^2 & r\psi\bar\lambda^2 \\ 0 & r\psi\bar\lambda^2 & \bar\lambda^2 \end{bmatrix},$$

where $\mathbf{B}_{\varkappa_0} = \mathbf{F}_{\varkappa_0}\mathbf{F}^T_{\varkappa_0}$ and $\mathbf{F}_{\varkappa_0}$ is given by Eq. (8.4). The stress distribution for $r_a \leqslant r \leqslant r_0$ is given by

$$(8.15)\qquad \mathbf{T} = -p\mathbf{I} + \zeta_1\mu\mathbf{B}_{\varkappa_0} + (1-\zeta_1)\mu\mathbf{B}_{\varkappa_1(r)}$$

$$= -p\mathbf{I} + \zeta_1\mu\begin{bmatrix} 1/\bar\lambda & 0 & 0 \\ 0 & (1/\bar\lambda) + (r\psi\bar\lambda)^2 & r\psi\bar\lambda^2 \\ 0 & r\psi\bar\lambda^2 & \bar\lambda^2 \end{bmatrix}$$

$$+ (1-\zeta_1)\mu\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + (r\psi - r_0\psi^*)^2 & r\psi - r_0\psi^* \\ 0 & r\psi - r_0\psi^* & 1 \end{bmatrix},$$

where $\mathbf{B}_{\varkappa_1(r)} = \mathbf{F}_{\varkappa_1(r)}\mathbf{F}^T_{\varkappa_1(r)}$ and $\mathbf{F}_{\varkappa_1(r)}$ is given by Eq. (8.13).

A discussion similar to that leading up to Eq. (7.13) shows that the equilibrium equations are satisfied when $p = p(r)$, and

$$(8.16)\qquad p(r) = \tilde\sigma_{rr} + \int_r^{r_0} [(\tilde\sigma_{\theta\theta} - \tilde\sigma_{rr})/r']\,dr',$$

where $T_{ii} = -p + \tilde\sigma_{ii}$, $ii = rr, \theta\theta, zz$. Using Eqs. (8.14), (8.15) and (8.16) the scalar field is found to be given by following expressions for $0 \leqslant r \leqslant r_a$,

$$(8.17)\qquad p(r)/\mu = \Big\{\tilde\sigma_{rr} + \int_r^{r_a} [(\tilde\sigma_{\theta\theta} - \tilde\sigma_{rr})/r']\,dr' + \int_{r_a}^{r_0} [(\tilde\sigma_{\theta\theta} - \tilde\sigma_{rr})/r']\,dr'\Big\}/\mu$$

$$= (1/\bar\lambda) + [(\psi\bar\lambda)^2/2](r_a^2 - r^2) + [\zeta_1(\psi\bar\lambda)^2/2](r_0^2 - r_a^2)$$

$$+ (1-\zeta_1)[(\psi^2/2)(r_0^2 - r_a^2) - 2\psi\psi^* r_0(r_0 - r_a) + (r_0\psi^*)^2\ln(r_0/r_a)],$$

and for $r_a \leqslant r \leqslant r_0$,

$$(8.18)\quad p(r)/\mu = \left\{\tilde{\sigma}_{rr} + \int_r^{r_0} [(\tilde{\sigma}_{rr} - \tilde{\sigma}_{\theta\theta})/r']dr'\right\}/\mu = \zeta_1(1/\bar{\lambda}) + (1-\zeta_1) + [\zeta_1(\psi\bar{\lambda})^2/2](r_0^2 - r^2) + (1-\zeta_1)[(\psi^2/2)(r_0^2-r^2) - 2\psi\psi^* r_0(r_0 - r) + (r_0\psi^*)^2 \ln(r_0/r)].$$

