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BRIEF NOTES

An optimal design problem
A nonexistence theorem

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WE PROVE the nonexistence of optimal design for the vibrating inhomogeneous membrane.

1. Statement of the problem

WE STUDY the design of a vibrating inhomogeneous membrane occupying a domain $D \subseteq R^2$, which is simply connected and has a sufficiently smooth boundary. The fundamental mode of vibration $u(x, y)$ satisfies the differential equation

$$\Delta u(x,y)+\lambda_1 \varrho(x,y)\cdot u(x,y) = 0 \text{ in } D, \quad \left(\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right), \quad u \equiv 0 \text{ on } \partial D, \tag{1.1}$$

where ∂D denotes the boundary of D, $\varrho(x, y)$, which is positive in D, is the material density, and λ_1 is the fundamental eigenvalue ($\lambda_1 = \omega_1^2$, where ω_1 is the fundamental frequency). We assume that $\varrho u \in H_0^1(D)$. Equation (1.1) can also represent a linear approximation to a neutron flux density in a critical operation of a nuclear reactor. An analysis of bounds for λ with fixed $\varrho(x, y)$ can be found in [8].

An almost natural problem of design theory is the following. Choose $\varrho(x, y) \in L^2(D) \cap L^1(D)$, $\varrho > 0$ (subject to a constraint such as $\int_D \varrho(x, y)\, dx\, dy = 1$), which maximizes the fundamental eigenvalue λ_1.

2. State space analysis

The problem of finding λ_1 for a given $\varrho(x, y)$ is equivalent to the variational problem of minimizing the Rayleigh quotient

$$\lambda_1(u) = \min, \quad \{||\nabla u||^2/\langle \varrho u, u\rangle\}, \quad (\nabla \equiv \text{grad}); \quad u \in H_0^1. \tag{2.1}$$

Here

$$\langle f, g\rangle = \int_D f(x,y) g(x,y) dx dy, \quad \text{and} \quad ||f||^2 = \langle f, f\rangle. \tag{2.2}$$

For physical reasons only continuous solutions $u(x, y)$ are considered. The vanishing of the Fréchet derivative of $\lambda_1(u)$ (for a fixed choice of $\varrho(x, y)$) which is a necessary condition for the minimum of λ_1 results in the Euler–Lagrange condition $\Delta u = -\lambda\varrho u$, which is the original equation (1.1); See Appendix 1 for computational details.

We are ready to state our main result.

We make first an observation that if $\varrho(x, y) \in C(D)$, then there exists a subset $K \subseteq H^1(D)$ such that for any $\varrho \in K \subseteq C(D)$ all eigenfunctions of Eq. (1.1) are of single multiplicity.

Here we could repeat the basic arguments of UHLENBECK [5]. The eingenvalue problem (1.1) can be restated in the form

$$(2.3) \qquad \big(I - \lambda_1 - \mu\big) G\big(\varrho(x, y); x, y\big)u = 0,$$

where I is the identity operator, $G(x, y)$ is the fundamental solution for the Laplace operator, and where μ is a constant $0 \leqslant \mu < \lambda_1$. The map $H^1(D) \times R \to L^2(D)$ is a Fredholm map of index zero. (see S. OZAWA [6] for the proof) Instead of perturbing the shape and keeping the boundary values constant, we can select a region Ω entirely contained inside the perturbation region D, but sufficiently close to the actual boundary, and regard the behavior of $u(x, y)$ only inside Ω and on the boundary of Ω as a corresponding boundary value perturbation problem.

Then the variational formulas for the Green function can be literally copied from [5] and [6].

Since this remark is only marginal to our results we shall refer the readers to papers of K. UHLENBECK, and D. FUJIWARA, M. TANIKAWA and S. YUKITA [7] for detailed arguments.

We make the following basic assumptions.

