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# General closed-form solutions of the equations of one-dimensional nonstationary isentropic motions of isotropic elastic media subject to finite deformations

E. WŁODARCZYK (WARSZAWA)

THE PAPER presents a relatively wide class of functions determining the internal energy of an isotropic elastic medium subject to one-dimensional large deformations. The functions make it possible to construct a closed-form, general solution of the equations of one-dimensional motion of a geometrically nonlinear isotropic elastic medium with constant entropy. By means of a special change of independent variables, the problem is reduced to a linear partial second-order differential equation with variable coefficients. Superposition of certain limitations upon the coefficients multiplying the first derivatives enables us to reduce the equation to the telegraph or Euler-Darboux equations. It is known [3, 4] that these equations have closed-form general solutions. The arbitrary functions appearing in the general solution are determined by the boundary and initiąl conditions of the problem considered.

Wyznaczono dość obszerną klasę funkcji dla określenia energii wewnętrznej izotropowego ośrodka sprężystego będącego w jednoosiowym stanie dużych odkształceń. Funkcje te pozwalają skonstruować zamknięte, ogólne rozwiązanie równań jednowymiarowego płaskiego ruchu, geometrycznie nieliniowego, izotropowego ośrodka sprężystego ze stałą entropią. Dokonując specyficznej zmiany zmiennych niezależnych, sprowadzono problem do liniowego równania cząstkowego drugiego rzędu o zmiennych współczynnikach. Przez nałożenie odpowiednich ograniczeń na współczynniki stojące przy pierwszych pochodnych, równanie to można sprowadzić do równania telegrafistów lub równania Eulera-Darboux. Jak wiadomo [3 i 4], równania te mają zamknięte ogólne rozwiązania. Dowolne funkcje występujące w rozwiązaniu ogólnym determinowane są przez warunki graniczne konkretnego problemu.

Найден достаточно широкий класс функций для определения внутренней энергии изотропной упругой среды в состоянии одноосных больших деформаций. Эти функции позволяют построить замкнутое, общее решение уравнений одномерного геометрически нелинейного плоского движения изотронной среды с постоянной энтропией. Путем специальной замены переменных задача сводится к линейному уравнению второго порядка в частных производных с переменными коэффициентами. Налагая соответсвующие ограничения на коэффициенты при первых производных это уравнение можно свести к уравнению телеграфистов либо к уровнению Эйлера-Дарбу. Как известно [3, 4] эти уровнения обладают замкнутыми общими решениями. Произвольные функции выступающие в общем решении определяются через граничные условия конкретной задачи.

## 1. Introduction

CERTAIN sets of quasi-linear differential hyperbolic equations may be reduced, by means of suitable changes of variables, to the linear second-order equation with variable coefficients [1, 2]. Such a change of variables proves to be particularly expedient in the case when the corresponding linear equation possesses an accurate, general solution, like in the cases of the telegraph or the Euler-Darboux equations [3, 4]. They have been utilized by the author in constructing closed-form solutions of the problems of propagation of one-dimensional stress waves in nonelastic media subject to small deformations [5–11].

In the present paper an analogous method will be used to construct the exact, general solution of equations governing the one-dimensional motion of a geometrically nonlinear, isotropic elastic medium with constant entropy. The corresponding class of functions will be found to determine the internal energy of the isotropic elastic medium subject to large one-dimensional strains. The functions make it possible to reduce the entire problem to a linear partial differential equation of second order with variable coefficients, which may be identified with the telegraph or Euler-Darboux equations possessing general closed-form solutions.

## 2. Equations of a one-dimensional plane isentropic motion of an isotropic elastic medium with finite deformations

Plane one-dimensional isentropic motion of a compressible isotropic elastic medium in the state of one-dimensional large deformations is governed by the equations [12, 13]

$$\frac{\partial v}{\partial t} = a^2(m)\frac{\partial m}{\partial x}, \qquad \frac{\partial m}{\partial t} = \frac{\partial v}{\partial x}. \tag{2.1}$$