The twisting moment on the cylinder is given by

$$(8.19)\quad M = 2\pi \int_0^{r_0} r^2 T_{z\theta} dr.$$

When $\psi \leqslant \psi^*$, $T_{z\theta}$ is given by Eq. (8.14) and

$$(8.20)\quad M = \mu J_0 \psi,$$

where $J_0 = \pi R_0^4/2$. When $\psi > \psi^*$, Eq. (8.19), $T_{z\theta}$ from Eq. (8.14) for $0 \leqslant r \leqslant r_a$, and Eq. (8.15) for $r_a \leqslant r \leqslant r_0$, and Eqs. (8.9) give

$$(8.21)\quad M/(\mu J_0 \psi) = (\psi^*/\psi)^4 + \zeta_1[1-(\psi^*/\psi)^4] + [(1-\zeta_1)/\bar{\lambda}^2]\{[1-(\psi^*/\psi)^4] - (4/3)(\psi^*/\psi)[1-(\psi^*/\psi)^3]\}.$$

A plot of $M/(\mu J_0)$ versus ψ is shown in Fig. 1 for the case in which $\lambda > 1$. When $\psi = \psi^*$,

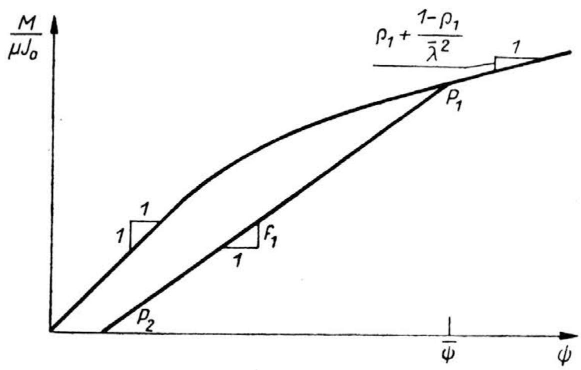

FIG. 1. Torque-twist curve; OP_1 — loading portion, P_1P_2 — unloading portion, during which $\hat{\lambda} = 1$.

the activation criterion is met at the outermost material elements. As was discussed following Eq. (5.6), there is a sudden decrease in the stress transmitted by these elements. As ψ increases there is a stress drop at material elements in the interior of the cylinder, and the stress in the outer material elements begins again to increase. This accounts for the decrease in slope.

The axial force on the cylinder is given by

$$(8.22)\quad N = 2\pi \int_0^{r_0} r T_{zz} dr,$$

in which T_{zz} is found from Eqs. (8.14), (8.15), (8.16) and (8.17). For purposes of brevity, this calculation will be omitted.

9. Untwisting and axial recovery of the cylinder

We now consider the response of the cylinder as the twisting is reversed and axial extension is reduced. Let $\bar{\psi}$ denote the maximum angle of twist and $\bar{\varkappa}$ denote the corresponding configuration. If $(\bar{r}, \bar{\theta}, \bar{z})$ are particle coordinates in $\bar{\varkappa}$, then by Eq. (8.3),

$$\begin{aligned} \bar{r} &= (1/\sqrt{\bar{\lambda}})R, \\ \bar{\theta} &= \Theta+\bar{\psi}\bar{\lambda}Z, \\ \bar{z} &= \bar{\lambda}Z. \end{aligned} \tag{9.1}$$

By Eqs. (8.9) the interaction radius is given by $R_a = (\psi^*/\bar{\psi})\, R_0$.

Configuration $\varkappa(t)$ at any state of deformation reversal is related to configuration $\bar{\varkappa}$ by

$$\begin{aligned} r &= (1/\sqrt{\hat{\lambda}})\bar{r}, \\ \theta &= \bar{\theta}+\hat{\psi}\hat{\lambda}\bar{z}, \\ z &= \hat{\lambda}\bar{z}, \end{aligned} \tag{9.2}$$

where $\hat{\lambda} < 1$ and $\hat{\psi} < 0$. The composite mapping which relates $\varkappa_0$ to $\varkappa(t)$ is found from Eqs. (9.1) and (9.2) to be

$$\begin{aligned} r &= (1/\sqrt{\hat{\lambda}\bar{\lambda}})R, \\ \theta &= \Theta+(\bar{\psi}\bar{\lambda}+\hat{\psi}\hat{\lambda}\bar{\lambda})Z, \\ z &= \hat{\lambda}\bar{\lambda}Z. \end{aligned} \tag{9.3}$$

