Arch. Mech., 33, 1, pp. 83—97, Warszawa 1981

Perturbation solution for viscoplastic beam under ideal impulse loading

W. WOJNO (WARSZAWA)

A PERTURBATION approach is proposed to get approximate solutions for a wide class of problems for dynamically-loaded rigid viscoplastic structures. The procedure consists in assuming a constant shape function for all values of a small parameter and then using the Galerkin weighted residual method combined with the general averaging technique for an amplitude. Such an approach admits nonhomogeneous forms of governing equations and does allow to apply the inhomogeneous rigid viscoplastic constitutive relation due to Perzyna. The method is illustrated by a problem of an impulsively-loaded fully clamped beam. A relatively simple approximate solution for large central deflections is obtained and compared to other existing perturbation solutions and available experimental data.

Zaproponowano metodę perturbacyjną poszukiwania rozwiązań przybliżonych dla szerokiej klasy problemów dynamicznie obciążonych konstrukcji lepkoplastycznych. W metodzie tej zakłada się stałą dla wszystkich wartości małego parametru funkcję kształtu, a następnie stosuje się metodę Galerkina pozostałości z wagą oraz uogólnioną technikę uśredniania dla wyznaczania amplitudy. Takie podejście dopuszcza niejednorodne równania opisujące materiał, a zwłaszcza równanie konstytutywne lepkoplastyczności sformułowane przez Perzynę. Sposobem tym rozwiązano zagadnienie obustronnie utwierdzonej belki, poddanej obciążeniu impulsowemu. Uzyskano stosunkowo proste rozwiązanie dla ugięć w środku belki, które porównano z innym, istniejącym w literaturze rozwiązaniem perturbacyjnym i z dostępnymi wynikami badań eksperymentalnych.

Предложен пертурбационный метод нахождения приближенных решений для широкого класса задач динамически нагруженных вязкопластических конструкций. В этом методе предполагается постоянная, для всех значений малого параметра, функция формы, а затем применяется метод Галеркина остатка с весом и обобщенная техника усреднения для определения амплитуды. Такой подход допускает неоднородные уравнения, описывающие материал, а особенно определяющее уравнение вязкопластичности сформулированное Пэжина. Этим способом решена задача закрепленной с обоих сторон балки, подвергнутой импульсной нагрузке. Получено сравнительно простое решение для прогибов в центре балки, которое сравнено с другим, существующим в литературе, пертурбационным решением и с доступными результатами экспериментальных исследований.

1. Introduction

RECENT literature on plastic structural dynamics put forward an interesting idea of applying perturbation methods to solve a class of nonlinear initial boundary-value problems for rate-sensitive structures. In the paper [5] the concept of perturbation around a rigid perfectly-plastic solution was introduced and explained on the basis of an example of the impulsively loaded thick-walled spherical container made of incompressible, homogeneous, rigid-viscoplastic material, obeying the constitutive relation due to Perzyna. Both the exact and the small parameter perturbation solutions, developed by means of the Linstedt–Poincaré technique, were then modified using Shank's transformation to extend the admissible range of the small parameter. Direct comparison of the exact solution with the perturbation solution revealed surprisingly good accuaracy of the latter within a rather wide range of the small parameter.

The idea of perturbation around a rigid-perfectly plastic solution was further extended in the papers [6] and [7] where the problem of the fully clamped viscoplastic beam and circular plate subjected to ideal impulses was considered. The main concept involved there was to apply a homogeneous viscous type of constitutive equation in order to get separable form solutions with stationary modes. What is worth emphasizing is the exponent in the constitutive relation which was identified as a small parameter. It was pointed out, see Ref. [6], that when the small parameter goes to zero, the power stress-strain rate constitutive law reduces to the equation describing perfectly plastic material with a constant, but raised, yield stress. This property was then used to obtain approximate solutions for viscous structures by perturbing the solutions of the corresponding reduced problems, that is problems for perfectly plastic structures. The perturbation solutions were worked out by means of the Rayleigh–Schrödinger method, and sufficiently good accuracy was obtained for a quite wide range of the small parameter, considering only first perturbations. A rather weak dependence of the shape functions on the small parameter was observed.

We shall make use of these important results to develop in the present paper an alternative perturbation approach for solving a wider class of problems for dynamically-loaded viscoplastic structures. The proposed procedure consists in assuming a constant shape function for all values of a small parameter and then reducing by means of Galerkin's weighted residual method the initial boundary-value problem to an initial-value problem for a nonlinear ordinary differential equation for the amplitude. Such a procedure admits a nonhomogeneous form of governing equations and does allow to apply the inhomogeneous rigid viscoplastic constitutive relations due to Perzyna.

This approach is used to solve the problem of the impulsively-loaded fully clamped beam which was previously considered in [6]. Assuming a sine shape function, a perturbation solution to the initial problem for the amplitude is developed by means of the general averaging technique frequently used in problems of weakly nonlinear oscillations. A relatively simple approximate solution for large central deflections is obtained and compared to other existing solutions and available experimental data.

2. Formulation of the problem

Consider a fully clamped beam of thickness h, length l and mass per unit lenght m, see Fig. 1, made of rigid-viscoplastic material obeying in a one-dimensional state of strain and stress the Cowper-Symonds power type constitutive equation

$$\sigma = \sigma_0\left[1+\left(\frac{\dot{\varepsilon}}{\gamma}\right)^{\frac{1}{n}}\right], \quad n = 1, 2, 3, \ldots. \tag{2.1}$$

Here σ, $\dot{\varepsilon}$ denote respectively stress and strain-rate, σ_0 is the static yield stress in tension,

γ and n are material constants. In particular, $\gamma = 40\ \mathrm{s}^{-1}$, $n = 5$ for mild steel and $\gamma = 120\ \mathrm{s}^{-1}$, $n = 9$ for titanium.

