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Steady linearised aerodynamics
III. Transonic

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IN THIS PAPER we discuss the application of the integral equation method to the study of steady transonic flow of inviscid fluid past an aerofoil. For the three-dimensional nonplanar aerofoil the solution of the problem is reduced to that of system of two singular nonlinear integral equations. In the plane case we succeded in obtaining an integral equation whose kernel is expressed in terms of elementary functions only.

Przedyskutowano zastosowanie metody równań całkowych do analizy ustalonych przepływów okołodźwiękowych płynów nielepkich wokół płata. W przypadku płata trójwymiarowego zagadnienie sprowadza się do układu dwóch nieliniowych równań całkowych osobliwych. W przypadku płaskim uzyskuje się równanie całkowe z jądrem wyrażonym przez funkcje elementarne.

Обсуждено применение метода интегральных уравнений для анализа установившихся околозвуковых течений невязких жидкостей вокруг крыла. В случае трехмерного крыла задача сводится к системе двух нелинейных особых интегральных уравнений. В плоском случае получается интегральное уравнение с ядром, выраженным через элементарные функции.

1. Introduction

THIS WORK continues the papers [1, 2] concerning the linearised motion of compressible fluid past thin aerofoils. While in papers [1] and [2] we considered the subsonic–supersonic regimes, the regime investigated here is the transonic one. A common feature of all these papers is the fact that they are based on fluid motion equations (and the corresponding jump conditions) only, without any supplementary hypothesis concerning the discontinuity surfaces inside the flow.

Thus, our analysis avoids the consideration of the sources and vortices on the body surface and on the vortex sheet (free vortices) behind the airfoil, as in other lifting line or lifting surface theories (e.g. [3]), and we do not need the replacement of the body surface by a continuous distribution of momentum as assumed in [4] either. This is important since it proves that it is of no use to look for other "models" to describe the motion of inviscid fluids in the limits of the small perturbation theories. Some terms in the obtained representation formulae can be associated with contribution of sources and vortices on the body surface and of a vortex sheet behind the body, but this is a *consequence* (therefore an aposteriori result) of the theory and not a *hypothesis* in developing it.

In all cases we succeeded in obtaining integral equations for determining the main parameter of interest — the local lift on the aerofoil. In the supersonic regime, for a large class of thin bodies this equation can be analytically solved; in the other cases the corresponding integral equations are integrated numerically or asymptotically.

The approach used herein is somewhat different from those considered in the previous papers [1] and [2]. So, unlike the line of attack of the papers mentioned, based on the form of distributions of the flow equations, here the direct use of the Fourier transforms proved to be sufficient for developing the complete theory. Theoretically the form of distributions of the flow equations is more general, they being valid for general nonlinear equations; from the practical point of view to solve these equations one must very often linearise them first. The fact that we left aside the form of distributions of the flow equations considerably simplyfyies the tratement. Now, the main mathematical apparatus used is that of integral transorms, in fact a "generalized operational calculus". By means of this calculus we obtain the integral representation of the solution and, further on, the integral equations of the problem.

The geometry of the aerofoil considered in this work is more general than that studied in previous papers; we consider here the case of nonplanar thin aerofoils at a small incidence angle. From the very begining the analysis of the problem puts into evidence the parameter of aerodynamic interest $l(x, y)$ (the local lift on the aerofoil). The usual mathematical tool, the Fourier transforms, enables us to obtain an integral representation of the velocity field in terms of local lift, of the aerofoil geometry and of the velocity component v_x along the undisturbed velocity direction. This representation holds at any point of the space including, possibly, vortex sheets and shock surfaces within the flow. By imposing the boundary condition that the fluid slides the airofoil, we obtain an integral relation over the lifting surface which involves the functions $l(x, y, z)$ and $v_x(x, y, z)$. By adding to this integral relation the expression of the component v_x resulting in the above-mentioned representation, a system of two integral equations (one at the lifting surface and the other in the whole space) is derived for determining the motion. The integral equations are singular and both of them contain nonlinear terms.

We note that a complete formulation of the problem includes the system of integral equations discussed above. The use of a single integral relation considered by some authors in the planar case is incomplete, and the numerical results obtained in this way depend directly on the type of approximation introduced for the nonlinear term. In the case of planar symmetrical aerofoil the first integral equation is identically satisfied and the other relation is just the Oswatitsch's integral equation.

In Sect. 4 we consider the case of plane flow. We obtain an integral representation for the complex velocity $w = v_x - iv_z$ and solution of the problem is reduced to the solution of an integral equation for the velocity component $v_x(x, z)$. The integral equation is still singular and nonlinear but its kernel is expressed in terms of elementary functions only. Thus, we were successful in incorporating more analytic steps into the theory in the problem of interest.

For easier reading of the paper Appendices A, B, C are included containing some formulae concerning singular integrals occurring in the paper and also some laborious calculations.

2. Basic equations

We consider the steady motion of an inviscid fluid past a thin aerofoil S. The body is supposed to be nearly contained in a cylindrical surface S_0 with fairly arbitrary cross-section. Let the origin of coordinates 0 be a point on S_0 and let the x-axis be parallel to the generatrices of S_0. We write $S_0(y, z) = 0$ for the equation of the cylindrical surface and $S_\pm(x, y, z, \varepsilon) = 0$ for the equations of the upper and lower sides of the aerofoil surfaces. Here ε stands for a small parameter characterising the thinness of the body. We denote by $\mathbf{n}_0$ the unit vector of the normal to the surface S_0 and by $\mathbf{n}$ — the unit vector of the normal to the surfaces $S_\pm$ directed as in Fig. 1.

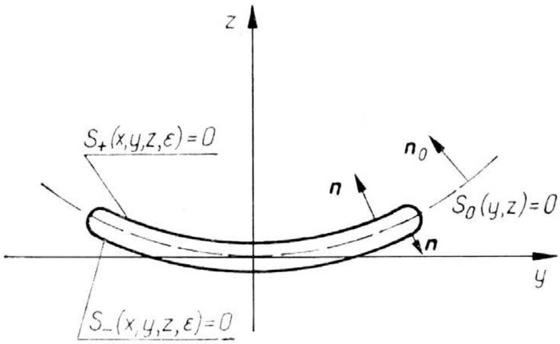

FIG. 1.

The far flow is characterised by density ϱ_∞, pressure p_∞ and velocity $\mathbf{V}_\infty$. We suppose that the incidence angle is small ($\mathbf{V}_\infty \cdot \mathbf{n}_0 = O(\varepsilon)$).