For notational convenience, let

$$\tilde{K} = \Psi\bar{\lambda}+\hat{\psi}\hat{\lambda}\bar{\lambda} = \bar{\lambda}(\psi+\hat{\psi}\hat{\lambda}). \tag{9.4}$$

By Eq. (8.4)

$$\mathbf{F}_{\varkappa_0} = \begin{bmatrix} 1/\sqrt{\hat{\lambda}\bar{\lambda}} & 0 & 0 \\ 0 & 1/\sqrt{\hat{\lambda}\bar{\lambda}} & r\tilde{K} \\ 0 & 0 & \hat{\lambda}\bar{\lambda} \end{bmatrix}. \tag{9.5}$$

The corresponding strain tensor and its invariants are

$$\mathbf{B}_{\varkappa_0} = \mathbf{F}_{\varkappa_0}\mathbf{F}_{\varkappa_0}^T = \begin{bmatrix} 1/\hat{\lambda}\bar{\lambda} & 0 & 0 \\ 0 & (1/\hat{\lambda}\bar{\lambda})+(r\tilde{K})^2 & r\tilde{K}\hat{\lambda}\bar{\lambda} \\ 0 & r\tilde{K}\hat{\lambda}\bar{\lambda} & (\hat{\lambda}\bar{\lambda})^2 \end{bmatrix}, \tag{9.6}$$

$$\begin{aligned} I_1 &= 2/(\hat{\lambda}\bar{\lambda})+(\hat{\lambda}\bar{\lambda})^2+r^2\bar{\lambda}^2(\bar{\psi}+\hat{\psi}\hat{\lambda})^2, \\ I_2 &= 2\hat{\lambda}\bar{\lambda}+1/(\hat{\lambda}\bar{\lambda})^2+r^2\bar{\lambda}(\bar{\psi}+\hat{\psi}\bar{\lambda})^2. \end{aligned} \tag{9.7}$$

Let I_1^0 and I_2^0 denote the first two terms in the expressions for I_1 and I_2, respectively. These correspond to pure uniaxial extension, and are monotonically increasing functions of the axial stretch ratio. Since $\hat{\lambda} < 1$, the values of I_1^0 and I_2^0 in Eqs. (9.7) are smaller than the corresponding values in Eqs. (8.6). Since $\hat{\psi} < 0$, the value of the last terms in I_1 and I_2 in Eqs. (9.7) are smaller than the corresponding terms in Eqs. (8.6). It follows that the values of I_1 and I_2 decrease during the reversal of deformation, as might have been expected.

In the configuration at maximum angle of twist $\bar{\psi}$, the activation radius, by Eqs. (8.9), is $R_a = (\psi^*/\bar{\psi})R_0$. The preceding discussion shows that for the elements of the original material in the inner core, $0 \leqslant R \leqslant R_a = (\psi^*/\bar{\psi})R_0$, the activation criterion will not be satisfied during deformation reversal. As it was assumed as a part of the constitutive model, the elements in the outer layer, $R_a \leqslant R \leqslant R_0$, undergo no further change. We conclude that the activation radius R_a does not change during deformation reversal.

Equation (9.5) represents the deformation gradient for the original material in the inner core and for the remaining portion of the inner material in the outer layer. In order to calculate the deformation gradient for the newly formed material elements in the outer layer, recall the discussion preceding Eq. (8.10). By Eqs. (8.11), (8.12) in which $r_0 = R_0/\sqrt{\lambda}$, and Eq. (9.5),

$$\text{(9.8)} \qquad \mathbf{F}_{\varkappa_1(r)} = \begin{bmatrix} 1/\sqrt{\hat{\lambda}} & 0 & 0 \\ 0 & 1/\sqrt{\hat{\lambda}} & \gamma \\ 0 & 0 & \hat{\lambda} \end{bmatrix}.$$

In Eq. (9.8),

$$\text{(9.9)} \qquad \gamma = [R(\bar{\psi}+\hat{\psi}\hat{\lambda}) - R_0\psi^*]/\sqrt{\hat{\lambda}\bar{\lambda}} = r(\bar{\psi}+\hat{\psi}\hat{\lambda}) - r_0\psi^*,$$

the latter following by Eqs. (9.3).

