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Supersonic nozzles without shocks

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TH METHOD of Bernoulli manifolds [2–5] provides a generalization of the hodograph method to three dimensional flows of ideal gases. Here using the Bernoulli manifolds technique we show that there exists a wide class of nozzles which transform one uniform supersonic flow into another one without shock.

Metoda rozmaitości Bernoulliego [2–5] jest uogólnieniem znanej metody hodografu na przepływy trójwymiarowe gazu idealnego. W niniejszej pracy metodę rozmaitości Bernoulliego wykorzystujemy dla wykazania, że istnieje szeroka klasa dysz przetwarzających jednorodny przepływ na wejściu na inny, również jednorodny, przepływ naddźwiękowy na wyjściu dyszy, bez przejścia przez falę uderzeniową.

Метод многообразии Бернулли [2—5] является обобщением на трехмерные течения идеального газа известного метода годографа. В настоящей работе метод многообразии Бернулли используется для того, чтобы показать существование обширного класса сопел, преобразующих однородное течение на входе на другое, также однородное, сверхзвуковое течение на выходе сопла без перехода через ударную волну.

1. Introduction

IT IS KNOWN that using the Busemann hodograph technique and Prandtl-Mayer flows, it is possible to construct nozzles which transform one uniform (i.e. constant velocity) supersonic flow into another one. However, this technique is limited by two factors: first, it is confined to plane nozzles; second, for plane nozzles there exists a one-to-one relation between initial and final velocities angles and between the velocity ratios at the ends of the nozzle.

Our aim is to extend the above technique to three-dimensional nozzles. We show that there is a wide class of nozzles having the demanded property and that there is some possibility to "regulate" the acceleration produced by the nozzles. Also the shape of the nozzle is not uniquely determined.

The method used throughout this paper is based on geometrical considerations and appriopriate theorems are proved. These theorems, however, can as those of Busemann provide a basis for a numerical method aimed at constructing nozzles with *a priori* given characteristics.

We consider the inviscid, isentropic, compressible flow, described by the system

$$(1.1)\qquad \begin{aligned} &u^1c_{x^1}+u^2c_{x^2}+u^3c_{x^3}+\frac{c}{k}\,\mathrm{div}u = 0,\\ &u^1u^s_{x^1}+u^2u^s_{x^2}+u^3u^s_{x^3}+kcc_{x^s} = 0, \quad s = 1, 2, 3.\end{aligned}$$

Our method, however, is applicable to more complicated systems when the motion is governed by the nonelliptic quasi-linear system of equations of the form

$$\sum_{j=1}^{l}\sum_{i=1}^{n} a_j^{si}(V^1, \dots, V^l)\, V^j_{x^i} = 0, \qquad s = 1, \dots, l. \tag{1.2}$$

For the system (1.1) $U = (c, u^1, u^2, u^3)$ where c is the speed of sound, $u = (u^1, u^2, u^3)$ is the flow velocity. We denote by C^1_w (D) the class of functions having continuous first derivative except perhaps on a finite number of smooth surfaces. On these surfaces the normal derivatives are not continuous.

We seek the solution of (1.1), $U \in C^1_w(D)$ satisfying the following two conditions:

1) The domain D contains two subdomains D_{I} and D_{II} such that for $x \in D_{\mathrm{I}}$

$$U(x) = U_{\mathrm{I}} = \text{const}$$

and for $x \in D_{\mathrm{II}}$

$$U(x) = U_{\mathrm{II}} = \text{const}.$$

2) There exists a stream line $t \subset D$ joining D_{I} and D_{II} (Fig. 1).

The solution $U \in C^1_w(D)$ which satisfies the conditions 1) and 2) defines the required nozzle.

The simplest solutions of that type may be found in the class of plane potential flows using the Busemann method [1]. The construction consists of two steps. First we construct

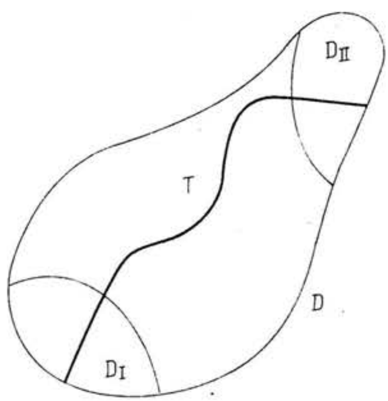

FIG. 1.

the Busemann epicycloids and choose the set H which is a part of one epicycloid bounded by the points u_{I} and u_{II} (Fig. 2) (the set of values of the solutions in the plane u^1, u^2). The second step is the parametrization of H by the independent variables x^1 and x^2.

The simplest possible parametrization is the conical one. To perform it we draw through the point x_0 the family of lines $\Sigma(u)$ where $\Sigma(u)$ is perpendicular to H in the point u (Fig. 2 and 3).

To obtain the conical parametrization we define the flow by the condition

$$u(x) = u \quad \text{for} \quad x \in \Sigma(u).$$

The flow may be prolonged by the constant values u_{I} and u_{II} chosen beforehand. Thus we obtain two kinds of domains D (Fig. 3) in which the flow obeys the conditions 1) and 2). One of them defined the accelerating nozzle (the stream line T^a), the other one gives the decelerating nozzle (the stream line T^b).

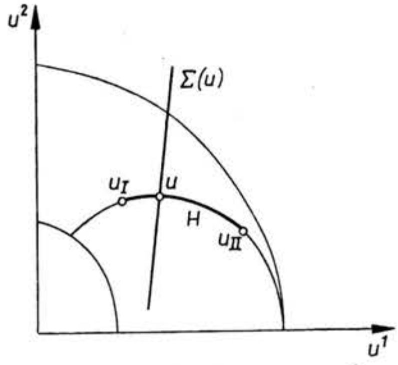

FIG. 2.

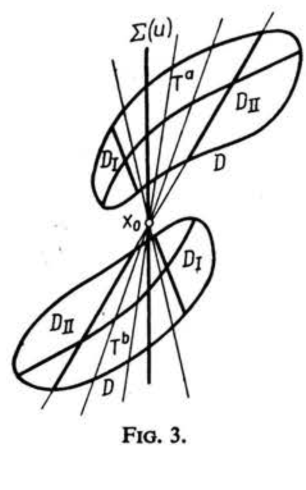

FIG. 3.