We seek a class of solution $\{\lambda_1, u(x, y)\} \in R \times H^1(D)$ corresponding to the choice of bounded, positive, piecewise continuous functions $\varrho(x, y) \leqslant M$; $x, y \in D$, satisfying $\int_D \varrho(x, y)\, dx\, dy = 1$. (Clearly each such function $\varrho(x, y)$ satisfies: $\varrho \in L^2(D)$ and $\varrho(x, y) \times \times u(x, y) \in H_0^1(D)$). Such functions $\varrho(x, y)$ shall be called admissible designs, if $u\big((x, y); \varrho(x, y)\big)$ is continuous in D. Our problem is to find a sequence of admissible designs $\varrho_i(x, y)$ such that $\varrho_i(x, y) \xrightarrow{L^2} \tilde{\varrho}(x, y)$ (which may not be an admissible design) and of corresponding pairs $\{\lambda_1(\varrho_i), u_i(\varrho_i)\}$ such that

$$\lambda_1(\tilde{\varrho}) = \sup_{\tilde{\varrho} \in L^2(D)} \{\min_{u \in H_0^1} \{\|\nabla u_i\|^2 / \langle \varrho_i u_i u_i \rangle\} = \lim_{i \to \infty} \big(\lambda_1(\varrho_i)\big).$$

We note that there exists $\lambda_1(\tilde{\varrho})$ and that a sequence of admissible designs can be chosen converging to the optimal design $\tilde{\varrho}$ since continuous functions are dense in $L^2(D)$ functions $\varrho(x, y)$, which in our case also are bounded in D by some number M.

We assume that the eigenvalue λ_1 is simple, hence, we assume Fréchet differentiability of λ_1.

Our purpose is to show that the limit function $\tilde{\varrho}(x, y)$ is not an admissible design.

THEOREM 1. *There does not exist an optimal design* $\tilde{\varrho}(x, y) \in L^1(D) \cap L^2(D)$ *and a corresponding displacement* $\tilde{u}(x, y) \in H_0^1(D)$ *such that* $\tilde{\varrho}\tilde{u} \in H^1(D)$ *and* $\tilde{\lambda}_1 = \max \lambda_1(\varrho, u)$

for all admissible choices of $\varrho(x, y)$ for either the unconstrained problem, or for choices of ϱ satisfying the constraint $\int_D \varrho\,(x, y)\,dx\,dy = 1$.

P r o o f. Let us consider the unconstrained problem first. Since $\lambda_1(\varrho)$ is a Fréchet differentiable function of ϱ (see the appendix and the preceding discussion), the necessary condition for an extremum of $\tilde{\lambda}_1$ is the vanishing of the Fréchet derivative of $\tilde{\lambda}_1$, i.e. $\dfrac{d\tilde{\lambda}(\varrho)}{d\varrho} = 0$.

Hence, in the unconstrained case we compute

$$\frac{d\tilde{\lambda}_1}{d\varrho} = \frac{\partial\tilde{\lambda}_1}{\partial u}\cdot\frac{\partial u}{\partial\varrho} + \frac{\partial\tilde{\lambda}}{\partial\varrho}.$$

Since $\dfrac{\partial\tilde{\lambda}_1}{\partial u} = 0$ (see Appendix 1), we obtain $\dfrac{\partial\tilde{\lambda}_1}{\partial\varrho} = -\tilde{u}^2\dfrac{\|\nabla\tilde{u}\|^2}{\langle\varrho\tilde{u}, \tilde{u}\rangle} = -\tilde{\lambda}_1\tilde{u}^2 = 0$, which is possible only if $\tilde{u} \equiv 0$, contradicting the fact that $\tilde{u}(x, y)$ is the eigenfunction corresponding to the eigenvalue $\tilde{\lambda}_1$; hence the trivial case $\tilde{u} \equiv 0$ is not permitted. In the constrained case we derive by a similar argument $\dfrac{d\Phi}{d\varrho} = 0$, where $\Phi = \tilde{\lambda}_1 + \mu\int\varrho\,dx\,dy$. Here μ is a (constant) Lagrangian multiplier. Hence $\dfrac{d\Phi}{d\varrho} = -\tilde{\lambda}_1\tilde{u}^2 + \mu = 0$ is the necessary condition for optimality of λ_1. This is possible only if $\tilde{u}(x, y) = \pm(\mu/\tilde{\lambda}_1)^{1/2}$.

But the continuity of $\tilde{u}(x, y)$ makes this solution inadmissible, and the proof is complete.