The equations characterising the fluid motion are

$$\operatorname{div}(\varrho \mathbf{V}) = 0, \tag{2.1}$$

$$\varrho(\mathbf{V}\,\mathrm{grad})\mathbf{V} + \mathrm{grad}\, P = 0, \tag{2.2}$$

$$\varrho = \{1 - 0.5(\gamma - 1) M_\infty^2 (V^2 - 1\}^{\frac{1}{\gamma - 1}}. \tag{2.3}$$

Also, on a discontinuity surface Σ inside the flow, to the above mentioned system of equations the jump conditions of Rankine–Hugoniot

$$[\varrho V_n] = 0, \tag{2.4}$$

$$\varrho V_n [\mathbf{V}] + [P]\mathbf{n} = 0 \quad \text{on } \Sigma \tag{2.5}$$

must be added. Here $[a]$ stands for the jump of the quantity a across the surface Σ.

Relation (2.3) is the Bernoulli's equation where the hypothesis of the isentropic flow was included. In the case of transonic flow we shall write this relation in the form

$$\varrho = 1 - 0.5\, M_\infty^2 (V^2 - 1) + 0.125\, M_\infty^4 (2 - \gamma)\, (V^2 - 1)^2 + O(M_\infty^6 (V^2 - 1)^4). \tag{2.6}$$

Here we used dimensionless variables by choosing the following characteristic quantities: L for space variables, ϱ_∞ for density, $V_\infty = |\mathbf{V}_\infty|$ for velocity, $\varrho_\infty V_\infty^2$ for pressure.

It is assumed that the presence of the body induces small perturbations into the basic quantities. Let us denote by small letters the perturbations. Correspondingly, the linearised form of the system (2.1)–(2.3) will be

$$\{1-M_\infty^2-M_\infty^2(3+(\gamma-2)M_\infty^2)v_x\}\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0, \tag{2.7}$$

$$\frac{\partial \mathbf{v}}{\partial x}+\operatorname{grad}p=0. \tag{2.8}$$

Now, we introduce the reduced quantities [5]

$$x=\tilde{x},\qquad \beta y=\tilde{y},\qquad \beta z=\tilde{z},$$

$$v_x=\frac{\beta^2}{\tilde{k}}\tilde{v}_x,\qquad v_y=\frac{\beta^3}{\tilde{k}}\tilde{v}_y,\qquad v_z=\frac{\beta^3}{\tilde{k}}\tilde{v}_z,\qquad p=\frac{\beta^2}{\tilde{k}}\tilde{p},$$

where $\beta^2=1-M^2$ and $\tilde{k}=M^2\cdot(3+(\gamma-2)M^2)$ is a transonic parameter.

Denoting again by $x, y, z, v_x, v_y, v_z, p$ the new variables we obtain the system of equations

$$\operatorname{div}\mathbf{v}-\frac{1}{2}\frac{\partial v_x^2}{\partial x}=0, \tag{2.9}$$

$$\frac{\partial \mathbf{v}}{\partial x}-\operatorname{grad}p=0. \tag{2.10}$$

The system (2.9)–(2.10) will be used at regular points inside the flow domain; on a discontinuity surface Σ we add the linearised form of the jump conditions (2.4), (2.5)

$$\frac{1}{2}[v_n^2]n_x-[v_n]=0, \tag{2.11}$$

$$[\mathbf{v}]n_x+[p]\mathbf{n}=0\qquad \text{on } \Sigma. \tag{2.12}$$

To solve the above system we need appropriate boundary conditions. Thus all perturbations must vanish far upstream. On the other hand, the fluid velocity must be tangent to the wing surface. The linearized form of this condition is

$$\mathbf{v}\cdot\mathbf{n}_0=-\mathbf{u}_0\cdot\mathbf{n}_0\mp n_x^\pm\qquad \text{on } \zeta_0^\pm, \tag{2.13}$$

where $\mathbf{u}_0$ is the unit vector along the undisturbed velocity direction; $\mathbf{n}^+=\mathbf{n}_0+n_x^+\mathbf{i}+O(\varepsilon^2)$, $\mathbf{n}^-=-\mathbf{n}_0+n_x^-\mathbf{i}+O(\varepsilon^2)$ refer to the upper and lower side of the aerofoil surface, respectively.

3. Transonic integral equations for nonplanar aerofoils

To solve the system (2.9)–(2.12) we apply the Fourier transform with respect to all space variables. We assume all dependent variables extended by zero values inside the body. Let

$$\hat{v}_x(k_1,k_2,k_3)\equiv\mathscr{F}[v_x]=\iiint v_x(x,y,z)\exp\{-i(k_1x+k_2y+k_3z)\}dx\,dy\,dz,$$

$$\hat{}_y=\mathscr{F}[v_y],\qquad \hat{v}_z=\mathscr{F}[v_z],\qquad \hat{p}=\mathscr{F}[p]$$

be the corresponding Fourier transforms. We have [6]:

$$\mathscr{F}[\operatorname{grad} p] = i\mathbf{k}\cdot\hat{p} - \iint_{\Sigma} [p]\mathbf{n}e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma - \iint_{S} p\mathbf{n}e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma,$$

$$\mathscr{F}[\operatorname{div}\mathbf{V}] = i\mathbf{k}\cdot\hat{\mathbf{V}} - \iint_{\Sigma} [\mathbf{V}]\cdot\mathbf{n}e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma - \iint_{S} \mathbf{V}\cdot\mathbf{n}e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma,$$

where $\mathbf{k} = k_1\mathbf{i}+k_2\mathbf{j}+k_3\mathbf{k}$, $\mathbf{x} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

The Fourier transform of the system (2.9) and (2.10) will be

$$\begin{aligned} &i\mathbf{k}\cdot\hat{\mathbf{v}} - 0.5ik_1\hat{v}_x^2 = \iint_{\Sigma}([v_n]-0.5[v_x^2]n_x)e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma + \iint_{S_0} me^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma, \\ &ik_1\hat{\mathbf{v}} + i\mathbf{k}\hat{p} = \iint_{\Sigma}([\mathbf{v}]n_x+[p]\mathbf{n})e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma + \iint_{S_0} l\mathbf{n}_0e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma, \end{aligned} \tag{3.1}$$

where

$$m = -(n_x^+ + n_x^-)$$

is a known quantity and

$$l = p^+ - p^-$$

remains to be determined. In fact this is the main aerodynamic term of interest in solving this problem. The integrals over the discontinuity surface Σ vanish due to jump conditions (2.11) and (2.12).