In the present paper we generalize the method of solution of the problem 1), 2) to the wide class of three-dimensional isentropic flows.

2. Bernoulli manifolds

The k-dimensional manifold $H_k \subset R^l$ will be called the Bernoulli manifold for the system (1.2) if for H_k there exists an infinite class of solutions $K(H_k)$ such that for $V \in K(H_k)$ the set of values $V(D)$ satisfies the conditions

$$V(D) \subset H_k, \quad \dim V(D) = k.$$

For example, the Bernoulli law for the system (1.1) reads $|u|^2+kc^2 = q^2 = \text{const}$, $|u|^2 = \sum_i u^i u^i$. It gives two kinds of Bernoulli manifolds: the three-dimensional manifold $H_3(q) \subset R^4$

$$H_3(q): |u|^2+kc^2-q^2 = 0$$

and the two-dimensional manifold $H_2(q) \subset R^4$

$$H_2(q): \quad \begin{aligned} &|u|^2+kc^2-q^2 = 0, \\ &u^3 = 0. \end{aligned}$$

The class $K(H_3(q))$ contains solutions describing the three-dimensional, potential, steady flows and the class $K(H_2(q))$ contains solutions describing the two-dimensional potential, steady flows. It should be noticed that the Busemann epicycloid H (Fig. 2) is, according to our definition, one-dimensional Bernoulli manifold.

Now returning to the system (1.2) we will construct one and two-dimensional Bernoulli manifolds H_1 and H_2. For this purpose we introduce two characteristic cones $\Lambda(V)$ and $\Gamma(V)$:

$$R^n \supset \Lambda(V) = \left\{\lambda = (\lambda_1 \dots \lambda_n)\colon \det\left|\sum_{i=1}^{n} a_j^{si}(V)\lambda_i\right| = 0\right\},$$

$$R^l \supset \Gamma(V) = \left\{\gamma = (\gamma^1 \dots \gamma)\colon \operatorname{rank}\left|\sum_{j=1}^{l} a_j^{si}(V)\gamma^j\right| < n\right\}.$$

We will write $\gamma \leftrightarrow \lambda$ (γ is knotted with λ) if γ is the right null vector for λ, i.e.

$$\sum_{i,j} a_j^{si}(V)\lambda_i\gamma^j = 0, \qquad s = 1, 2, \dots, l.$$

If the system (1.2) is nonelliptic (i.e. the cones $\Lambda(V)$ and $\Gamma(V)$ are not empty), then the curves $H_1\colon V = V(\mu)$, $H_1 \subset R^l$, are one-dimensional Bernoulli manifolds (Bernoulli curves) if the following condition is satisfied:

$$\frac{dV(\mu)}{d\mu} \in \Gamma(V(\mu)).$$

The geometrical meaning of this condition is that the curve H_1 is at the point $V(\mu)$ tangent to the cone $\Gamma(V(\mu))$.

Let $\lambda(\mu) \leftrightarrow \gamma(\mu)$ and $\gamma(\mu) = dV/d\mu$. Further we introduce the plane Π_μ by

$$\Pi_\mu\colon \left(x, \lambda(\mu)-x_0(\mu)\right) = 0,$$

where $x_0(\mu)$ is an arbitrary regular function.

The class of solutions $K(H_1)$ is called simple waves (see [2, 3]). It contains functions obeying the condition

$$V(x) = V(\mu) \quad \text{for} \quad x \in \Pi_\mu.$$

(This statement generalizes the Busemann relation for a much wider class of equations).

For different $x_0(\mu)$ we obtain different parametrization of H_1 by the independent variables $x^1, x^2, \dots, x^n$. If $x_0(\mu) = \text{const}$, the parametrization is conical.

Now we pass to two-dimensional manifolds. The two-dimensional Bernoulli manifold $H_2: V = V(\mu)$, $\mu = (\mu^1, \mu^2)$, $H_2 \in R^l$, is defined by the conditions

$$\text{(I)} \qquad V_{\mu^i} = \underset{i}{\gamma}(\mu) \in \Gamma(V(\mu)), \quad i = 1, 2,$$

i.e. the tangent plane to H_2 is spanned by two vectors which we denote, as usual

$$T_{V(\mu)}(H_2) = [\underset{1}{\gamma}(\mu), \underset{2}{\gamma}(\mu)].$$

(II) There exists two linearly independent vectors:

$$\underset{i}{\lambda}(\mu) \in \Lambda(V(\mu)), \quad i = 1, 2,$$

such that

$$\underset{i}{\lambda}(\mu) \leftrightarrow \underset{i}{\gamma} = V_{\mu^i}$$

and

$$\partial \underset{i}{\lambda} / \partial \mu^j \in [\underset{1}{\lambda}(\mu), \underset{2}{\lambda}(\mu)] \quad \text{for} \quad i \neq j, \quad i, j = 1, 2.$$

We denote by $\Sigma(\mu)$ the n-2 dimensional space spanned by the vectors

$$\underset{1}{\sigma}, \underset{2}{\sigma}, \ldots, \underset{n-2}{\sigma}, \quad (\underset{i}{\sigma}(\mu) \underset{j}{\lambda}(\mu)) = 0, \quad i = 1, 2, \ldots, n-2, \quad j = 1, 2.$$

The solutions belonging to the class $K(H_2)$ are constant on n-2 dimensional planes parallel to $\Sigma(\mu)$ [5].

To obtain solutions of class $K(H_2)$ the parametrization of H_2 by $x^1, \ldots, x^n$ is used [4]. The simplest possible parametrization is the conical one [5] and it can be performed for the conical Bernoulli manifolds defined as follows.

The manifold $H_2: V = V(\mu^1, \mu^2)$ is called conical Bernoulli manifold if the mapping $(t^1, \ldots, t^{n-2}, \mu^1, \mu^2) \to (x^1, \ldots, x^n)$ given by the equation

$$\text{(2.1)} \qquad x = \sum_{i=1}^{n-2} t^i \underset{i}{\sigma}(\mu^1, \mu^2)$$

is one-to-one for $t \neq 0$.