The stress distribution for $0 \leqslant r \leqslant r_a$ is, by Eq. (9.5),

$$\text{(9.10)} \qquad \mathbf{T} = -p\mathbf{I} + \mu\mathbf{B}_{\varkappa_0} = -p\mathbf{I} + \mu \begin{bmatrix} 1/\hat{\lambda}\bar{\lambda} & 0 & 0 \\ 0 & (1/\hat{\lambda}\bar{\lambda}) + r^2\tilde{K}^2 & r\tilde{K}\hat{\lambda}\bar{\lambda} \\ 0 & r\tilde{K}\hat{\lambda}\bar{\lambda} & (\hat{\lambda}\bar{\lambda})^2 \end{bmatrix}.$$

The stress distribution for $r_a \leqslant r \leqslant r_0$ is, by Eqs. (9.5) and (9.8),

$$\text{(9.11)} \qquad \mathbf{T} = -p\mathbf{I} + \zeta_1\mu\mathbf{B}_{\varkappa_0} + (1-\zeta_1)\mu\mathbf{B}_{\varkappa_1(r)} = -p\mathbf{I} + \zeta_1\mu \begin{bmatrix} 1/\hat{\lambda}\bar{\lambda} & 0 & 0 \\ 0 & (1/\hat{\lambda}\bar{\lambda}) + (r\tilde{K})^2 & r\tilde{K}\hat{\lambda}\bar{\lambda} \\ 0 & r\tilde{K}\hat{\lambda}\bar{\lambda} & (\hat{\lambda}\bar{\lambda})^2 \end{bmatrix}$$

$$+ (1-\zeta_1)\mu \begin{bmatrix} 1/\hat{\lambda} & 0 & 0 \\ 0 & (1/\hat{\lambda}) + \gamma^2 & \gamma\hat{\lambda} \\ 0 & \gamma\hat{\lambda} & \hat{\lambda}^2 \end{bmatrix}.$$

As discussed earlier, the stress field given by Eqs. (9.10) and (9.11) will satisfy the equilibrium equations if the scalar field p is given by Eq. (8.16). This is found to be, for $0 \leqslant r \leqslant r_a$,

$$\text{(9.12)} \qquad p(r)/\mu = (1/\hat{\lambda}\bar{\lambda}) + (1/2)\tilde{K}^2[A^2r_0^2 - r^2] + (1/2)\zeta_1\tilde{K}^2r_0^2[1-A^2]$$

$$+ (1-\zeta_1)\,[(1/2)(K/\lambda)^2r_0^2(1-A^2) - (2K\bar{\psi}^*r_0^2/\lambda)(1-A) - (r_0\psi^*)^2\ln A],$$

where $A = \psi^*/\bar{\psi}$. For $r_a \leqslant r \leqslant r_0$,

$$p(r)/\mu = \zeta_1(1/\hat{\lambda}\bar{\lambda})+(1-\zeta_1)(1/\hat{\lambda})+(1/2)\zeta_1\tilde{K}^2(r_0^2-r^2) + (1-\zeta_1)[(1/2)(\tilde{K}/\bar{\lambda})^2(r_0^2-r^2)-(2\tilde{K}\psi^* r_0/\bar{\lambda})(r_0-r)+(r_0\psi^*)^2\ln(r_0/r)]. \tag{9.13}$$

The axial force on the cylinder is given by

$$N = 2\pi\int_0^{r_0} rT_{zz}dr = 2\pi\Big[\int_0^{r_a} rT_{zz}dr+\int_{r_a}^{r_0} rT_{zz}dr\Big], \tag{9.14}$$

in which

$$T_{zz} = -p+\mu(\hat{\lambda}\bar{\lambda})^2, \quad 0 \leqslant r \leqslant r_a, \tag{9.15}$$

with p given by Eq. (9.12) and

$$T_{zz} = -p+\zeta_1\mu(\hat{\lambda}\bar{\lambda})^2+(1-\zeta_1)\mu\hat{\lambda}^2, \quad r_a \leqslant r \leqslant r_0, \tag{9.16}$$

with p given by Eq. (9.13).