It is seen that by letting $\gamma \to \infty$ with σ and $\dot{\varepsilon}$ kept fixed, Eq. (2.1) is reduced to one describing a rigid plastic material with static yield stress σ_0, see Fig. 2. This limit transition will be used for developing an approximate solution to the title problem.

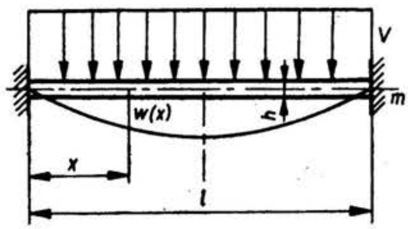

FIG. 1.

Let the beam be subjected to an ideal impulse uniformly distributed over the length so that at an initial time $t = 0$ a constant velocity V is prescribed over the beam. Our task is to analyse the subsequent motion of the beam.

Following the approach taken in [6] we will disregard entirely the flexural resistance of the beam so that the only component of the generalized strain rate vector is the exten-

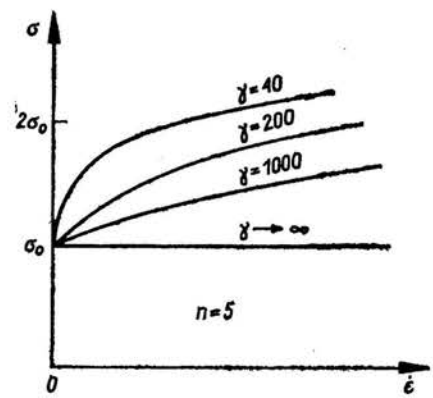

FIG. 2.

sion rate $\partial_t \lambda$, but we will retain nonlinear geometrical effects. Under these assumptions the von Kármán equations of moderately large deflections of beams, involving strain-displacement relations and equations of motion, are reduced respectively to

(2.2) $$\partial_t \lambda = \partial_x w \partial_{xt} w,$$

(2.3) $$\partial_x (N \partial_x w) = m \partial_{tt} w,$$

where w is a vertical deflection, N denotes an axial force and x,t stand respectively for space coordinate and time.

By integrating Eq. (2.1) over the beam thickness, we obtain the constitutive relation

$$N = N_0\left[1+\left(\frac{\partial_t \lambda}{\gamma}\right)^{\frac{1}{n}}\right], \tag{2.4}$$

in which

$$N_0 = \sigma_0 h \tag{2.5}$$

is the reference plastic axial force.

Assuming that the beam is initially undeformed, the initial conditions take the form

$$w(x, 0) = 0, \tag{2.6}$$

$$\partial_t w(x, 0) = V \quad \text{for} \quad 0 < x < l. \tag{2.7}$$

For clamped ends beam the boundary conditions are

$$w(0, t) = 0, \tag{2.8}$$

$$w(l, t) = 0 \quad \text{for} \quad t \geqslant 0. \tag{2.9}$$

Let us introduce the following nondimensional independent variables

$$x^* = \frac{x}{l}, \quad t^* = \sqrt{\frac{N_0}{m}}\,\frac{t}{l}, \tag{2.10}$$

and the nondimensional dependent ones

$$w^* = \frac{w}{h}, \quad N^* = \frac{N}{N_0}, \quad \lambda^* = \frac{l^2}{h^2}\lambda. \tag{2.11}$$

Then, Eqs. (2.2)–(2.4) together with the auxiliary conditions (2.6)–(2.9) can be rewritten in the form

$$\partial_{t^*} \lambda^* = \partial_{x^*} w^* \partial_{x^* t^*} w^*, \tag{2.12}$$

$$\partial_{x^*}(N^* \partial_{x^*} w^*) = \partial_{t^* t^*} w^*, \tag{2.13}$$

$$N^* = 1+\beta_n(\partial_{t^*} \lambda^*)^{\frac{1}{n}}, \tag{2.14}$$

$$w^*(x^*, 0) = 0, \tag{2.15}$$

$$\partial_{t^*} w^*(x^*, 0) = \sqrt{K} \quad \text{for} \quad 0 < x < 1; \tag{2.16}$$

$$w^*(0, t^*) = 0, \tag{2.17}$$

$$w^*(1, t^*) = 0 \quad \text{for} \quad t^* \geqslant 0, \tag{2.18}$$

where the nondimensional and non-negative parameter β_n is defined by

$$\beta_n = \left(\frac{h^2}{\gamma l^3}\sqrt{\frac{N_0}{m}}\right)^{\frac{1}{n}}, \tag{2.19}$$

and

$$K = \frac{mV^2 l^2}{N_0 h^2} \tag{2.20}$$

is a nondimensional kinetic energy of the beam at the initial time $t^* = 0$. From now on the "star" will be omitted for the sake of simplicity.

3. Stationary mode solution

Eliminating $\partial_t\lambda$ and N between Eqs. (2.12)–(2.14) we obtain the following equation:

$$\partial_{xx}w-\partial_{tt}w+\beta_n\partial_x[(\partial_x w\partial_{xt}w)^{\frac{1}{n}}\partial_x w]=0, \tag{3.1}$$

which, together with the conditions (2.15)–(2.18) furnish an initial boundary-value problem for the beam deflection $w(x, t)$. It can be easily checked that Eq. (3.1) does not admit for any x and t stationary mode solutions, that is solutions in a separable form. Consequently, contrary to the technique used in [6], the set of equations (3.1), (2.15)–(2.18) cannot be reduced to a nonlinear eigenvalue problem for an ordinary differential equation describing the mode of deflection. We shall, however, seek an approximate stationary mode solution, denoted in what follows by w_a, making use of the Galerkin's weighted residual method (see for example [1]).