In the conical parametrization of the conical Bernoulli manifold we put

$$V(x) = V(\mu^1, \mu^2)$$

for points x such that

$$x = \sum_{i=1}^{n-2} t^i \underset{i}{\sigma}(\mu^1, \mu^2), \quad -\infty < t^i < +\infty.$$

This $V(x)$ is a solution in the class $K(H_2)$.

Now we will use the concepts introduced above to the system (1.1) and for this particular system in the supersonic case we will give the form of corresponding cones $\Lambda(U)$ and $\Gamma(U)$.

Let us denote

$$\gamma = (\gamma^0, \gamma^1, \gamma^2, \gamma^3), \quad \bar{\gamma} = (\gamma^1, \gamma^2, \gamma^3),$$
$$\lambda = (\lambda_1, \lambda_2, \lambda_3).$$

The cone Λ is given by

$$\Lambda(U) = \overset{1}{\Lambda}(U) \cup \overset{2}{\Lambda}(U) \subset R^3,$$

where

$$\overset{1}{\Lambda}\colon (u, \lambda)^2 - c^2|\lambda|^2 = 0; \quad \overset{2}{\Lambda}\colon (u, \lambda) = 0.$$

The cone $\overset{1}{\Lambda}$ is perpendicular to the Mach cone and $\overset{2}{\Lambda}$ is a plane.

The cone Γ in our case is given by

$$\Gamma(U) = \overset{1}{\Gamma}(U) \cup \overset{2}{\Gamma}(U) \subset R^4,$$

where

(2.2)
$$\overset{1}{\Gamma}\colon \begin{cases} (u, \overline{\gamma})^2 - c^2|\overline{\gamma}|^2 = 0 \\ kc\gamma^0 + (u, \overline{\gamma}) = 0 \end{cases}; \quad \overset{2}{\Gamma}\colon \gamma^0 = 0.$$

It can be shown that

(2.3)
$$\overset{1}{\Gamma} \ni \gamma \leftrightarrow \lambda = \overline{\gamma} \in \overset{1}{\Lambda}$$

and

(2.4)
$$\overset{2}{\Gamma} \ni \gamma \leftrightarrow \lambda = \underset{1}{e}\varphi + \underset{2}{e}\psi \in \overset{2}{\Lambda}, (^1)$$

where $\underset{1}{e} = (u^2, -u^1, 0)$, $\underset{2}{e} = (u^3, 0, -u^1)$ and φ and ψ are arbitrary real numbers.

For the above form of cones it follows that the system (1, 1) possesses the following three types of Bernoulli manifolds: (i) $\overset{1,1}{H_2}$, characterized by the property that the tangent plane to this manifold is spanned by vectors belonging to $\overset{1}{\Gamma}$ only, i.e.

$$T(\overset{1,1}{H_2}) = [\underset{+}{\overset{1}{\gamma}}, \underset{-}{\overset{1}{\gamma}}], \quad \underset{+}{\overset{1}{\gamma}}, \underset{-}{\overset{1}{\gamma}} \in \overset{1}{\Gamma}.$$

The corresponding class of solutions $K(\overset{1,1}{H_2})$ consists of potential flows. The classical Bernoulli manifolds $H_2(q)$ are of this type. (ii) $\overset{1,2}{H_2}$ for which

$$T(\overset{1,2}{H_2}) = [\overset{1}{\gamma}, \overset{2}{\gamma}], \quad \overset{1}{\gamma} \in \overset{1}{\Gamma}, \quad \overset{2}{\gamma} \in \overset{2}{\Gamma}.$$

This class contain both potential and nonpotential flows. (iii) $\overset{2,2}{H_2}$, described by the relation

$$T(\overset{2,2}{H_2}) = [\underset{+}{\overset{2}{\gamma}}, \underset{-}{\overset{2}{\gamma}}], \quad \underset{+}{\overset{2}{\gamma}}, \underset{-}{\overset{2}{\gamma}} \in \overset{2}{\Gamma}.$$

The flows belonging to this class have the property that the sound velocity at each point of the flow field is the same. As no solutions of our problem 1), 2) belong to this class, it will not be considered here.

(^{1}) The right form of this formula is $\overset{2}{\Gamma} \ni \gamma = (0, \overline{\gamma}) \leftrightarrow \lambda = \underset{1}{e}\varphi + \underset{1}{e}\psi, (\lambda, \overline{\gamma}) = 0$. Our considerations is part 4 must be correspondingly changed.

3. The conical Bernoulli manifolds $\overset{1,1}{H_2}$ for the system (1.1)

First we shall consider the problem of existence of the conical $\overset{1,1}{H_2}$ Bernoulli manifolds. If $\overset{1,1}{H_2}: U = U(\mu^1, \mu^2)$, then due to Eq. (2.2) the condition I

$$U_{\mu^i}(\mu) \in \overset{1}{\Gamma}(U(\mu)), \quad i = 1, 2,$$

has the form

$$(u, u_{\mu^i})^2 - c^2|u_{\mu^i}|^2 = 0, \quad kcc_{\mu^i} + (u, u_{\mu^i}) = 0, \quad i = 1, 2. \tag{3.1}$$

For

$$\gamma = U_{\mu^i} = (c_{\mu^i}, u^1_{\mu^i}, u^2_{\mu^i}, u^3_{\mu^i})$$

(2.3) may be written

$$\overset{1}{\Gamma} \ni U_{\mu^i} = \gamma \leftrightarrow \lambda(\mu) = u_{\mu^i} \in \overset{1}{\Lambda}. \tag{3.2}$$

Hence the condition II takes the form

$$(u_{\mu^1\mu^2}, u_{\mu^1} \wedge u_{\mu^2}) = 0,$$

where $\wedge$ denotes the vector product and we assume that U is a twice continuous differentiable function.

According to the notation introduced already $u = (u^1, u^2, u^3)$ and $U = (c, u) = (c, u^1, u^2, u^3)$; by projection $\downarrow W \subset R^3$ of the set $W \subset R^4$ we understand the set of u, such that there exists at least one c for which $U = (c, u) \in W$.