The solution of the system (3.1) can be written in the form

$$\hat{\mathbf{V}} = A\cdot\mathbf{T}, \tag{3.2}$$

where

$$\hat{\mathbf{V}}^T = [\hat{v}_x, \hat{v}_y, \hat{v}_z, \hat{p}],$$

$$\mathbf{T}^T = \Big[0, \iint_{S_0} l\,n_{0y}e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma, \iint_{S_0} l\,n_{0z}e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma, 0.5ik_1\mathscr{F}[v_x^2] + \iint_{S_0} m\,e^{-i\mathbf{k}\cdot\mathbf{x}}d\sigma$$

and A is the matrix

$$\begin{bmatrix} \dfrac{1}{ik_1}+\dfrac{ik_1}{k^2} & \dfrac{ik_2}{k^2} & \dfrac{ik_3}{k^2} & -\dfrac{ik_1}{k^2} \\ \dfrac{ik_2}{k^2} & \dfrac{1}{ik_1}+\dfrac{(ik_2)^2}{ik_1\cdot k^2} & \dfrac{ik_2\cdot ik_3}{ik_1\cdot k^2} & -\dfrac{ik_2}{k^2} \\ \dfrac{ik_3}{k^2} & \dfrac{ik_2\cdot ik_3}{ik_1\cdot k^2} & \dfrac{1}{ik_1}+\dfrac{(ik_3)^2}{ik_1\cdot k^2} & -\dfrac{ik_3}{k^2} \\ -\dfrac{ik_1}{k^2} & -\dfrac{ik_2}{k^2} & -\dfrac{ik_3}{k^2} & \dfrac{ik_1}{k^2} \end{bmatrix}.$$

To determine the inverse Fourier transforms of the above relations we use the formula

$$\mathscr{F}^{-1}\left[\frac{1}{ik_1\cdot k^2}\right] = -\frac{1}{4\pi}\ln|x-r|, \tag{3.3}$$

where $r^2 = x^2+y^2+z^2$. This formula takes into account the vanishing of all perturbations far upstream.

Outside the surface S_0 we have

$$
(3.4)\qquad \mathbf{v}(x, y, z) = \operatorname{grad}\Bigg\{\frac{1}{4\pi}\iint\limits_{S_0} l'\frac{\partial}{\partial n_0'}\ln|x-x'-R|\,d\sigma' - \frac{1}{4\pi}\iint\limits_{S_0}\frac{m'}{R}\,d\sigma' - \frac{1}{8\pi}\iiint\limits_{\mathbf{R}^3} v_x^2(x', y', z')\frac{\partial}{\partial x}\left(\frac{1}{R}\right)dv'\Bigg\},
$$

$$
p(x, y, z) = -v_x(x, y, z).
$$

We have denoted

$$
R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2},\qquad l = l(x', y', z'),\quad dv' = dx'dy'dz' \qquad \text{etc.}
$$

Formula (3.4) is just the representation relation looked for. If the last term in it is neglected, this formula may also be used to develop a nonplanar wing theory in subsonic flow. Here it will be used to obtain the system of integral equations to study transonic flow over aerofoils.

To obtain the integral equation for determining the function $l(x, y, z)$ we shall impose boundary condition (2.13). We get

$$
(3.5)\qquad -\frac{\partial}{\partial n_0}\frac{1}{4\pi}\iint\limits_{S_0} l'\left(1+\frac{x-x'}{R}\right)\frac{(y-y')n_{0y}'+(z-z')n_{0z}'}{(y-y')^2+(z-z')^2}\,d\sigma' \pm \frac{m}{2}
$$
$$
-\frac{1}{4\pi}\iint\limits_{S_0} m'\frac{\partial}{\partial n_0}\left(\frac{1}{R}\right)d\sigma' - \frac{3}{8\pi}\iiint\limits_{\mathbf{R}^3}{}^{*}\, v_x^2(x', y', z')\frac{(x-x')\{(y-y')n_{0y}+(z-z')n_{0z}\}}{R^5}\,dv'
$$
$$
= -\mathbf{u}_0\cdot\mathbf{n}_0 \mp n_x^{\pm} \qquad \text{for}\qquad (x, y, z)\in S_0^{\pm}.
$$

Since the integral equation contains an unknown function v_x^2 we have to consider also the first relation (3.4) in the form

$$
(3.6)\qquad v_x(x, y, z) = -\frac{1}{4\pi}\iint\limits_{S_0} l'\frac{\partial}{\partial n_0'}\left(\frac{1}{R}\right)d\sigma' - \frac{1}{4\pi}\iint\limits_{S_0} m'\frac{\partial}{\partial x}\left(\frac{1}{R}\right)d\sigma'
$$
$$
+\lambda v_x^2(x, y, z) - \frac{1}{8\pi}\iiint\limits_{\mathbf{R}^3}{}^{*}\, v_x^2(x', y', z')\frac{2(x-x')^2-(y-y')^2-(z-z')^2}{R^5}\,dv'.
$$

In relations (3.5) and (3.6) symbol $\iiint^{*}$ denotes a regularization of the corresponding integral discussed in Appendix A. When the finite part of the integral is defined by eliminating a small sphere from the domain of definition we have $\lambda = 1/6$, in case of the Oswatitsch's definition of the principal value of the integral we have to put $\lambda = 1/2$.

The two relations (3.5) and (3.6) form a system of two nonlinear singular integral equations for determining the functions $l(x, y, z)$ and $v_x(x, y, z)$.

We can eliminate the terms containing the function $m(x, y, z)$ in (3.5), (3.6) by means of Prandtl's linearised solution $\tilde{l}(x, y, z)$, $\tilde{v}_x(x, y, z)$. These quantities satisfy relations (3.5), (3.6) where v_x^2 is formally assumed to be zero.

$$(3.5') \qquad -\frac{\partial}{\partial n_0}\frac{1}{4\pi}\iint\limits_{S_0}\tilde{l}'\left(1+\frac{x-x'}{R}\right)\frac{(y-y')n'_{0y}+(z-z')n'_{0z}}{(y-y')^2+(z-z')^2}\,d\sigma' \pm \frac{m}{2}$$
$$-\frac{1}{4\pi}\iint\limits_{S_0} m'\frac{\partial}{\partial n_0}\left(\frac{1}{R}\right)d\sigma' = -\mathbf{u}_0\cdot\mathbf{n}_0 \mp n_x^{\pm} \quad \text{for} \quad (y, x, z) \quad \text{on } S_0^{\pm},$$

$$(3.6') \qquad \tilde{v}_x(x, y, z) = -\frac{1}{4\pi}\iint\limits_{S_0}\tilde{l}'\frac{\partial}{\partial n'_0}\left(\frac{1}{R}\right)d\sigma' - \frac{1}{4\pi}\iint\limits_{S_0} m'\frac{\partial}{\partial x}\left(\frac{1}{R}\right)d\sigma'.$$