We observe that in the sine Fourier series representation of the constant initial velocity distribution of the beam over the interval $0 < x < 1$, the greatest Fourier coefficient corresponds to the term $\sin\pi x$. Hence we take this function as the "trial function" and following the Galerkin's technique we assume the sought solution to be of the form

$$\begin{aligned} w_a &= A(t)\sqrt{2}\sin\pi x,\\ \frac{\partial w_a}{\partial t} &= \dot{A}(t)\sqrt{2}\sin\pi x, \end{aligned} \tag{3.2}$$

where $A(t)$ is an amplitude function to be found.

The solution $(3.2)_1$ satisfies the boundary conditions (2.17) and (2.18) and the requirement of the symmetry with respect to the beam center. It is worth noting that the orthonormal mode shape $\sqrt{2}\sin\pi x$ chosen here does not differ much from the one obtained in the paper [6] by considering a nonlinear viscous constitutive equation.

In view of Eq. $(3.2)_{1,2}$, the differential equation residual R_1 is

$$R_1(w_a) = -(\ddot{A}+\pi^2 A)\sqrt{2}\sin\pi x-\beta_n 2^{\frac{2+n}{2n}}\frac{2+n}{n}\pi^{\frac{2(n+1)}{n}}A^{\frac{1+n}{n}}\dot{A}^{\frac{1}{n}}\sin\pi x\cos^{\frac{2}{n}}\pi x, \tag{3.3}$$

and the initial condition residuals R_2 and R_3, respectively, are

$$R_2(w_a) = A(0)\sqrt{2}\sin\pi x, \tag{3.4}$$

$$R_3(w_a) = \sqrt{K}-\dot{A}(0)\sqrt{2}\sin\pi x. \tag{3.5}$$

Now, taking $\sqrt{2}\sin\pi x$ as the weighting function and setting the weighted integrals $\int_0^1 R_j\sqrt{2}\sin\pi x\,dx$ $(j = 1, 2, 3)$ equal to zero, we arrive at the nonlinear initial value problem

$$\ddot{A}+\pi^2 A = -\beta_n 2^{\frac{1}{n}}\pi^{\frac{3n+4}{2n}}n\frac{n+2}{n+1}\frac{\Gamma\left(\frac{n+2}{2n}\right)}{\Gamma\left(\frac{1}{n}\right)}A^{\frac{n+1}{n}}\dot{A}^{\frac{1}{n}}, \tag{3.6}$$

$$A(0) = 0, \tag{3.7}$$

$$\dot{A}(0) = \frac{2\sqrt{2}}{\pi}\sqrt{K}, \tag{3.8}$$

for the sought amplitude function $A(t)$, where $\Gamma(\cdot)$ is the Gamma function. Note that because of the uniform distribution of mass, the condition $\int\limits_0^1 R_3(w_a)\sqrt{2}\sin\pi x\,dx = 0$ represents the balance of the weighted initial momenta in the exact and approximate solutions.

4. Perturbation solution for amplitude

In most practical applications to metals the value of the constant in the constitutive equation (3.1) does not exceed 9. Hence the numerical coefficient on the right hand side of Eq. (3.6) can be treated as not greater than 14.82 β_n. Moreover, in the limit case when $\gamma \to \infty$, that is when the material becomes rigid-plastic, the parameter β_n, given by Eq. (2.19), tends to zero and, consequently, the right hand side of Eq. (3.6) disappears reducing Eqs. (3.6)–(3.8) to the linear problem

$$\ddot{A}_0 + \pi^2 A_0 = 0, \tag{4.1}$$

$$A_0(0) = 0, \tag{4.2}$$

$$\dot{A}_0(0) = \frac{2\sqrt{2}}{\pi}\sqrt{K}, \tag{4.3}$$

whose solution is

$$A_0 = \frac{2\sqrt{2}}{\pi^2}\sqrt{K}\sin\pi t, \tag{4.4}$$

$$\dot{A}_0 = \frac{2\sqrt{2}}{\pi}\sqrt{K}\cos\pi t. \tag{4.5}$$

Since the motion of the plastic beam has a purely dissipative character, the phase πt in the solution (4.4) and (4.5) should be limited to the segment $\left[0, \frac{\pi}{2}\right]$ which corresponds to the time interval $\left[0, \frac{1}{2}\right]$. At time $t = \frac{1}{2}$ the plastic beam comes to rest.

For appropriately small values of $\beta_n \neq 0$ Eqs. (3.6)–(3.8) of the rigid viscoplastic beam can be treated as the regularly perturbed problem (4.1)–(4.3) for a rigid plastic one. In what follows we shall make use of this property and find a perturbation solution to the problem (3.6)–(3.8) applying the approach developed by Krylov and Bogoliubov, see [2] or [3].

Thus, in accordance with this approach, for small β we assume the sought solution to have the form

$$A = \frac{2\sqrt{2}}{\pi^2}\sqrt{K}\,a(t)\sin\phi(t), \tag{4.6}$$

$$\dot{A} = \frac{2\sqrt{2}}{\pi}\sqrt{K}\,a(t)\cos\phi(t), \tag{4.7}$$

with an unknown time-varying amplitude $a(t)$ and a not known phase $\phi(t)$, such that

$$a(0) = 1, \tag{4.8}$$

$$\phi(0) = 0, \tag{4.9}$$

to fulfill the initial conditions (3.7) and (3.8).