The second set of equations (3.1) is equivalent to

$$kc^2 + |u|^2 = \text{const} = q^2.$$

Hence to construct $\overset{1,1}{H_2}$ we need only to find $\downarrow \overset{1,1}{H_2}: u = u(\mu^1, \mu^2)$ satisfying the system

$$\begin{gathered} (u, u_{\mu^i})^2 - c^2(u)|u_{\mu^i}|^2 = 0, \quad i = 1, 2, \\ (u_{\mu^1\mu^2}, u_{\mu^1} \wedge u_{\mu^2}) = 0, \end{gathered} \tag{3.3}$$

where $c^2(u) = (q^2 - |u|^2)\dfrac{1}{k}$. Differentiating the first equation with respect to μ^2 and the second with respect to μ^1 the system (3.3) can be reduced to the hyperbolic system

$$u_{\mu^1\mu^2} = f(u, u_{\mu^1}, u_{\mu^2}). \tag{3.4}$$

It is known that for differentiable functions, $\phi(\mu^1, \mu^2) = 0$ iff

$$\phi_{\mu^1}(\mu^1, \mu^2) = 0 \quad \text{and} \quad \phi(0, \mu^2) = 0,$$

or

$$\phi_{\mu^2}(\mu^1, \mu^2) = 0 \quad \text{and} \quad \phi(\mu^1, 0) = 0.$$

Applying this fact we conclude that the solution of Eq. (3.4) satisfies Eq. (3.3) iff

$$(u(\mu^1, 0), \mu_{\mu^1}(\mu^1, 0))^2 - c^2(u(\mu^1, 0))|u_{\mu^1}(\mu^1, 0)|^2 = 0,$$
$$(u(0, \mu^2), u_{\mu^2}(0, \mu^2))^2 - c^2(u(0, \mu^2))|u_{\mu^2}(0, \mu^2)|^2 = 0.$$

This means that for the solution $u = u(\mu^1, \mu^2)$ of Eq. (3.4) the curves $u = u(\mu^1, 0)$ and $u = u(0, \mu^2)$ are Beroulli manifolds $H_\perp$.

Let us consider for the system (3.4) the boundary problem with the following boundary conditions:

$$(3.5) \qquad u(\mu^1, 0) = \underset{1}{u}(\mu^1), \quad u(0, \mu^2) = \underset{2}{u}(\mu^2),$$

where $\underset{1}{u}(\mu^1)$, $\underset{2}{u}(\mu^2)$ are arbitrary functions satisfying the relation

$$\underset{1}{u}(0) = \underset{2}{u}(0).$$

The following theorem for the system (3.4) holds [6, 7]:

THEOREM 1. *If $f \in C^1$, $\underset{i}{u} \in C^1$, $i = 1, 2$, then there exist two numbers, $\alpha > 0$, $\beta > 0$ such that in $M = (0, \alpha) \times (0, \beta)$ there exists only one solution*

$$u \in C^2(M) \cap C^1(\bar{M})$$

of the system (3.4) *satisfying* (3.5).

From this theorem and the above considerations it follows:

THEOREM 2. *Through each pair of smooth curves h_1' and h_1'' of the kind $\downarrow H_\perp$ such that $h_1' \cap h_1'' = \underset{0}{u}$ and such that the vectors $\bar{\gamma}'$, $\bar{\gamma}''$ tangent to h_1' and h_1'' at $\underset{0}{u}$ are linearly independent, there passes exactly one surface h_2 of the kind $\downarrow \overset{1,1}{H_2}$.*

This means that through each pair of $H_\perp$ curves there passes exactly one H_2 Bernoulli manifold. The remaining question is when this manifolds are conical.

For $H_2 : U = U(\mu^1, \mu^2)$ we have, according to Eq. (3.2),

$$u_{\mu^i} = \lambda(\mu) \leftrightarrow \gamma = U_{\mu^i}, \quad i = 1, 2.$$

We may put

$$\sigma(\mu) = \frac{u_{\mu^1} \wedge u_{\mu^2}}{|u_{\mu^1} \wedge u_{\mu^2}|}$$

and the resulting $\Sigma(\mu)$ are one-dimensional spaces.

Let (Fig. 4) $H_1', H_1'' \subset \overset{1,1}{H_2}$, H_1': $U = U(\mu^1, \underset{0}{\mu^2})$, H_1'': $U = U(\underset{0}{\mu^1}, \mu^2)$, where $\underset{0}{\mu^1}$, $\underset{0}{\mu^2}$

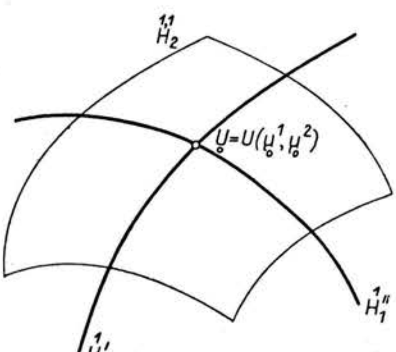

FIG. 4.

are constants. By $P'(\underset{0}{\mu^1}, \underset{0}{\mu^2})$ and $P''(\underset{0}{\mu^1}, \underset{0}{\mu^2})$ we denote contact planes of the curves $\downarrow H_1$ and $\downarrow H_1''$ at the point $\underset{0}{u} = u(\underset{0}{\mu^1}, \underset{0}{\mu^2})$. Using this notation we can express the condition for Bernoulli manifold to be conical in the form of the following:

THEOREM 3. *The Bernoulli manifold $H_2 : U = U(\mu^1, \mu^2)$ is in the neighbourhood* of $\underset{0}{U} = U(\underset{0}{\mu^1}, \underset{0}{\mu^2})$ *conical if $\sigma(\underset{0}{\mu^1}, \underset{0}{\mu^1})$ is not perpendicular to any of contact planes $P'(\underset{0}{\mu^1}, \underset{0}{\mu^2})$ and $P''(\underset{0}{\mu^1}, \underset{0}{\mu^2})$.*