By subtracting relation (3.5') and (3.6') from (3.5) and (3.6), respectively, we get

$$(3.7) \qquad -\frac{\partial}{\partial n_0}\frac{1}{4\pi}\iint\limits_{S_0}(l'-\tilde{l}')\left(1+\frac{x-x'}{R}\right)\frac{(y-y')n'_{0y}+(z-z')n'_{0z}}{(y-y')^2+(z-z')^2}\,d\sigma'$$
$$-\frac{3}{8\pi}\iiint\limits_{\mathbf{R}^3}{}^{*}\, v_x^2(x', y', z')\frac{(x-x')\{(y-y')n_{0y}+(z-z')n_{0z}\}}{R^5}\,dv' = 0, \quad (x\ \ y, z)\in S_0,$$

$$(3.8) \qquad v_x(x, y, z) = \tilde{v}_x(x, y, z) - \frac{1}{4\pi}\iint\limits_{S_0}(l'-\tilde{l}')\frac{\partial}{\partial n'_0}\left(\frac{1}{R}\right)d\sigma' + \lambda v_x^2(x, y, z)$$
$$-\frac{1}{8\pi}\iiint\limits_{\mathbf{R}^2}{}^{*}\, v_x^2(x', y', z')\frac{2(x-x')^2-(y-y')^2-(z-z')^2}{R^5}\,dv', \quad (x, y, z)\in\mathbf{R}^3.$$

This system of two integral equations may be solved by the method of succesive approximations, Prandtl's linearized solution being assumed as the zero'th approximation.

The relation (3.8) over S_0 also gives

$$(3.9) \qquad v_x^{\pm}(x, y, z) = \tilde{v}_x^{\pm}(x, y, z) \mp \frac{1}{2}\{l(x, y, z)-\tilde{l}(x, y, z)\}$$
$$-\frac{1}{4\pi}\iint\limits_{S_0}(l'-\tilde{l}')\frac{\partial}{\partial n'_0}\left(\frac{1}{R}\right)d\sigma' + \lambda v_x^2(x, y, z)$$
$$-\frac{1}{8\pi}\iiint\limits_{\mathbf{R}^3} v_x^2(x', y', z')\frac{2(x-x')^2-(y-y')^2-(z-z')^2}{R^5}\,dv', \quad (x, y, z)\in S_0.$$

In the case of planar aerofoil ($S_0 \equiv \{z = 0\}$) the above relations are somewhat simplyfying. Thus the integral containing m in Eq. (3.5) disapears as well as the integral containing $l-\tilde{l}$ in Eq. (3.9).

In this case we have

$$-\frac{\partial}{\partial n_0}\frac{1}{4\pi}\iint\limits_{S_0} l'\left(1+\frac{x-x'}{R}\right)\frac{(y-y')n'_{0y}+(z-z')n'_{0z}}{(y-y')^2+(z-z')^2}\,d\sigma'$$
$$= -\frac{1}{4\pi}\iint\limits_{S_0}{}^{*}\frac{l'}{(y-y')^2}\left(1+\frac{x-x'}{\sqrt{(x-x')^2+(y-y')^2}}\right)dx'dy' \quad \text{on } z = 0,$$
$$-\frac{1}{4\pi}\iint\limits_{S_0} l'\frac{\partial}{\partial n'_0}\left(\frac{1}{R}\right)d\sigma' = -\frac{z}{4\pi}\iint\limits_{S_0} l'\frac{dx'dy'}{\{(x-x')^2+(y-y')^2+z^2\}^{3/2}}, \quad z \neq 0.$$

4. The plane flow

For the plane flow relations (2.9) become

$$
\begin{aligned}
&\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}-\frac{1}{2}\frac{\partial v_x^2}{\partial x}=0,\\
(4.1)\qquad &\frac{\partial v_x}{\partial x}+\frac{\partial p}{\partial x}=0,\\
&\frac{\partial v_z}{\partial x}+\frac{\partial p}{\partial z}=0.
\end{aligned}
$$

Likewise the boundary conditions will be

$$(4.2)\qquad v_z(x,\ \pm 0)=-u_{0z}\mp n_{x_3}^{\pm}(x)\quad \text{for}\quad x\in(-1,\ 1).$$

By taking the Fourier transform with respect to both space variables the system (4.1) becomes

$$
\begin{aligned}
&ik_1\hat{v}_x+ik_3\hat{v}_z=\frac{1}{2}ik_1\hat{v}_x^2+\int_\Gamma\left\{[v_xn_x+v_zn_z]-\frac{1}{2}[v_x^2]n_x\right\}e^{-i\mathbf{k}\cdot\mathbf{x}}ds\\
(4.3)\qquad &\qquad\qquad+\int_C(v_xn_x+v_zn_z-0.5v_x^2n_x)e^{-i\mathbf{k}\cdot\mathbf{x}}ds,\\
&ik_1\hat{v}_z+ik_3\hat{p}=\int_\Gamma\{[v_z]n_x+[p]n_z\}e^{-i\mathbf{k}\cdot\mathbf{x}}ds+\int_\Gamma(v_zn_x+pn_z)e^{-i\mathbf{k}\cdot\mathbf{x}}ds.
\end{aligned}
$$

Here $\mathbf{k}\cdot\mathbf{x}=k_1x+k_3z$, Γ is a discontinuity line within the fluid and C is the profile curve. Due to jump conditions (2.11) and (2.12) the integrals on Γ vanish and by using also the linearised boundary conditions we get

$$
\begin{aligned}
&ik_1\hat{v}_x+ik_3\hat{v}_z=0.5\,ik_1\hat{v}_x^2+\int_{-1}^{1}m(x')e^{-ik_1x'}dx',\\
(4.4)\qquad &ik_1\hat{v}_z-ik_3\hat{v}_x=\int_{-1}^{1}l(x')e^{-ik_1x'}dx'.
\end{aligned}
$$

Hence

$$(4.5)\qquad \hat{v}_x-i\hat{v}_z=\frac{1}{2}\frac{ik_1}{i(k_1+ik_3)}\hat{v}_x^2-\frac{1}{k_1+ik_3}\int_{-1}^{1}\{l(x')+im(x')\}e^{-ik_1x'}dx'.$$

To determine the inverse Fourier transform of this relation we use the formula

$$(4.6)\qquad \mathscr{F}^{-1}\left\{\frac{1}{k_1+ik_3}\right\}=\frac{i}{2\pi(x+iz)}.$$

By considering the complex variables $\zeta=x+iz$, $\zeta'=x'+iz'$, the inverse Fourier transform of relation (4.5) becomes

$$(4.7)\qquad v_x-iv_z=\frac{1}{2\pi i}\int_{-1}^{1}\frac{l(x')+im(x')}{\zeta-x'}dx'+\frac{1}{4\pi}\frac{\partial}{\partial x}\iint_{R^2}\frac{v_x^2(x',z')}{\zeta-\zeta'}dx'dz',$$

where $\zeta' = x'+iz'$. The last term in this relation can be written alternatively as

$$\frac{1}{4\pi}\frac{\partial}{\partial x}\iint\limits_{R^2}\frac{v_x^2(x',z')}{\zeta-\zeta'}\,dx',dz' = \lambda\frac{v_x^2(x,z)}{4}-\frac{1}{4\pi}\iint\limits_{R^2}\frac{v_x^2(x',z')}{(\zeta-\zeta')^2}\,dx'dz'$$

at every point where the function $v_x^2(x, z)$ is continuous; here $\lambda = 1$ if the principal value of the integral is obtained by removing the singularity by a small circle, and $\lambda = 2$ if we use the Oswatitsch's definition of the principal value. The representation formula (4.7) is somehow similar to the formula given in [8].