Upon combining Eqs. (9.12)–(9.16) we find

$$\begin{aligned}[N(\hat{\lambda}\bar{\lambda})]/(\mu\pi R_0^2) &= [(\hat{\lambda}\bar{\lambda})^2-(1/\hat{\lambda}\bar{\lambda})]A^2+\{\zeta_1[(\hat{\lambda}\bar{\lambda})^2-(1/\hat{\lambda}\bar{\lambda})]+(1-\zeta_1)[\hat{\lambda}^2-(1/\hat{\lambda})]\}(1-A^2) \\ &\quad -[(\tilde{K}r_0)^2/4][A^4+(\zeta_1+(1-\zeta_1)/\bar{\lambda}^2)(1-A^4)] \\ &\quad +(1-\zeta_1)\{(\tilde{K}/\bar{\lambda})(2/3)A(1-A^3)\bar{\psi}-(1/2)A^2(1-A^2)\bar{\psi}^2\}r_0^2.\end{aligned} \tag{9.17}$$

The shear stress distribution is, from Eqs. (9.10) and (9.11),

$$\begin{aligned}T_{z\theta} &= \mu r\tilde{K}\hat{\lambda}\bar{\lambda}, \quad 0 \leqslant r \leqslant r_a, \\ &= \zeta_1\mu r\tilde{K}\hat{\lambda}\bar{\lambda}+(1-\zeta_1)\mu\gamma\hat{\lambda}, \quad r_a \leqslant r \leqslant r_0.\end{aligned} \tag{9.18}$$

Upon substituting into the twisting moment

$$M = 2\pi\int_0^{r_a} r^2T_{z\theta}+2\pi\int_{r_a}^{r_0} r^2T_{z\theta}dr, \tag{9.19}$$

and integrating, it is found that

$$[M(\hat{\lambda}\bar{\lambda})^2]/(\mu J_0) = (\hat{\psi}\hat{\lambda}+\bar{\psi})\hat{\lambda}\bar{\lambda}^2\{A^4+(1-A^4)[\zeta_1+(1-\zeta_1)/\bar{\lambda}^2]\} -\bar{\psi}\hat{\lambda}(4/3)(1-\zeta_1)A(1-A^3). \tag{9.20}$$

There are many possible sequences of untwisting and axial stretch recovery that can be considered. We will consider the case in which the cylinder is allowed to untwist while at maximum axial stretch, $\hat{\lambda} = 1$. The M-ψ relation during deformation recovery is then

$$M/(\mu J_0) = (\hat{\psi}+\bar{\psi})\{A^4+(1-A^4)[\zeta_1+(1-\zeta_1)/\bar{\lambda}^2]\}-(4/3)[(1-\zeta_1)/\bar{\lambda}^2](1-A^3)A\bar{\psi}. \tag{9.21}$$

The linear relation between M and $\hat{\psi}$ is a consequence of both the original and newly formed material being neo-Hookean in response.

The loading and unloading curves are shown in Fig. 1. The point at maximum deformation is denoted as P_1. It has coordinates $(\bar{M}, \bar{\psi})$ which are related by

$$\bar{M}/(\mu J_0) = \bar{\psi}f_1-f_2, \tag{9.22}$$

where

$$f_1 = A^4+(1-A^4)\,[\zeta_1+(1-\zeta_1)/\bar{\lambda}^2], \tag{9.23}$$
$$f_2 = (4/3)\,[(1-\zeta_1)/\bar{\lambda}^2]\,(1-A^3)A\bar{\psi}.$$

Note that since $A < 1$, $\zeta_1 < 1$, then $f_1 > 0, f_2 > 0$. It can be shown that if $\bar{\lambda} > 1$, then $f_1 < 1$.