It should be noticed that for the system (1.1) at each point $\underset{0}{U} = (\underset{0}{c}, \underset{0}{u}, \underset{0}{u^2}, \underset{0}{u^3})$, $\underset{0}{c} > |u|$ we may construct infinitely many distinct pairs of Bernoulli curves H_1', H_1'' such that

$$H_1' \cap H_1'' = \underset{0}{U},$$

and $\downarrow H_1' : u = u_1(\mu)$, $\downarrow H_1'' : u = \underset{2}{u}(\mu)$ satisfy the conditions of Theorem 2 and that the planes $P'(\underset{0}{\mu})$, $P''(\underset{0}{\mu})$ contact to $\downarrow H_1'$, $\downarrow H_1''$ are not perpendicular to

$$\sigma(\underset{0}{\mu}) = \frac{\dfrac{d}{d\mu} \underset{1}{u}(\underset{0}{\mu}) \wedge \dfrac{d}{d\mu} \underset{2}{u}(\underset{0}{\mu})}{\left| \dfrac{d}{d\mu} \underset{1}{u}(\underset{0}{\mu}) \wedge \dfrac{d}{d\mu} \underset{2}{u}(\underset{0}{\mu}) \right|}.$$

Hence, making use of Theorems 2 and 3 we may construct a wide class of conical Bernoulli manifolds $\overset{1,1}{H_2}$.

The classical Bernoulli manifold $H_2(q)$ is of the kind $\overset{1,1}{H_2}$ and, obviously, does not satisfy the conditions of Theorem 3. Indeed $\downarrow H_2(q)$ is the plane $u^3 = 0$; it is a contact plane for each Bernoulli curve $\downarrow H_1 \subset \downarrow H_2(q)$ and $\sigma = (0, 0, 1)$ is perpendicular to it. Therefore $H_2(q)$ is not a conical manifold.

P r o o f of Theorem 3. For $\underset{0}{\mu} = (\underset{0}{\mu^1}, \underset{0}{\mu^2})$ we have

$$P'(\underset{0}{\mu}) = [u_{\mu^1}(\underset{0}{\mu}), u_{\mu^2\mu^1}(\underset{0}{\mu})],$$
$$P''(\underset{0}{\mu}) = [u_{\mu^2}(\underset{0}{\mu}), u_{\mu^2\mu^2}(\underset{0}{\mu})]$$

and $(\sigma, u_{\mu^i}) = 0$, $i = 1, 2$. Hence the conditions of Theorem 3 may be formulated as

$$\big(\sigma(\underset{0}{\mu}), u_{\mu^i\mu^i}(\underset{0}{\mu})\big) \neq 0, \quad i = 1, 2. \tag{3.6}$$

The manifold $\overset{1,1}{H_2}$ is conical if the mapping (2.1) is one-to-one and this is equivalent to the condition that the vectors

$$\sigma(\underset{0}{\mu}), \sigma_{\mu^1}(\underset{0}{\mu}), \sigma_{\mu^2}(\underset{0}{\mu}) \tag{3.7}$$

are linearly independent. Differentiating the equation $(\sigma, u_{\mu^i}) = 0$ we obtain $(\sigma_{\mu^i}, u_{\mu^i}) + (\sigma, u_{\mu^i\mu^i}) = 0$, $i = 1, 2$. Hence from Eq. (3.6) it follows that

(3.8) $$\sigma_{\mu^i}(\underset{0}{\mu}) \neq 0, \quad i = 1, 2.$$

Differentiating $\sigma(\mu)$ we get

$$\sigma_{\mu^1} = (u_{\mu^1\mu^1} \wedge u_{\mu^2} + u_{\mu^1} \wedge u_{\mu^1\mu^2})|u_{\mu^1} \wedge u_{\mu^2}|^{-1} + u_{\mu^1} \wedge u_{\mu^2} \frac{\partial}{\partial \mu^1}(|u_{\mu^1} \wedge u_{\mu^2}|^{-1}).$$

But $u_{\mu^1} \wedge u_{\mu^1\mu^2} || \sigma$ and therefore

$$\sigma_{\mu^1} = \underset{1}{\alpha}\sigma + \underset{1}{\beta} u_{\mu^1\mu^1} \wedge u_{\mu^2}$$

and similarly

$$\sigma_{\mu^2} = \underset{2}{\alpha}\sigma + \underset{2}{\beta} u_{\mu^2\mu^2} \wedge u_{\mu^1},$$

where $\underset{i}{\alpha}, \underset{i}{\beta}$ depend on μ.

Due to Eq. (3.8) and the condition $(\sigma, \sigma_{\mu^i}) = 0$ we have $\underset{i}{\beta} \neq 0$. Hence the vectors (3.7) are linearly independent if the vectors

(3.9) $$\sigma, u_{\mu^1\mu^1} \wedge u_{\mu^2}, \quad u_{\mu^2\mu^2} \wedge u_{\mu^1}$$

are linearly independent. For the planes

$$R^3 \supset \Pi_i: (x, u_{\mu^i}) = 0, \quad i = 1, 2,$$

we have

$$\underset{1}{\varrho} = u_{\mu^2\mu^2} \wedge u_{\mu^1} \in \Pi_1, \quad \underset{2}{\varrho} = u_{\mu^1\mu^1} \wedge u_{\mu^2} \in \Pi_2,$$
$$\sigma \in \Pi_1 \cap \Pi_2,$$

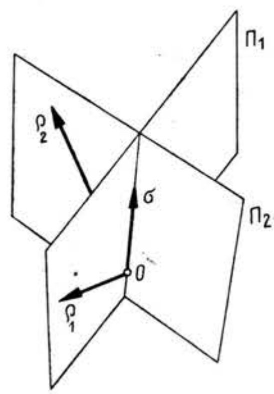

FIG. 5.

(Fig. 5). Therefore the linear dependence of the vectors (3.9) means that at least one of the following conditions is obeyed

$$\underset{1}{\varrho} = u_{\mu^2\mu^2} \wedge u_{\mu^1} || \sigma \Rightarrow (\sigma, u_{\mu^2\mu^2}) = 0,$$
$$\underset{2}{\varrho} = u_{\mu^1\mu^1} \wedge u_{\mu^2} || \sigma \Rightarrow (\sigma, u_{\mu^1\mu^1}) = 0,$$

but these conditions contradict Eqs. (3.6). Thus the theorem is proved.