On the segment $(-1, 1)$ of the real axis we have

$$(4.8)\quad v_x(x,\pm 0)-iv_z(x,\pm 0) = \mp\frac{1}{2}\{l(x)+im(x)\} + \frac{1}{2\pi i}\int\limits_{-1}^{1}\frac{l(x')+im(x')}{x-x'}\,dx'+\frac{1}{4\pi}\frac{\partial}{\partial x}\iint\limits_{R^2}\frac{v_x^2(x',z')}{x-\zeta'}\,dx'dz'.$$

The boundary conditions (4.2) give

$$(4.9)\quad \frac{1}{\pi}\int\limits_{-1}^{1}\frac{l(x')}{x-x'}\,dx' = -2u_{0z}-\left(n_x^+(x)-n_x^-(x)\right) + \frac{1}{2\pi}\frac{\partial}{\partial x}\iint\limits_{R^2}v_x^2(x',z')\frac{z'dx'dz'}{(x-x')^2+z'^2}.$$

Let us suppose, for the moment, the r.h.s. of relation (4.9) as being known. Then, the solution of the integral equation (4.9) is

$$(4.10)\quad l(x) = \frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\int\limits_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left\{2u_{0z}+n_x^+(t)-n_x^-(t) - \frac{1}{\pi}\frac{d}{dt}\iint\limits_{R^2}v_x^2(x',z')\frac{z'dx'dz'}{(t-x')^2+z'^2}\right\}\frac{dt}{x-t}.$$

The Kutta–Joukowsky condition was mutually included during the inversion of equation (4.9).

The real part of relations (4.7), (4.8) and (4.10) give up to some sign changes the Nixon's integral relations for lifting profile in transonic flow [9].

Let us substitute $l(x)$ given by relation (4.10) in formula (4.7). We get

$$(4.11)\quad v_x-iv_z = \frac{1}{2\pi}\int\limits_{-1}^{1}\frac{m(x')}{\zeta-x'}\,dx'+\frac{1}{4\pi}\frac{\partial}{\partial x}\iint\limits_{R^2}v_x^2(x',z')\frac{dx'dz'}{\zeta-\zeta'}$$
$$\frac{1}{2\pi i}\sqrt{\frac{\zeta-1}{\zeta+1}}\int\limits_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left(\frac{1}{2\pi}\frac{d}{dt}\iint\limits_{R^2}v_x^2(x',z')\frac{z'dx'dz'}{(x'-t)^2+z'^2}\right)$$
$$\times\frac{dt}{t-\zeta}+\frac{1}{2\pi i}\sqrt{\frac{\zeta-1}{\zeta+1}}\int\limits_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\,\frac{2u_{0z}+n_x^+(t)-n_x^-(t)}{t-\zeta}\,dt.$$

It is next possible to perform analytically the integration in the last term of relation (4.11). The details of calculation are given in Appendix C. We obtain

$$(4.12)\quad v_x - iv_z = \frac{1}{2\pi}\int_{-1}^{1}\frac{m(x')}{\zeta - x'}dx' + \frac{1}{2\pi i}\sqrt{\frac{\zeta-1}{\zeta+1}}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{2u_{0z}+n_x^+(t)-n_x^-(t)}{t-\zeta}dt$$
$$+\lambda\frac{v_x^2(x,z)}{4} - \frac{1}{8\pi}\iint_{R^2}\{v_x^2(x',z')+v_x^2(x',-z')\}\frac{dx'dz'}{(\zeta-\zeta')^2}$$
$$+\frac{1}{8\pi}\iint_{R^2}\{v_x^2(x',z')-v_x^2(x',-z')\}K(\zeta,\zeta')\,dx'dz',$$

where

$$(4.13)\quad K(\zeta,\zeta') = \sqrt{\frac{\zeta-1}{\zeta+1}}\sqrt{\frac{\zeta'+1}{\zeta'-1}}\left\{\frac{1}{(\zeta'^2-1)(\zeta-\zeta')} - \frac{1}{(\zeta-\zeta')^2}\right\}.$$

The real part of relation (4.12) gives the integral equation of the lifting profile problem in transonic flow [10]

$$(4.14)\quad v_x(x,z) = \frac{1}{4\pi}\int_{-1}^{1}m(t)\left\{\frac{1}{\zeta-t}+\frac{1}{\bar{\zeta}-t}\right\}dt + \frac{1}{4\pi i}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\{2u_{0z}+n_x^+(t)-n_x^-(t)\}$$
$$\cdot\left\{\sqrt{\frac{\zeta-1}{\zeta+1}}\frac{1}{t-\zeta} - \sqrt{\frac{\bar{\zeta}-1}{\bar{\zeta}+1}}\frac{1}{t-\bar{\zeta}}\right\}dt + \lambda\frac{v_x^2(x,z)}{4}$$
$$-\frac{1}{16\pi}\iint_{R^2}\{v_x^2(x',z')+v_x^2(x',-z')\}\left\{\frac{1}{(\zeta-\zeta')^2}+\frac{1}{(\bar{\zeta}-\bar{\zeta}')^2}\right\}dx'dz'$$
$$+\frac{1}{16\pi}\iint_{R^2}\{v_x^2(x'z')-v_x^2(x',-z')\}\cdot\{K(\zeta,\zeta')+K(\bar{\zeta},\bar{\zeta}')\}dx'dz',$$

where

$$\zeta = x+iz,\quad \zeta' = x'+iz',\quad m(t) = v_z(x,+0)-v_z(x,-0),$$
$$2u_{0z}+n_x^+-n_x^- = -(v_z(x_1+0)+v_z(x-0)).$$

This integral equation holds at every point of the plane including, possibly, the discontinuity surface (shocks) inside the flow.

For the symmetrical profile and zero incidence angle we have $v_x(x+0) = v_x(x-0)$ and the integral equation (4.14) reduces to the Oswatisch's integral equation [11, 12].