Equation (9.21) can be written in terms of Eqs. (9.23) in the form

$$M/(\mu J_0) = \hat{\psi}f_1+\bar{\psi}f_1-f_2. \tag{9.24}$$

According to Eq. (8.20) the loading portion of the curve in Fig. 1 has a slope whose value is one. It is seen from Eq. (9.24) that the slope of the unloading portion of the curve is $f_1 < 1$ and is thus less than that of the loading portion.

According to Eqs. (9.3) the angle of twist of the cylinder at any stage of deformation reversal is $(\bar{\psi}+\hat{\psi}\hat{\lambda})\bar{\lambda}$. From Eq. (9.20) the residual angle of twist when $M = 0$, for any axial elongation recovery $\hat{\lambda}$, is given by

$$\hat{\psi}\hat{\lambda}+\bar{\psi} = \frac{4}{3}\,\frac{[(1-\zeta_1)\,(1-A^3)\,A]}{[\bar{\lambda}^2A^4+(1-A^4)\,(\zeta_1\bar{\lambda}^2+1-\zeta_1)]}\,\bar{\psi}, \tag{9.25}$$

which is positive since $A < 1$, $\zeta_1 < 1$. The recovered angle of twist from the configuration of maximum deformation is $\hat{\psi}\hat{\lambda}$, by Eqs. (9.2). This can be calculated from Eq. (9.25) to be

$$\hat{\psi}\hat{\lambda} = -\,\frac{[(1-\zeta_1)/\bar{\lambda}^2]\,[1-(4/3)A+(1/3)A^4]+[\zeta_1+(1-\zeta_1)A^4]}{A^4+(1-A^4)\,[\zeta_1+(1-\zeta_1)/\bar{\lambda}^2]}\,\bar{\psi}. \tag{9.26}$$

It is straightforward to show that $\hat{\psi}\hat{\lambda} < 0$ and $|\hat{\psi}\hat{\lambda}| < \hat{\psi}$.

When $M = 0$, the axial force is calculated by recalling that $\tilde{K} = \bar{\lambda}(\bar{\psi}+\hat{\psi}\hat{\lambda})$, and then substituting from Eq. (9.25) into Eq. (9.17). The result is

$$[N(\hat{\lambda}\bar{\lambda})]/(\mu J_0) = (\hat{\lambda}\bar{\lambda})^2\{A^2+[\zeta_1+[(1-\zeta_1)/\bar{\lambda}^2]](1-A^2)\} \tag{9.27}$$
$$-\,[1/(\hat{\lambda}\bar{\lambda})]\{A^2+[\zeta_1+\bar{\lambda}(1-\zeta_1)](1-A^2)+\alpha\},$$

where α is defined by the expression,

$$\alpha = [\bar{\psi}^2R_0^2A^2/2] \tag{9.28}$$
$$\times\,\frac{[(1-\zeta_1)/\bar{\lambda}]^2[(1-A^2)-(8/9)\,(1-A^3)^2]+(1-\zeta_1)\,(1-A^2)\,[\zeta_1+A^4(1-\zeta_1)]}{A^4+(1-A^4)\,[\zeta_1+(1-\zeta_1)/\bar{\lambda}^2]}.$$

The expression α, which depends on ζ_1, $\bar{\lambda}$, $\bar{\psi}$, A can be shown to be positive when $A < 1$. When $N = 0$, the residual axial stretch ratio is given by

$$\hat{\lambda}\bar{\lambda} = \left\{\frac{[A^2+\big(\zeta_1+\bar{\lambda}(1-\zeta_1)\big)\,(1-A^2)+\alpha]}{A^2+[\zeta_1+(1-\zeta_1)/\bar{\lambda}^2]\,(1-A^2)}\right\}^{1/3}. \tag{9.29}$$

When $\bar{\lambda} > 1$, it is seen that $\hat{\lambda}\bar{\lambda} > 1$.

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