4. The conical Bernoulli manifolds $\overset{1,2}{H_2}$

The $\overset{1,2}{H_2}$ Bernoulli manifolds can be constructed similarly as $\overset{1,1}{H_2}$ in Sect. 3. If $\overset{1,2}{H_2}$: $U = U(\mu^1, \mu^2)$, then the condition I takes the form

$$U_{\mu^1} = \underset{1}{\gamma} \in \overset{1}{\Gamma}(U(\mu)), \quad U_{\mu^2} = \underset{2}{\gamma} \in \overset{2}{\Gamma}(U(\mu))$$

that is

$$(u, u_{\mu^1})^2 - c^2|u_{\mu^1}|^2 = 0, \quad kcc_{\mu^1} + (u, u_{\mu^1}) = 0 \tag{4.1}$$

and

$$c_{\mu^2} = 0. \tag{4.2}$$

Moreover, the vectors $\underset{i}{\gamma} \leftrightarrow \underset{i}{\lambda}$ entering the condition II are

$$\underset{1}{\gamma}(\mu) = (c_{\mu^1}, u^1_{\mu^1}, u^2_{\mu^1}, u^3_{\mu^1}) \leftrightarrow \underset{1}{\lambda}(\mu) = \underset{1}{\overline{\gamma}}(\mu) = u_{\mu^1} \in \overset{1}{\Lambda}(U(\mu)),$$

$$\underset{2}{\gamma}(\mu) = (c_{\mu^2}, u^1_{\mu^2}, u^2_{\mu^2}, u^3_{\mu^2}) \leftrightarrow \underset{2}{\lambda}(\mu) = \underset{1}{e}\varphi(\mu) + \underset{2}{e}\psi(\mu) \in \overset{2}{\Lambda}(U(\mu)),$$

where $\varphi(\mu)$, $\psi(\mu)$ are arbitrary functions.

According to II

$$\underset{1}{\lambda}_{\mu^2} = u_{\mu^1\mu^2} \in [\underset{1}{\lambda}, \underset{2}{\lambda}], \quad \underset{2}{\lambda}_{\mu^1} \in [\underset{1}{\lambda}, \underset{2}{\lambda}]$$

and thus

$$\left(u_{\mu^1\mu^2} \wedge u_{\mu^1}(\underset{1}{e}\varphi + \underset{2}{e}\psi)\right) = 0, \tag{4.3}$$

where, as before $\underset{1}{e} = (u^2, -u^1, 0)$, $\underset{2}{e} = (u^3, 0, -u^1)$ and

$$\begin{vmatrix} (u^2\varphi + u^3\varphi)_{\mu^1}, & u^1_{\mu^1}, & u^2\varphi + u^3\psi \\ -(u^1\varphi)_{\mu^1}, & u^2_{\mu^1}, & -u^1\varphi \\ -(u^1\psi)_{\mu^1}, & u^3_{\mu^1}, & -u^1\psi \end{vmatrix} = 0.$$

This determinant leads to the expression

$$\varphi_{\mu^1}\psi - \psi_{\mu^1}\varphi = f(u, u_{\mu^1}, \varphi, \psi). \tag{4.4}$$

We shall present the system (4.1), (4.2), (4.3) and (4.4) in a form similar to (3.4). This form allows us to prove the existence of $\overset{1,2}{H_2}$ manifolds and is well suited to numerical computations of corresponding nozzles.

Differentiating Eqs. (4.1) and (4.4) in respect to μ^2 and Eq. (4.2) in respect to μ^1 we obtain

$$\left(u_{\mu^1\mu^2}, (u, u_{\mu^1})u - c^2 u_{\mu^1}\right) = (u_{\mu^1}, u_{\mu^2})(u, u_{\mu^1}), \tag{4.5}$$

$$(u_{\mu^1\mu^2}, u) = -(u_{\mu^1}, u_{\mu^2}), \tag{4.6}$$

$$\varphi_{\mu^1\mu^2} = D(u, u_{\mu^1}, u_{\mu^2}, \varphi, \psi, \varphi_{\mu^1}, \varphi_{\mu^2}, \psi_{\mu^1}, \psi_{\mu^2}, \psi_{\mu^1\mu^2}), \tag{4.7}$$

$$c_{\mu^1\mu^2} = 0. \tag{4.8}$$

The system (4.3) (4.5) and (4.6) is equivalent to

(4.9) $$u_{\mu^1\mu^2} = F(u, u_{\mu^1}, u_{\mu^2}, \varphi, \psi)$$

provided that

$$\dim[(u, u_{\mu^1})u - c^2 u_{\mu^1}, u, u_{\mu^1} \wedge (\underset{1}{e}\varphi + \underset{2}{e}\psi)] = 3.$$

Introducing Eq. (4.9) to Eq. (4.7) we get

(4.10) $$\varphi_{\mu^1\mu^2} = E(u, u_{\mu^1}, u_{\mu^2}, \varphi, \psi, \varphi_{\mu^1}, \varphi_{\mu^2}, \psi_{\mu^1}, \psi_{\mu^2}, \psi_{\mu^1\mu^2}).$$

The system (4.8), (4.9) and (4.10) has the same form as the system (3.4) with an arbitrary function ψ. This system is equivalent to the system (4.1)–(4.4) if we assume that the unknown functions U and φ obey the boundary conditions

(4.11) $$\frac{d}{d\mu^1} U(\mu^1, 0) \in \overset{1}{\Gamma}(U(\mu^1, 0)),$$

(4.12) $$\varphi_{\mu^1}(\mu^1, 0)\psi(\mu^1, 0) - \psi_{\mu^1}(\mu^1, 0)\varphi(\mu^1, 0) = f(u(\mu^1, 0), u_{\mu^1}(\mu^1, 0), \varphi(\mu^1, 0), \psi(\mu^1, 0)),$$

(4.13) $$U(0, \mu^2) = (\overset{0}{c}, \overset{0}{u}(\mu^2))$$

with an arbitrary constant $\overset{0}{c}$ and arbitrary functions $\overset{0}{u}(\mu^2)$.