To obtain the lift on the profile we can use the velocity component $v_x(x, \pm 0)$. By using relation (B.6) we get

$$(4.15)\quad v_x(x\pm 0) = \frac{1}{2\pi}\int_{-1}^{1}\frac{m(x')}{x-x'}dx' \pm \frac{1}{2\pi}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\frac{2u_{0z}+n_x^+(t)+n_x^-(t)}{t-x}dt$$
$$+\frac{\lambda}{8}\{v_x^2(x,+0)+v_x^2(x,-0)\}$$

(4.15) [cont.]
$$-\frac{1}{8\pi}\iint\limits_{R^2}\{v_x^2(x',z')+v_x^2(x',-z')\}\frac{(x-x')^2-z'^2}{\{(x-x')^2+z'^2\}^2}\,dx'dz'$$
$$\mp\frac{1}{8\pi i}\iint\limits_{R^2}\{v_x^2(x',z')-v_x^2(x',-z')\}K_1(x,\zeta')\,dx'd\dot{z}',$$

where

$$K_1(x,\zeta')=\sqrt{\frac{1-x}{1+x}}\sqrt{\frac{\zeta'+1}{\zeta'-1}}\left\{\frac{1}{(\zeta'^2-1)(x-\zeta')}-\frac{1}{(x-\zeta')^2}\right\}.$$

Appendix A

We shall consider now the definition of principal part of the integrals appearing in Sects. 3–4. Let $\varphi\in\mathscr{D}(R^3)$ be a test function [7]. We have

$$\left\langle\frac{\partial^2}{\partial x^2}\frac{1}{4\pi r},\varphi(x,y,z)\right\rangle=-\left\langle\frac{\partial}{\partial x}\frac{1}{4\pi r},\frac{\partial\varphi}{\partial x}\right\rangle=\iiint\limits_{R^3}\frac{x}{4\pi r^3}\frac{\partial\varphi}{\partial x}\,dv. \tag{A.1}$$

The last term can also be written as

$$\iiint\limits_{R^3}\frac{x}{4\pi r^3}\frac{\partial\varphi}{\partial x}\,dv=\lim_{\varepsilon\to0}\left\{\iiint\limits_{R^3-D_\varepsilon}\frac{\partial}{\partial x}\left(\varphi\frac{x}{4\pi r^3}\right)dv-\iiint\limits_{R^3-D_\varepsilon}\varphi\frac{\partial}{\partial x}\left(\frac{x}{4\pi r^3}\right)dv\right\} \tag{A.2}$$
$$=\lim_{\varepsilon\to0}\left\{-\iint\limits_{S_\varepsilon}\varphi\frac{x}{4\pi r^3}n_x\,d\sigma+\iiint\limits_{R^3-D_\varepsilon}\frac{2x^2-y^2-z^2}{4\pi r^5}\varphi\,dv\right\}.$$

Here D_ε is a domain containing the origin, whose volume vanishes as $\varepsilon\to0$, S_ε is the boundary surface of this domain and $\mathbf{n}$—the external with respect to the domain D_ε normal to the surface S_ε. Since the last integral in relation (A.2) is semiconvergent, we obtain different expressions for different domains D_ε. Let D_ε be the sphere $r<\varepsilon$. We have

$$\iint\limits_{S\varepsilon}\varphi(x,y,z)\frac{x}{4\pi r^3}n_x\,d\sigma=\frac{\varphi(0,0,0)}{4\pi\varepsilon^3}\iint\limits_{S_\varepsilon}xn_x\,d\sigma$$
$$+\frac{1}{4\pi\varepsilon^3}\iint\limits_{S_\varepsilon}\{\varphi(x,y,z)-\varphi(0,0,0)\}\,xn_x\,d\sigma.$$

But

$$\left|\iint\limits_{S\varepsilon}\{\varphi(x,y,z)-\varphi(0,0,0)\}\,xn_x\,d\sigma\right|$$
$$\leqslant\varepsilon^3\int\limits_0^{2\pi}d\varphi\int\limits_{-\pi/2}^{\pi/2}|\varphi(\varepsilon\cos\theta\cos\varphi,\,\varepsilon\cos\theta\sin\varphi,\,\varepsilon\sin\theta)-\varphi(0,0,0)|\,d\theta,$$
$$\iint\limits_{S}xn_x\,d\sigma=\iiint\limits_{D}dv=\frac{4\pi\varepsilon^3}{3},$$

such that we have

$$\text{(A.3)} \qquad \iiint\limits_{R^3} \frac{x}{4\pi r^3}\frac{\partial\varphi}{\partial x}\,dv = -\frac{\varphi(0,0,0)}{3} + \iiint\limits_{\odot} \frac{2x^2-y^2-z^2}{4\pi r^5}\,\varphi dv.$$

Relations (A.1) and (A.3) give

$$\text{(A.4)} \qquad \frac{\partial^2}{\partial x^2}\left(\frac{1}{4\pi r}\right) = -\frac{1}{3}\delta(x,y,z) + \left\{\frac{2x^2-y^2-z^2}{4\pi r^5}\right\}_{\odot}.$$

The sign $\odot$ indicates the regularization of the corresponding distribution by means of spheres.

Let now

$$D_\varepsilon = \{(x,y,z) \mid -\varepsilon < x < \varepsilon\}.$$

We have

$$\text{(A.5)} \qquad \iint\limits_{S_\varepsilon} \varphi(x,y,z)\frac{x}{4\pi r^3}\,n_x\,d\sigma = \frac{\varepsilon}{4\pi}\iint\limits_{R^2} \frac{\varphi(\varepsilon,y,z)-\varphi(-\varepsilon,y,z)}{(\varepsilon^2+y^2+z^2)^{3/2}}\,dy\,dz$$

$$= \frac{1}{4\pi}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\infty} \frac{\varphi(\varepsilon,\varepsilon\varrho\cos\varphi,\varepsilon\varrho\sin\varphi)+\varphi(-\varepsilon,\varepsilon\varrho\cos\varphi,\varepsilon\varrho\sin\varphi)}{(1+\varrho^2)^{3/2}}\,\varrho\,d\varrho.$$

We have also

$$\iint\limits_{S_\varepsilon} \frac{x}{4\pi r^3}\,n_x\,d\sigma = \frac{\varphi(0,0,0)}{2\pi}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\infty} \frac{\varrho\,d\varrho}{(1+\varrho^2)^{3/2}}$$

$$+\frac{1}{4\pi}\int\limits_0^{2\pi} d\varphi \int\limits_0^{\infty} \frac{\varphi(\varepsilon,\varepsilon\varrho\cos\varphi,\varepsilon\varrho\sin\varphi)+\varphi(-\varepsilon,\varepsilon\varrho\cos\varphi,\varepsilon\varrho\sin\varphi)-2\varphi(0,0,0)}{(1+\varrho^2)^{3/2}}\,\varrho\,d\varrho.$$