We also postulate that, as in the plastic case, the phase $\phi(t)$ changes from 0 to $\frac{\pi}{2}$.

Treating the relations (4.6) and (4.7) as a transformation from the variables A and $\dot{A}$ into the new variables a and ϕ, by a standard procedure described in detail in Ref. [2] we obtain the system of two equations of the first order:

$$\frac{da}{dt} = -\beta_n \alpha a^{\frac{n+2}{n}} \sin^{\frac{n+1}{n}}\phi \cos^{\frac{n+1}{n}}\phi, \tag{4.10}$$

$$\frac{d\phi}{dt} = \pi + \beta_n \alpha a^{\frac{2}{n}} \sin^{\frac{2n+1}{n}}\phi \cos^{\frac{1}{n}}\phi, \tag{4.11}$$

where

$$\alpha = 16^{\frac{1}{n}} \pi^{\frac{n-2}{2n}} n \frac{n+2}{n+1} \frac{\Gamma\left(\frac{n+2}{2n}\right)}{\Gamma\left(\frac{1}{n}\right)} K^{\frac{1}{n}}, \tag{4.12}$$

which, together with the conditions (4.8) and (4.9), furnish a nonlinear initial-value problem for the amplitude a and the phase ϕ. This problem will now be solved by a general averaging method, see for example [3].

With this purpose let us introduce a new time variable

$$\bar{t} = \alpha t, \tag{4.13}$$

and expand the terms $\sin^{\frac{n+1}{n}}\phi \cos^{\frac{n+1}{n}}\phi$ and $\sin^{\frac{2n+1}{n}}\phi \cos^{\frac{1}{n}}\phi$ into the cosine trigonometric series in the interval $0 < \phi < \frac{\pi}{2}$. Then, Eqs. (4.10) and (4.11) take the form

$$\frac{da}{d\bar{t}} = -\beta_n a^{\frac{n+2}{n}} \left(\frac{c_0}{2} + \sum_{m=1}^{\infty} c_m \cos 2m\phi\right), \tag{4.14}$$

$$\frac{d\phi}{d\bar{t}} = \frac{\pi}{\alpha} + \beta_n a^{\frac{2}{n}} \left(\frac{d_0}{2} + \sum_{m=1}^{\infty} d_m \cos 2m\phi\right), \tag{4.15}$$

in which the Fourier coefficients are

$$c_k = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin^{\frac{n+1}{n}}\phi \cos^{\frac{n+1}{n}}\phi \cos 2k\phi \, d\phi, \tag{4.16}$$

$$d_k = \frac{4}{\pi}\int\limits_0^{\frac{\pi}{2}} \sin^{\frac{2n+1}{n}}\phi \cos^{\frac{1}{n}}\phi \cos 2k\phi \, d\phi, \quad k = 0, 1, 2, 3, \ldots. \tag{4.17}$$

To integrate approximately the system of Eqs. (4.14) and (4.15) for a small $\beta_n \neq 0$, we introduce a near identity transformation

$$a = \bar{a} + \beta_n a_1(\bar{a}, \bar{\phi}) + \beta_n^2 a_2(\bar{a}, \bar{\phi}) + \ldots, \tag{4.18}$$

$$\phi = \bar{\phi} + \beta_n \phi_1(\bar{a}, \bar{\phi}) + \beta_n^2 \phi_2(\bar{a}, \phi) + \ldots, \tag{4.19}$$

from the variables a, ϕ to other variables, $\bar{a}$, $\bar{\phi}$ such that the transform of the system of Eqs. (4.14), (4.15) is of the form

$$\frac{d\bar{a}}{d\bar{t}} = \beta_n A_1(\bar{a}) + \beta_n^2 A_2(\bar{a}) + \ldots, \tag{4.20}$$

$$\frac{d\bar{\phi}}{d\bar{t}} = \frac{\pi}{\alpha} + \beta_n B_1(\bar{a}) + \beta_n^2 B_2(\bar{a}) + \ldots, \tag{4.21}$$

with the functions $A_1, A_2, \ldots, B_1, B_2, \ldots$ dependent only on $\bar{a}$.

Let us now substitute (4.18)–(4.21) into Eqs. (4.14) and (4.15) and expand the result in the powers of the small parameter β_n. Then, equating the coefficients of β_n we arrive at the set of two equations

$$\frac{\pi}{\alpha}\frac{\partial a_1}{\partial \bar{\phi}} + A_1 = -\bar{a}^{\frac{n+2}{n}}\left(\frac{c_0}{2} + \sum_{m=1}^{\infty} c_m \cos 2m\bar{\phi}\right), \tag{4.22}$$

$$\frac{\pi}{\alpha}\frac{\partial \phi_1}{\partial \bar{\phi}} + B_1 = \bar{a}^{\frac{2}{n}}\left(\frac{d_0}{2} + \sum_{m=1}^{\infty} d_m \cos 2m\bar{\phi}\right), \tag{4.23}$$

for the functions a_1, ϕ_1.