For our system we can now prove the following:

THEOREM 4. *For each pair of functions*

$$\underset{i}{U}: R^1 \to R^4, \quad \underset{i}{U}(s) = (\underset{i}{c}(s), \underset{i}{u}(s)), \quad \underset{i}{u}: R^1 \to R^3, \quad \underset{i}{U} \in C^2, \quad i = 1, 2,$$

$\underset{1}{U}(0) = \underset{2}{U}(0)$ *satisfying the conditions*

$$\frac{d\underset{1}{U}}{ds} \in \overset{1}{\Gamma}(\underset{1}{U}(s)), \quad \frac{d\underset{1}{u}}{ds} \neq 0$$

and $\underset{2}{U} = (\overset{0}{c}, \underset{2}{u}(s))$, *where c is an arbitrary number and* $\underset{2}{u}(s)$ *is an arbitrary function such that*

$$\frac{d\underset{2}{u}}{ds} \neq 0$$

there exists an infinite family of different Bernoulli manifolds $\overset{1,2}{H_2}: U = U(\mu^1, \mu^2) \in C^2$ *which are conical in the neighbourhood of* $U(0, 0)$ *and such that*

$$U(\mu^1, 0) = \underset{1}{U}(\mu^1), \quad U(0, \mu^1) = \underset{2}{U}(\mu^2).$$

Proof

Solutions of the class $K(\overset{1,2}{H_2})$, $(\overset{1,2}{H_2}): U = U(\mu^1, \mu^2)$ are constant along the lines parallel to the vectors

$$\sigma(\mu) = u_{\mu^1} \wedge (\underset{1}{e}\varphi(\mu) + \underset{2}{e}\psi(\mu)).$$

Hence it is sufficient to show that we can choose $\varphi(\mu^1, 0)$, $\psi(\mu^1, 0)$, $\varphi(0, \mu^2)$, $\psi(0, \mu^2)$

such that Eq. (4.12) is satisfied and the mapping $(t, \mu^1, \mu^2) \to (x^1, x^2, x^3)$ given by the relation

$$x = t\sigma(\mu^1, \mu^2),$$

$t \neq 0$, is one-to-one in the neighbourhood of $(t, 0, 0)$.

Therefore, for any solution $U(\mu)$ of the system (4.8)–(4.10) with the boundary conditions (4.11)–(4.13) we must check that the vectors

$$\sigma(0,0), \quad \sigma_{\mu^1}(0,0), \quad \sigma_{\mu^2}(0,0) \tag{4.14}$$

are linearly independent.

First we show that $u_{\mu^1\mu^1} \neq 0$. Indeed, differentiating $(4.1)_1$ we obtain

$$(u, u_{\mu^1})[|u_{\mu^1}|^2 + (u, u_{\mu^1\mu^1})] - cc_{\mu^1}|u_{\mu^1}|^2 - c^2(u_{\mu^1}, u_{\mu^1\mu^1}) = 0.$$

But the assumption $u_{\mu^1\mu^1} = 0$ together with Eq. $(4.1)_2$ leads to

$$(1+k)(u, u_{\mu^1})|u_{\mu^1}|^2 = 0,$$

which is not possible because $k > 0$ and $(u, u_{\mu^1}) \neq 0$.

Second, we check that

$$(\sigma_{\mu^1}(0,0), u_{\mu^1}(0,0)) \neq 0. \tag{4.15}$$

From the definition of σ we have

$$(\sigma, u_{\mu^1}) = 0 \tag{4.16}$$

and hence

$$(\sigma_{\mu^1}, u_{\mu^1}) + (\sigma, u_{\mu^1\mu^1}) = 0.$$

As $u_{\mu^1\mu^1} \neq 0$ we may choose $\varphi(0, 0)$ and $\psi(0, 0)$ such that

$$(\sigma(0,0), u_{\mu^1\mu^1}(0,0)) \neq 0$$

and therefore the required inequality (4.15) follows.

Third, we need to show that $\sigma_{\mu^2}(0,0)$ is linearly independent of both $\sigma(0, 0)$ and $\sigma_{\mu^1}(0, 0)$.

The vector $\sigma_{\mu^2}(0,0)$ depends on $\varphi_{\mu^2}(0, 0)$ and $\psi_{\mu^2}(0, 0)$. According to the definition of σ we have

$$\sigma_{\mu^2}(0,0) = u_{\mu^1\mu^2} \wedge d(0,0) + u_{\mu^1}(0,0) \wedge d_{\mu^2}(0,0),$$

where

$$d(\mu^1, \mu^2) = \underset{2}{e}(\mu^1, \mu^2)\varphi(\mu^1, \mu^2) + \underset{2}{e}(\mu^1, \mu^2)\psi(\mu^1, \mu^2).$$

We can write

$$d_{\mu^2} = W(\varphi, \psi) + \underset{1}{e}\varphi_{\mu^2} + \underset{2}{e}\psi_{\mu^2}. \tag{4.17}$$

Any point on the plane

$$(x - W(\varphi, \psi), u(0,0)) = 0$$

can be presented in the form (4.17), where $\mu = (0, 0)$, by an appropriate choice of $\varphi_{\mu^2}(0, 0)$ and $\psi_{\mu^2}(0, 0)$. Hence the vector $\alpha\, u_{\mu^1}(0, 0) \wedge d_{\mu^2}(0, 0)$, $\alpha \in R^1$, is an arbitrary point on the plane $(x, u_{\mu^1}(0, 0)) = 0$.

If we change $\varphi_{\mu^2}(0, 0)$, $\psi_{\mu^2}(0, 0)$, then $d(0, 0)$ and $u_{\mu^1\mu^2}(0, 0)$ remain unaltered as follows from Eq. (4.9). Hence $\sigma_{\mu^2}(0, 0)$ is an arbitrary vector on the plane

$$(4.18)\qquad \big(x-u_{\mu^1\mu^2}(0,0)\wedge d(0,0), u_{\mu^1}(0,0)\big) = 0.$$

Because of Eqs. (4.15), (4.16) and (4.18), the linear independence of the vectors (4.14) follows.