The last integral vanishes as $\varepsilon \to 0$ so that we have

$$\text{(A.6)} \qquad \iiint\limits_{R^3} \frac{x}{4\pi r^3}\frac{\partial\varphi}{\partial x}\,dv = -\varphi(0,0,0) + \iiint\limits_{|\cdot|} \frac{2x^2-y^2-z^2}{4\pi r^5}\,\varphi dv.$$

Finally, we can write

$$\text{(A.7)} \qquad \frac{\partial^2}{\partial x^2}\left(\frac{1}{4\pi r}\right) = -\delta(x,y,z) + \left\{\frac{2x^2-y^2-z^2}{4\pi r^5}\right\}_{|\cdot|},$$

where now last term indicates the distribution

$$\text{(A.8)} \qquad \left\langle\left\{\frac{2x^2-y^2-z^2}{r^5}\right\}_{|\cdot|}, \varphi(x,y,z)\right\rangle \equiv \iiint\limits_{|\cdot|} \frac{2x^2-y^2-z^2}{r^5}\,\varphi(x,y,z)\,dv$$

$$= \lim_{\varepsilon\to 0}\iiint\limits_{|x|>\varepsilon} \frac{2x^2-y^2-z^2}{r^5}\,\varphi(x,y,z)\,dv.$$

Likewise, in the plane case we obtain

(A.9)
$$\frac{\partial}{\partial x}\frac{1}{2\pi\zeta} = \frac{1}{2}\,\delta(x, z) - \left\{\frac{1}{2\pi\zeta^2}\right\}_{\odot},$$
$$\frac{\partial}{\partial x}\frac{1}{2\pi\zeta} = \delta(x, z) - \left\{\frac{1}{2\pi\zeta^2}\right\}_{|\cdot|}.$$

Appendix B

In this appendix we shall calculate the derivative

$$\frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{R^2}\frac{f(x', z')}{\zeta-\zeta'}\,dx'dz'$$

at a discontinuity point of the function $f(x, z)$. Let $z' = z$ be a discontinuity line of the function $f(x', z')$, and let us put

(B.1)
$$\lim_{\substack{z_1\to z\\ z_1<z}} f(x, z_1) = f(x, z+0), \qquad \lim_{\substack{z_1\to z\\ z_1<z}} f(x, z_1) = f(x, z-0).$$

We have

(B.2)
$$\frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{R^2}\frac{f(x', z')}{\zeta-\zeta'}\,dx'dz' = \frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{R_2-D_\varepsilon}\frac{f(x', z')}{\zeta-\zeta'}\,dx'dz'$$
$$+\frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{D_\varepsilon^+}\frac{f(x', z')}{\zeta-\zeta'}\,dx'dz' + \frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{D_\varepsilon^-}\frac{f(x', z')}{\zeta-\zeta'}\,dx'dz',$$

where

$$D_\varepsilon^+ = \{(x', z')|x-\varepsilon < x' < x+\varepsilon, z' > z\},$$
$$D_\varepsilon^- = \{(x'z')|x-\varepsilon < x' < x+\varepsilon, z' < z\}, \qquad D_\varepsilon = D_\varepsilon^+\cup D_\varepsilon^-.$$

But

(B.3)
$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{R^2-D_\varepsilon}\frac{f(x', z')}{\zeta-\zeta'}\,dx'dz'$$
$$= \lim_{\varepsilon\to 0} -\frac{1}{\pi}\iint\limits_{R^2-D_\varepsilon}\frac{f(x', z')}{(\zeta-\zeta')^2}\,dx'dz' = -\frac{1}{\pi}\iint\limits_{|\cdot|}\frac{f(x', z')}{(\zeta-\zeta')^2}\,dx'dz'.$$

We have also

(B.4)
$$\frac{1}{\pi}\iint\limits_{D_\varepsilon^+}\frac{f(x', z')}{\zeta-\zeta'}\,dx'dz' = -\frac{1}{\pi}\iint\limits_{D_\varepsilon^+}\frac{\partial}{\partial x'}\{f(x', z')\ln(\zeta-\zeta')\}dx'dz'$$
$$+\frac{1}{\pi}\iint\limits_{D_\varepsilon^+}\frac{\partial f}{\partial x'}\ln(\zeta-\zeta')dx'dz' = -\frac{1}{\pi}\int\limits_{C_\varepsilon^+} f(x', z')\ln(\zeta-\zeta')dz'$$
$$+\frac{1}{\pi}\iint\limits_{D_\varepsilon}\frac{\partial f}{\partial x'}\ln(\zeta-\zeta')\,dx'dz',$$

where C_ε^+ is the boundary curve of domain D_ε^+. Hence

$$\text{(B.5)} \qquad \frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{D_\varepsilon^+}\frac{f(x',z')}{\zeta-\zeta'}\,dx'dz' = -\frac{1}{\pi}\int\limits_{C_\varepsilon^+}\frac{f(x',z')}{\zeta-\zeta'}\,dz'+\frac{1}{\pi}\iint\limits_{D_\varepsilon^+}\frac{\partial f}{\partial x'}\,\frac{dx'dz'}{\zeta-\zeta'}\,.$$

The last term in (B.5) vanishes as $\varepsilon\to 0$ and the other gives

$$-\frac{1}{\pi}\int\limits_{C_\varepsilon^+}\frac{f(x',z')}{\zeta-\zeta'}\,dz' = \frac{1}{\pi}\int\limits_{z}^{\infty}\frac{f(x+\varepsilon,z')dz'}{\varepsilon-i(z-z')}+\frac{1}{\pi}\int\limits_{z}^{\infty}\frac{f(x-\varepsilon,z')}{\varepsilon+i(z-z')}\,dz'$$

$$= \frac{2}{\pi}\int\limits_{z}^{\infty}\frac{\varepsilon}{\varepsilon^2+(z+z')^2}\,f(x',z')\,dz'+\mathcal{O}(\varepsilon).$$

As $\varepsilon/\{\pi(\varepsilon^2+z^2)\}$ is a $\delta(z)$ sequence, we obtain [7]

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial x}\frac{1}{\pi}\iint\limits_{D_\varepsilon^+}\frac{f(x',z')}{\zeta-\zeta'}\,dx'dz' = f(x,z+0).$$

The integral on D_ε^- can be calculated similarly. Finally we get

$$\text{(B.6)} \qquad \frac{\partial}{\partial x}\frac{1}{2\pi}\iint\limits_{R^2}\frac{f(x',z')}{\zeta-\zeta'}\,dx'dz' = \frac{f(x,z+0)+f(x,z-0)}{2}-\frac{1}{2\pi}\iint\limits_{|\cdot|}\frac{f(x',z')}{(\zeta-\zeta')^2}\,dx'dz'.$$

In the case of the continuous $f(x,z)$, formula (B.6) coincides with (A.10).