We make the observation that the entire argument can be repeated without major changes for the more general equation $\Delta(T(x, y)\,u(x, y)) + \lambda\varrho(x, y)\,u(x, y) = 0$ where $T(x, y) > 0$ in Ω, with an identical boundary condition $u(x, y) \equiv 0$ on $\partial\Omega$.

3. Suboptimal designs

Numerical analysis reveals that the known difficulties leading to singularities in the design of optimal vibrating beams and plates (see N. OLHOFF [1], or E. F. MASUR [2]) persist in this much simpler case. The design develops a zero cross-section and in the manner resembling the unconstrained optimal column design of Olhoff. To optimize λ_1 the membrane design exhibits a definite development of singularities and of almost discontinuous corresponding diplacements (the slope is almost vertical at the thin points in the membrane). The author did not attempt to duplicate the numerical analysis of N. OLHOFF and S. H. RASMUSSEN in encorporating a minimum cross-sectional area constraint, which produces a definite optimal design (without singularities) in the case of a vibrating beam or a column clamped at both ends.

However, the simple result offered in this paper illustrates the lack of smoothness encountered in the more difficult optimization problems.

Appendix 1

Existence of the Fréchet derivatives, and computation of their values. It is well known that the existence of a minimum (or maximum) of a Fréchet differentiable functional Φ in a cone S of a Hilbert space H does not necessarily imply that the derivative vanishes at the point where Φ attains a minimum (maximum), or even that it satisfies the Hesteness condition $\Phi_u \geqslant 0$ at that point. However, if the minimum occurs at an interior point of an open ball in H, then this assertion is correct. (See M. M. VAINBERG [4])

In this paper we need the Fréchet differentiability of the eigenvalue λ_1. In general this requirement is too stringent. But if we can ascertain that λ_1 is a simple eigenvalue, and the design problems are solved in a stable region where the changes in design do not cause coalescing of the first and second fundamental frequencies, in that case the Fréchet derivative $\frac{\partial \lambda_1}{\partial u}$ exists and it can be explicitly computed. This assertion is true in linear vibration problems.

For $u(x, y)$ considered as an interior point of an open ball in $H_0^1(D)$, and for a fixed $\varrho(x, y) \in L^2(D) \cap L^1(D)$, we select an arbitrary vector $\eta \in H_0^1(D)$, $\eta \equiv 0$ on ∂D, a sufficiently small (in absolute value) real number t, and we compute the Gateaux difference:

$$\delta_\eta \lambda_1 = \lambda_1(u+t\eta) - \lambda(u) = \frac{\langle \nabla(u+t\eta), \nabla(u+t\eta)\rangle}{\langle \varrho(u+t\eta\rangle, (u+t\eta)\rangle} - \frac{\langle \nabla u, \nabla u\rangle}{\langle \varrho u, u\rangle}.$$

After some manipulation we reduce $\delta_\eta \lambda_1$ to the form

$$\frac{2t\{\langle \varrho u, u\rangle \cdot \langle \nabla u, \nabla \eta\rangle - \langle \nabla u, \nabla u\rangle \langle \varrho u, \eta\rangle\} + o(t^2)}{\langle \varrho u, u\rangle \{\langle \varrho u, u\rangle + 2t\langle \varrho u, \eta\rangle + o(t^2)\}}.$$

Using Green's formula we replace $\langle \nabla u, \nabla \eta\rangle$ by $-\langle \Delta u, \eta\rangle$. The necessary condition for the stationary behaviour of λ_1 is given by the vanishing of the Gateaux derivative, which becomes the vanishing of the Fréchet derivative if the corresponding form is a continuous functional of η. Hence we assert that for all admissible $\eta \in H^1(D)$, the following equation must be satisfied:

$$\langle \eta, (-\langle \varrho u, u\rangle \Delta u - \langle \nabla u, \nabla u\rangle \varrho u)\rangle = 0;$$

this is equivalent to $\Delta u + \lambda_1 \varrho u = 0$, by the fundamental theorem of calculus of variations, that is to the original equation describing the configuration of the membrane corresponding to the fundamental eigenvalue λ_1.

Our conclusion that λ_1 is Fréchet differentiable follows from our definition of the inner product $\langle , \rangle$ (which is defined as an integral) from $L^2(D)$ property of η, and from the previously made assumptions concerning λ_1.

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