5. Construction of nozzles without shock

Now we intend to parametrize the conical Bernoulli manifold H_2, where H_2 denotes either $\overset{1,1}{H_2}$ or $\overset{1,2}{H_2}$, $H_2: U = U(\mu^1, \mu^2)$, $\mu \in M = \langle 0, \alpha\rangle \times \langle 0, \beta\rangle$. This procedure leads to the solution $U \in C^1_W(D)$ satisfying the conditions 1) and 2).

We define the lines $l^+(\mu)$ and $l^-(\mu)$ by

$$l^+(\mu):\ x = t\sigma(\mu), \quad t > 0,$$
$$l^-(\mu):\ x = t\sigma(\mu), \quad t < 0,$$
$$Z^+ = \bigcup_{\mu\in M} l^+(\mu), \quad Z^- = \bigcup_{\mu\in M} l^-(\mu), \quad Z = Z^+ \cup Z^-.$$

Using the conical parametrization (2.1) we obtain the solution

$$U_{\text{con}} \in C^1(Z) \cap K(H_2).$$

Now we shall prolong this solution to the neighbourhood of Z through the lines $l^+(\mu)$, $l^-(\mu)$, $\mu \in \partial M$ using simple waves and constant solutions. To this end we use characteristic vectors $\underset{i}{\lambda}(\mu) \leftrightarrow \underset{i}{\gamma}(\mu) = U_{\mu^i}(\mu)$ and the planes

$$\underset{i}{\Pi}(\mu):\ \big(x, \underset{i}{\lambda}(\mu)\big) = 0.$$

According to our definition we have

$$l^\pm(\mu^1, 0) \subset \underset{1}{\Pi}(\mu^1, 0), \quad l^\pm(\mu^1, \beta) \subset \underset{1}{\Pi}(\mu^1, \beta).$$

The solution defined already on lines $l^\pm$, we prolong it in such a way that it remains constant on the corresponding Π-plane (Fig. 6). Therefore, in the region adjacent to Z we obtain solutions of the type of a simple wave.

In the corner bounded by the planes $\underset{1}{\Pi}(0, 0)$ and $\underset{2}{\Pi}(0, 0)$ (Fig. 6) we put

$$U_{\text{con}}(X) = U(0, 0) = \text{const}$$

and, similarly, in the corner bounded by $\underset{1}{\Pi}(\alpha, \beta)$ and $\underset{2}{\Pi}(\alpha, \beta)$ we put

$$U_{\text{con}}(x) = U(\alpha, \beta) = \text{const}.$$

For the conical solution U_{con} we have $U_{\text{con}} \in C^1_W(D)$, where $D \supset Z, D = D^+ \cup D^-$, and $D^\pm$ are the neighbourhoods of $Z^\pm$.

Now we can choose $u(0, 0)$, $|u(0, 0)| > c(0, 0) > 0$, such that the stream lines of the solutions $U_{\text{con}}(x)$ crossing $l^+(0, 0)$ enter the set Z^+.

Intersecting the domains D^+ and D^- by planes Q^+, Q^- which are perpendicular to $\sigma(0, 0)$,

$$Q^-:\ \big(x-\sigma(0,0), \sigma(0,0)\big) = 0, \quad Q^+:\ \big(x+\sigma(0,0), \sigma(0,0)\big) = 0,$$

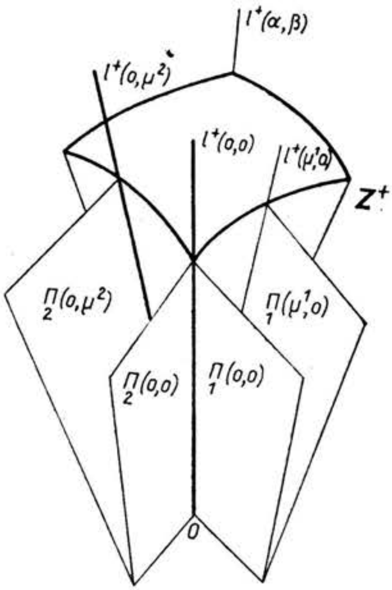

FIG. 6.

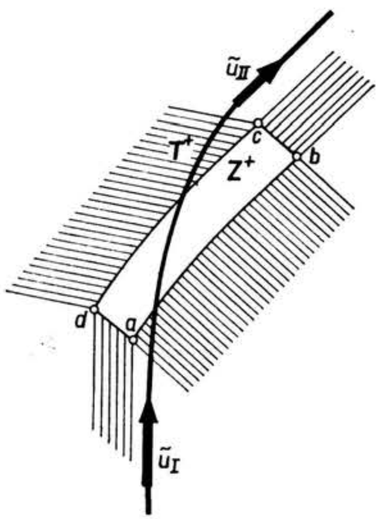

FIG. 7.

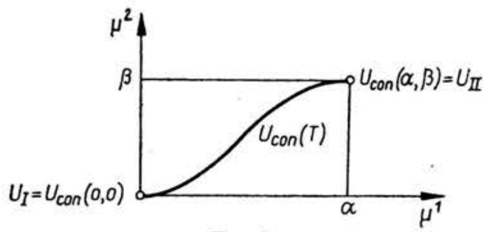

FIG. 8.

we get the picture shown in Fig. 7 (for D^+). Here $\tilde{u}_{\mathrm{I}}$ and $\tilde{u}_{\mathrm{II}}$ denote the projections of $u_{\mathrm{I}} = u(0, 0)$ and $u_{\mathrm{II}} = u(\alpha, \beta)$ on Q^+ and T^+ denotes the trajectory tangent to the field $\tilde{u}(x)$.

The existence of stream lines $T^\pm$ joining the regions of the constant solution depends on the velocity distribution in Z. But if b, c (Fig. 7) is sufficiently small, then such stream lines do exist. This condition is fulfilled if we put β in $M = \langle 0, \alpha\rangle \times \langle 0, \beta\rangle$ small enough. If the nozzle in D^+ accelerates the flow, then the nozzle in D^- slows it down and vice versa. The μ^1, μ^2 map of the set of values of the nozzle is shown in Fig. 8.

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