According to the method of averaging we choose A_1 and B_1 to be equal to the long-period terms on the right hand side of Eqs. (4.22) and (4.23). That is we put

$$A_1 = -\frac{c_0}{2}\bar{a}^{\frac{n+2}{n}}, \quad B_1 = \frac{d_0}{2}\bar{a}^{\frac{2}{n}}, \tag{4.24}$$

and as a result we get the equations

$$\frac{\partial a_1}{\partial \bar{\phi}} = -\frac{\alpha}{\pi}\bar{a}^{\frac{n+1}{n}}\left(\sum_{m=1}^{\infty} c_m \cos 2m\bar{\phi}\right), \tag{4.25}$$

$$\frac{\partial \phi_1}{\partial \bar{\phi}} = \frac{\alpha}{\pi}\bar{a}^{\frac{2}{n}}\left(\sum_{m=1}^{\infty} d_m \cos 2m\bar{\phi}\right), \tag{4.26}$$

which, by integrating, give

$$a_1 = -\frac{\alpha}{2\pi}\bar{a}^{\frac{n+2}{n}}\left(\sum_{m=1}^{\infty} \frac{c_m}{m} \sin 2m\bar{\phi}\right), \tag{4.27}$$

$$\phi_1 = \frac{\alpha}{2\pi}\bar{a}^{\frac{2}{n}}\left(\sum_{m=1}^{\infty}\frac{d_m}{m}\sin 2m\bar{\phi}\right). \tag{4.28}$$

Thus, to the second order, the perturbation solution of Eqs. (4.14) and (4.15) is

$$a = \bar{a} - \beta_n\frac{\alpha}{2\pi}\bar{a}^{\frac{n+2}{n}}\left(\sum_{m=1}^{\infty}\frac{c_m}{m}\sin 2m\bar{\phi}\right) + 0(\beta_n^2), \tag{4.29}$$

$$\phi = \bar{\phi} + \beta_n\frac{\alpha}{2\pi}\bar{a}^{\frac{2}{n}}\left(\sum_{m=1}^{\infty}\frac{d_m}{m}\sin 2m\bar{\phi}\right) + 0(\beta_n^2), \tag{4.30}$$

where the functions $\bar{a}$ and $\bar{\phi}$ are defined by the system of the differential equations

$$\frac{d\bar{a}}{d\bar{t}} = -\beta_n\frac{c_0}{2}\bar{a}^{\frac{n+2}{n}} + 0(\beta_n^2), \tag{4.31}$$

$$\frac{d\bar{\phi}}{d\bar{t}} = \frac{\pi}{\alpha} + \beta_n\frac{d_0}{2}\bar{a}^{\frac{2}{n}} + 0(\beta_n^2). \tag{4.32}$$

If on the right hand sides of Eqs. (4.29)–(4.32) we retain only the terms up to the order $0(\beta_n)$, then, bearing in mind the transformation (4.13), we have the following approximate solution for the functions $\bar{a}$ and $\bar{\phi}$:

$$\bar{a} \approx \left(D + \beta_n\frac{c_0\alpha}{n}t\right)^{-\frac{n}{2}}, \tag{4.33}$$

$$\bar{\phi} \approx \pi t + \frac{n}{2}\frac{d_0}{c_0}\ln\left(1 + \beta_n\frac{c_0\alpha}{n}t\right) + E, \tag{4.34}$$

where D and E stand for constants of integration. Next, substituting Eqs. (4.33) and (4.34) into the simplified relations (4.29) and (4.30) we conclude that in order to satisfy the initial conditions (4.8) and (4.9) there can be $D = 1$, $E = 0$ and the approximate expressions for the amplitude a and the phase ϕ are

$$a \approx \left(1 + \beta_n\frac{c_0\alpha}{n}t\right)^{-\frac{n}{2}} - \beta_n\frac{\alpha}{2\pi}\left(1 + \beta_n\frac{c_0\alpha}{n}t\right)^{-\frac{n+2}{2}}\left\{\sum_{m=1}^{\infty}\frac{c_m}{m}\sin 2m\left[\pi t + \frac{n}{2}\frac{d_0}{c_0}\ln\left(1 + \beta_n\frac{c_0\alpha}{n}t\right)\right]\right\}, \tag{4.35}$$

$$\phi \approx \pi t + \frac{n}{2}\frac{d_0}{c_0}\ln\left(1 + \beta_n\frac{c_0\alpha}{n}t\right) + \beta_n\frac{\alpha}{2\pi}\left(1 + \beta_n\frac{c_0\alpha}{n}t\right)^{-1}\left\{\sum_{m=1}^{\infty}\frac{d_m}{m}\sin 2m\left[\pi t + \frac{n}{2}\frac{d_0}{c_0}\ln\left(1 + \beta_n\frac{c_0\alpha}{n}t\right)\right]\right\}. \tag{4.36}$$

5. Approximate mode solution

On behalf of applications we simplify further the above relations by retaining only the first non-zero terms in the sums on the right hand sides of Eqs. (4.35) and (4.36). Let a_s and ϕ_s denote the simplified amplitude and phase. Then, in view of the expressions (4.16) and (4.17) we have

$$
\begin{gathered}
c_0 = \frac{1}{2\pi(n+1)} \frac{\Gamma^2\left(\dfrac{1}{2n}\right)}{\Gamma\left(\dfrac{1}{n}\right)}, \quad c_1 = 0, \quad c_2 = -\frac{n+1}{3n+1} c_0, \\
d_0 = \frac{n}{\pi} \frac{\Gamma^2\left(\dfrac{n+1}{2n}\right)}{\Gamma\left(\dfrac{1}{n}\right)}, \quad d_1 = -\frac{n}{2n+1} d_0,
\end{gathered} \tag{5.1}
$$

and the simplified formulae for the amplitude and phase become

$$
\begin{aligned}
a_s = \left(1+\beta_n \frac{c_0 \alpha}{n} t\right)^{-\frac{n}{2}} + \beta_n \frac{c_0 \alpha(n+1)}{4\pi(3n+1)} \left(1+\beta_n \frac{c_0\alpha}{n} t\right)^{-\frac{n+2}{n}} \sin 4\Bigg[\pi t \\
+ \frac{n}{2} \frac{d_0}{c_0} \ln\left(1+\beta_n \frac{c_0 \alpha}{n} t\right)\Bigg],
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
\phi_s \approx \pi t + \frac{n}{2} \frac{d_0}{c_0} \ln\left(1+\beta_n \frac{c_0\alpha}{n} t\right) - \beta_n \frac{d_0 \alpha n}{2\pi(2n+1)} \left(1+\beta_n \frac{c_0 \alpha}{n} t\right)^{-1} \sin 2\Bigg[\pi t \\
+ \frac{n}{2} \frac{d_0}{c_0} \ln\left(1+\beta_n \frac{c_0\alpha}{n} t\right)\Bigg].
\end{aligned} \tag{5.3}
$$