Appendix C

Now, we shall transform the term

$$\text{(C.1)} \qquad I = \frac{1}{2\pi i}\sqrt{\frac{\zeta-1}{\zeta+1}}\int\limits_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left(\frac{d}{dt}\iint\limits_{R^2}v_x^2(x',z')\,\frac{z'dx'dz'}{(x'-t)^2+z'^2}\right)\frac{dt}{t-\zeta}$$

occurring in relation (4.11).

By using the result of Appendix B we have

$$\text{(C.2)} \qquad I = \frac{1}{2\pi i}\sqrt{\frac{\zeta-1}{\zeta+1}}\int\limits_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left(\frac{1}{2\pi}\iint\limits_{R^2}v_x^2(x',z')\,\frac{\partial}{\partial t}\,\frac{z'}{(x'-t)^2+z'^2}\,dx'dz'\right)\frac{dt}{t-\zeta}$$

$$= -\frac{1}{2i\pi}\sqrt{\frac{\zeta-1}{\zeta+1}}\int\limits_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left(\frac{1}{2\pi}\iint\limits_{R^2}v_x^2(x',z')\,\frac{\partial}{\partial x'}\,\frac{z'}{(x'-t)^2+z'^2}\,dx'dz'\right)\frac{dt}{t-\zeta}.$$

Hence

$$\text{(C.3)} \qquad I = -\frac{1}{2\pi i}\sqrt{\frac{\zeta-1}{\zeta+1}}\iint\limits_{R^2}v_x^2(x',z')z'\,\frac{\partial}{\partial x'}\left(\frac{1}{2\pi}\int\limits_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\right.$$

$$\left.\times\,\frac{1}{(x't-t)^2+z'^2}\,\frac{dt}{t-\zeta}\right)dx'dz'.$$

The inner integral in relation (C.3) can be estimated by means of the residue theorem. Let

$$F(Z) = \frac{1}{2\pi i}\sqrt{\frac{Z+1}{Z-1}}\,\frac{1}{(Z-\zeta')(Z-\bar{\zeta}')}\,\frac{1}{Z-\zeta}, \tag{C.4}$$

where $Z = X+iY$, $\zeta' = x'+iz'$ are complex variables and the square root is equal to 1 at infinity. We shall integrate this function along the contour given in Fig. 2.

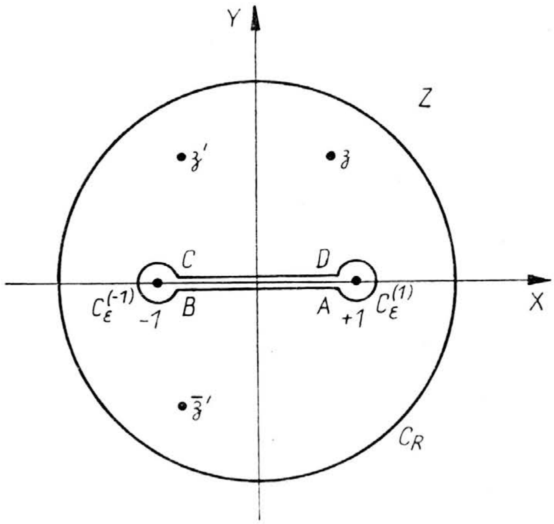

FIG. 2.

We have

$$\int\limits_{C_R} F(Z)dZ + \int\limits_{\overline{AB}} F(Z)dZ + \int\limits_{C_\varepsilon^{(-1)}} F(Z)dZ + \int\limits_{\overline{CD}} F(Z)dZ + \int\limits_{C_\varepsilon^{(1)}} F(Z)dZ \tag{C.5}$$

$$= \sqrt{\frac{\zeta+1}{\zeta-1}}\,\frac{1}{(\zeta-\zeta')(\zeta-\bar{\zeta}')} + \sqrt{\frac{\zeta'+1}{\zeta'-1}}\,\frac{1}{(\zeta'-\bar{\zeta}')(\zeta'-\zeta)} + \sqrt{\frac{\bar{\zeta}'+1}{\bar{\zeta}'-1}}\,\frac{1}{(\bar{\zeta}'-\zeta')(\bar{\zeta}'-\zeta)}.$$

The integrals along C_R, $C_\varepsilon^{(-1)}$, $C_\varepsilon^{(1)}$ are vanishing as R and $\varepsilon \to 0$, and the other two give

$$-\frac{1}{\pi}\int\limits_{-1}^{1}\sqrt{\frac{1+X}{1-X}}\,\frac{1}{(X-x')^2+z'^2}\,\frac{dX}{X-\zeta} = \sqrt{\frac{\zeta+1}{\zeta-1}}\,\frac{1}{2iz'}\left(\frac{1}{\zeta-\zeta'}-\frac{1}{\zeta-\bar{\zeta}'}\right) \tag{C.6}$$

$$+\frac{1}{2iz'}\left(\sqrt{\frac{\zeta'+1}{\zeta'-1}}\,\frac{1}{\zeta'-\zeta}-\sqrt{\frac{\bar{\zeta}'+1}{\bar{\zeta}'-1}}\,\frac{1}{\bar{\zeta}'-\zeta}\right).$$

By using this result we obtain

$$(C.7)\quad I = -\frac{1}{8\pi}\iint_{R^2} v_x^2(x', z')\left(\frac{1}{(\zeta-\zeta')^2} - \frac{1}{(\zeta-\bar{\zeta}')^2}\right) dx'dz'$$

$$-\frac{1}{8\pi}\iint_{R^2} v_x^2(x', z')\sqrt{\frac{\zeta-1}{\zeta+1}}\left\{\sqrt{\frac{\zeta'+1}{\zeta'-1}}\left(\frac{1}{(\zeta'^2-1)(\zeta-\zeta')} - \frac{1}{(\zeta-\zeta')^2}\right)\right.$$

$$\left. -\sqrt{\frac{\bar{\zeta}'+1}{\bar{\zeta}'-1}}\left(\frac{1}{(\bar{\zeta}'^2-1)(\zeta-\bar{\zeta}')} - \frac{1}{(\zeta-\bar{\zeta}')^2}\right)\right\} dx'dz'.$$

Finally we can write

$$(C.8)\quad I = -\frac{1}{8\pi}\iint_{R^2} \{v_x^2(x', z') - v_x^2(x', -z')\}\frac{dx'dz'}{(\zeta-\zeta')^2} - \frac{1}{8\pi}\iint_{R^2}\{v_x^2(x', z')$$

$$-v_x^2(x', -z')\}\sqrt{\frac{\zeta-1}{\zeta+1}}\sqrt{\frac{\zeta'+1}{\zeta'-1}}\left(\frac{1}{(\zeta'^2-1)(\zeta-\zeta')} - \frac{1}{(\zeta-\zeta')^2}\right) dx'dz'.$$

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