Next, with the help of these relations and Eqs. (3.2), (4.6) and (4.7) as well, we get the approximate expressions

$$
w_a = \frac{4\sqrt{K}}{\pi^2} a_s(t) \sin\phi_s(t) \sin\pi x, \tag{5.4}
$$

$$
\frac{\partial w_a}{\partial t} = \frac{4\sqrt{K}}{\pi} a_s(t) \cos\phi_s(t) \sin \pi x, \tag{5.5}
$$

respectively, for the displacement and velocity of the beam. By setting $x = 1/2$ in Eqs. (5.4) and (5.5) we obtain the formulae for the displacement $w_a\left(\frac{1}{2}\right)$ and velocity $\dfrac{\partial w_a\left(\frac{1}{2}\right)}{\partial t}$ in the mid-span

$$
w_a\left(\frac{1}{2}\right) = \frac{4\sqrt{K}}{\pi^2} a_s(t) \sin\phi_s(t), \tag{5.6}
$$

$$
\frac{\partial w_a\left(\dfrac{1}{2}\right)}{\partial t} = \frac{4\sqrt{K}}{\pi} a_s(t) \cos\phi_s(t). \tag{5.7}
$$

Finally, denoting by t_f the response time, that is the time at which the phase ϕ_s reaches the value $\frac{\pi}{2}$, and by a_f the corresponding value of a_s, we obtain the following expression for the permanent deflection w_{af} of the mid-span:

$$w_{af}\left(\frac{1}{2}\right) = \frac{4\sqrt{K}}{\pi^2} a_f. \tag{5.8}$$

6. Ilustrative example and conclusions

We shall correlate now the results predicted by the present "string" solution with that obtained in [6] and also with the experiments on impulsively-loaded mild steel clamped beams reported in [4]. To this end we shall consider the same numerical values for geometrical and mechanical constants as in [6], namely

$$\begin{aligned} h &= 0{,}1 \text{ in}, & \sigma_0 &= 30.5\ 10^3 \text{ lb in}^{-2},\\ l &= 5 \text{ in}, & n &= 5,\\ \varrho &= 0.732\ 10^{-3} \text{ lb s}^2 \text{ in}^{-4}, & \gamma &= 40 \text{ s}^{-1}. \end{aligned}$$

The corresponding value of the small parameter is $\beta_5 = 0.41897$, the quantity $\alpha = 4.04843\ \mathrm{K}^{0.2}$ and the first Fourier coefficients are equal to $c_0 = 0.52294$, $d_0 = 0.76884$.

The approximate formulae (5.6) and (5.7) for the mid-span displacement and velocity take the form

$$w_a\left(\frac{1}{2}\right) = 0.40528\sqrt{K}a_s(t)\sin\phi_s(t), \tag{6.1}$$

$$\frac{\partial w_a\left(\frac{1}{2}\right)}{\partial t} = 1.27324\sqrt{K}a_s(t)\cos\phi_s(t), \tag{6.2}$$

where, according to Eq. (5.2) and (5.3), the quantities $a_s(t)$, $\phi_s(t)$ are expressed as follows:

$$a_s = (1+0.17740\ \mathrm{K}^{0.2}t)^{-2.5}+0.02647\ \mathrm{K}^{0.2}(1+0.17740\ \mathrm{K}^{0.2}t)^{-3.5}\sin 4[\pi t + 3.67557\ln(1+0.17740\ \mathrm{K}^{0.2}t)], \tag{6.3}$$

$$\phi_s = \pi t+3.67557\ln(1+0.17740\ \mathrm{K}^{0.2}t)-0.09434\ \mathrm{K}^{0.2}(1+0.17740\ \mathrm{K}^{0.2}t)^{-1}\sin 2[\pi t + 3.67557\ln(1+0.17740\ \mathrm{K}^{0.2}t)]. \tag{6.4}$$

Plots of the functions $w_a\left(\frac{1}{2}\right)$ and $\frac{\partial w_a\left(\frac{1}{2}\right)}{\partial t}$ versus time for the initial energies $\mathrm{K} = 75, 200, 400, 700$ are represented, respectively, in Figs. 3 and 4. The dashed lines in these figures denote the approximate perturbation solution whereas the solid lines refer to the exact solution obtained directly from the numerical solution of the initial-value problem (3.6)–(3.8)

$$\ddot{A}+\pi^2 A = -6.98702A^{1.2}\dot{A}^{0.2}, \quad A(0) = 0, \quad \dot{A}(0) = 0.90032\ \mathrm{K}^{0.5}. \tag{6.5}$$

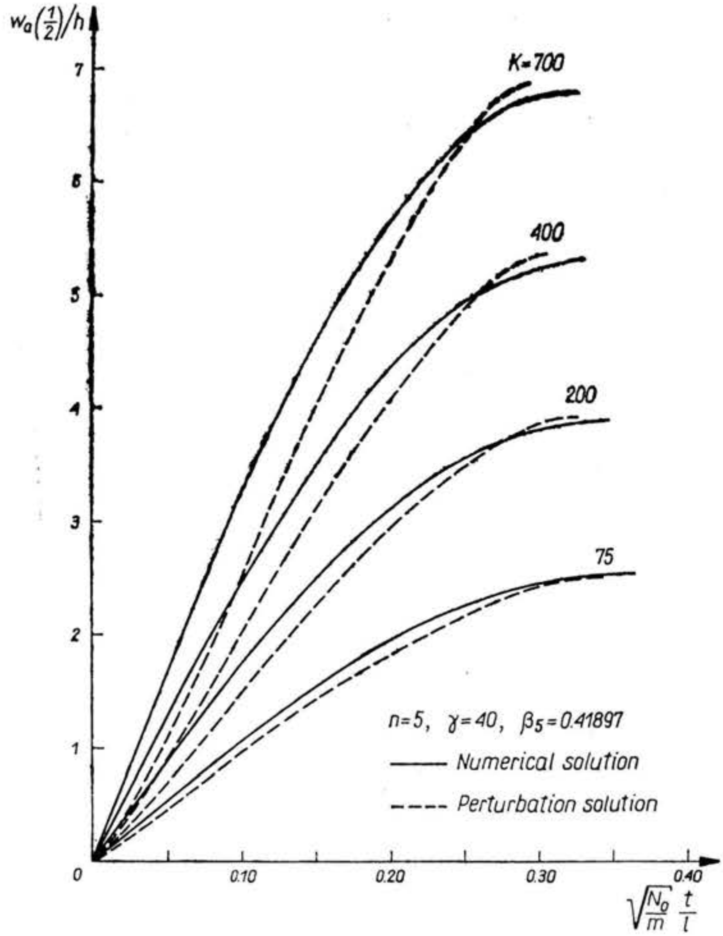

FIG. 3.

It is seen from these figures that the results of both solutions are generally in satisfactory agreement. As the value of the initial kinetic energy decreases, this agreement become more pronounced. The response time obtained from the approximate perturbation solution is always smaller than that resulting from the numerical solution. This is illustrated more clearly in Fig. 5. However, the most important conclusion is that in the considered range of the initial kinetic energy the final mid-span deflections $w_{af}\left(\frac{1}{2}\right)$, $w_{afn}\left(\frac{1}{2}\right)$(1) predicted by both solutions are practically the same, see Fig. 6. More exactly, the percentage relative error $100\%\left|w_{af}\left(\frac{1}{2}\right)-w_{afn}\left(\frac{1}{2}\right)\right|\Big/ w_{afn}\left(\frac{1}{2}\right)$ does not exceed 1.7%.

(1) $w_{afn}\left(\frac{1}{2}\right)$ denotes the final mid-span deflection obtained numerically.

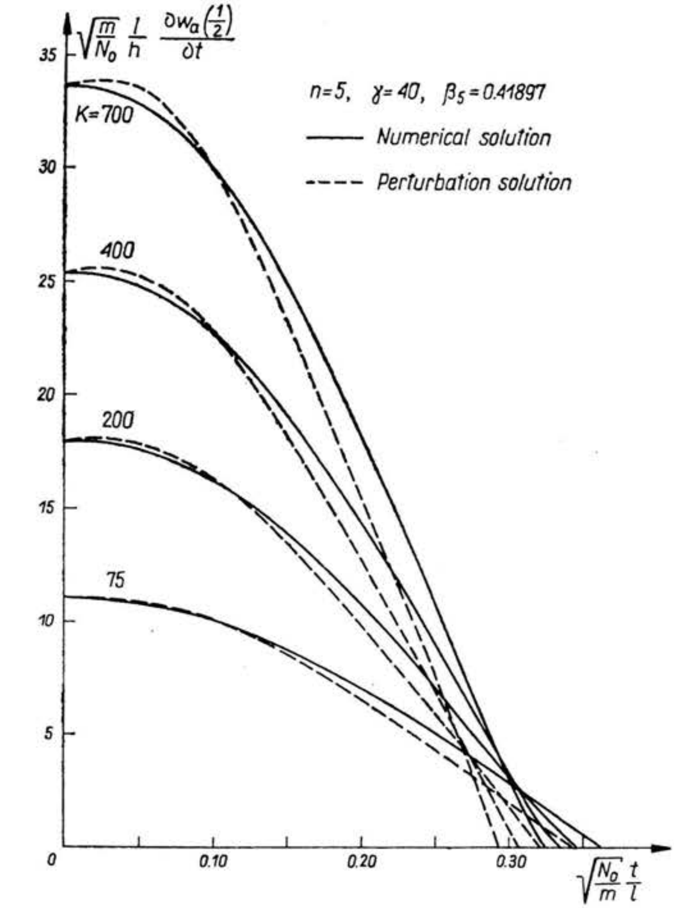
$\sqrt{\frac{m}{N_0}}\,\frac{l}{h}\,\frac{\partial w_a\left(\frac{1}{2}\right)}{\partial t}$
$n=5,\ \gamma=40,\ \beta_5=0.41897$
Numerical solution
Perturbation solution
K=700
400
200
75
35
30
25
20
15
10
5
0
0.10
0.20
0.30
$\sqrt{\frac{N_0}{m}}\,\frac{t}{l}$

FIG. 4.

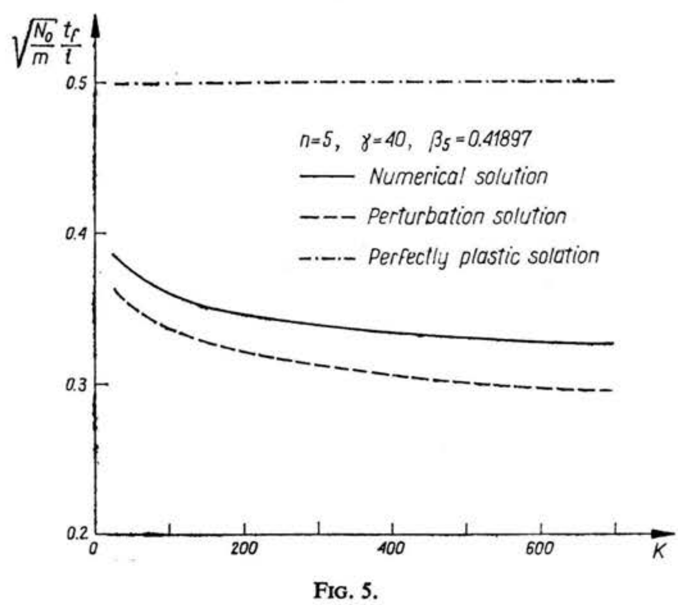
$\sqrt{\frac{N_0}{m}}\,\frac{t_f}{l}$
0.5
0.4
0.3
0.2
$n=5,\ \gamma=40,\ \beta_5=0.41897$
Numerical solution
Perturbation solution
Perfectly plastic solation
0
200
400
600
K

FIG. 5.

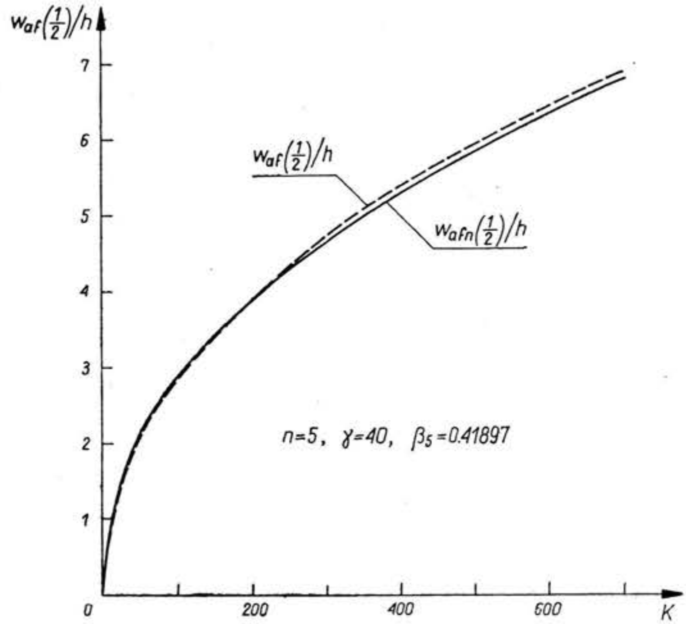

FIG. 6.

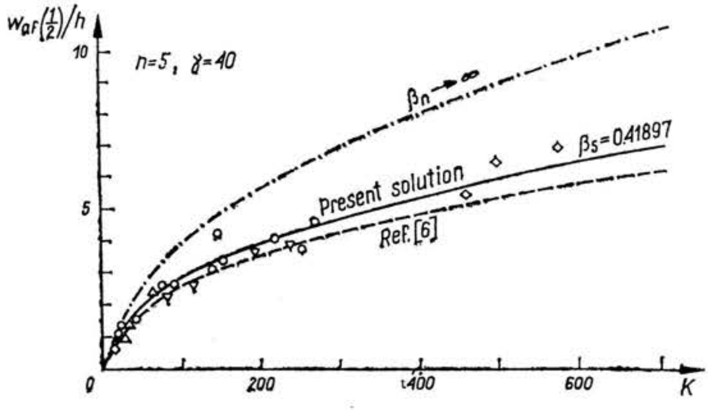

FIG. 7.

Fig. 7 reveals fairly good agreement between the experimental results reported by SYMONDS and JONES in [4] and the final mid-span deflections obtained from Eqs. (6.1), (6.3) and (6.4). These permanent deflections seem to fit better the trend of experimental points than the final deflections predicted by the solution obtained in the paper [6].

The advantage of the proposed approach is that it can be easily extended to more complicated two-dimensional viscoplastic structures subjected to impulsive and general pulse loading.

Reference

1. W. F. AMES, *Nonlinear partial differential equations in engineering*, Mathematics in science and engineering, **18**, Academic Press, New York and London 1965.
2. N. N. BOGULIUBOW and Y. A. MITROPOLSKI, *Asymptotic methods in the theory of nonlinear oscillations*, Gordon and Breach, New York 1961.
3. A. NAYFEH, *Perturbation methods*, Pure and Appl. Math., J. Wiley and Sons, New York—London—Sydney—Toronto, 1973.
4. P. S. SYMONDS and N. JONES, *Impulsive loading of fully clamped beams with finite plastic deformations and strain-rate sensitivity*, Int. J. Mech. Sci., **14**, 49–69, 1972.
5. W. WOJNO, *Perturbation solution for rigid-viscoplastic spherical container*, Arch. Mech., **31**, 407–422, 1979.
6. W. WOJNO and T. WIERZBICKI, *On perturbation solutions for impulsively loaded viscoplastic structures*, ZAMP, **30**, 41–55, 1979.
7. W. WOJNO and T. WIERZBICKI, *Perturbation solution for impulsively loaded viscoplastic plates*, International Journal of Non-Linear Mechanics, **15**, 211–223, 1980.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received March 14, 1980.