Here

$$v = \frac{\partial u}{\partial t}, \qquad m = \frac{\partial u}{\partial x},$$

$u$ — displacement, $v$ — velocity of motion, $m$ — deformation of the medium; $x$ and $t$ are Lagrange coordinates, and $a(m)$ — velocity of propagation of the disturbances. The latter magnitude may be expressed by the following formula:

$$a(m) = \left[\frac{1}{\varrho_0}\frac{d^2 W(m)}{dm^2}\right]^{\frac{1}{2}}, \tag{2.2}$$

where $W(m)$ is the internal energy of the dynamic deformation of the medium, and $\varrho_0$ denotes its initial density. In the case of an elastic isotropic medium subject to isentropic deformations, $W(m)$ is usually represented in the form of a power series [13],

$$W(m) = \frac{1}{2}E_0 m^2 + \frac{1}{3}E_1 m^3 + 0(m^4). \tag{2.3}$$

Here

$$E_0 = \lambda + 2\mu, \qquad E_1 = 3\left(\frac{\lambda}{2} + \mu + \gamma + \vartheta + \nu\right),$$

$\lambda$ and $\mu$ are Lamé's constants, and $\gamma$, $\vartheta$, $\nu$ denote the third-order elasticity moduli.

The set of Eqs. (2.1) may be replaced by an equivalent system of two ordinary equations

$$dv = \pm a(m)dm \tag{2.4}$$

provided

$$dx = \pm a(m)dt. \tag{2.5}$$

It is seen that Eqs. (2.4) may be integrated along the characteristics (2.5). Then

$$v-F(m) = \text{const} \quad \text{if} \quad dx = a(m)dt \tag{2.6}$$

and

$$v+F(m) = \text{const} \quad \text{if} \quad dx = -a(m)dt, \tag{2.7}$$

where

$$F(m) = \int_0^m a(m_1)dm_1. \tag{2.8}$$

## 3. Reduction of Eqs. (2.1) to the Euler-Darboux equation

Let us introduce new independent variables $\xi$ and $\eta$ in the following manner:

$$v-F(m) = 2\xi = \text{const}, \tag{3.1}$$

$$v+F(m) = 2\eta = \text{const}. \tag{3.2}$$

The Lagrange variables $x, t$ occurring in this relation will be treated as functions of the characteristic variables $\xi$, $\eta$, that is

$$\begin{aligned} x &= x(\xi, \eta), \\ t &= t(\xi, \eta). \end{aligned} \tag{3.3}$$

Because along the first family of characteristics (2.5) the variable $\xi = \text{const}$ (cf. Eqs. (2.6) and (3.1)), variables $x$ and $t$ at these characteristics are functions of the only variable $\eta$,

$$dx = \frac{\partial x}{\partial \eta} d\eta, \quad dt = \frac{\partial t}{\partial \eta} d\eta. \tag{3.4}$$

Consequently, on the characteristics of the first family (positive), we have

$$\frac{\partial x}{\partial \eta} = a(m) \frac{\partial t}{\partial \eta}. \tag{3.5}$$

An analogous expression is obtained on the characteristics of the second family (negative),

$$\frac{\partial x}{\partial \xi} = -a(m) \frac{\partial t}{\partial \xi}. \tag{3.6}$$

Equations (3.1) and (3.2) yield

$$F(m) = \eta - \xi. \tag{3.7}$$

Moreover, from Eqs. (2.2), (2.3) and (2.8) we obtain, by disregarding the terms $0(m^4)$, the formulae

$$\begin{aligned} F(m) &= \frac{E_0}{3E_1} \left[ \left( 1+2\frac{E_1}{E_0} m \right)^{\frac{3}{2}} - 1 \right] a_0, \\ a(m) &= \left( 1+2\frac{E_1}{E_0} m \right)^{\frac{1}{2}} a_0, \quad a_0 = \sqrt{\frac{E_0}{\varrho_0}}. \end{aligned} \tag{3.8}$$

On comparing Eqs. (3.7) and (3.8) we obtain

$$a(m) = \left[\frac{3E_1}{a_0 E_0}(\eta-\xi)+1\right]^{\frac{1}{3}} a_0. \tag{3.9}$$

Finally, substitution of Eq. (3.9) into Eqs. (3.5) and (3.6) yields the necessary system of equations which is equivalent to the set (2.4),

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= -\left[\frac{3E_1}{a_0 E_0}(\eta-\xi)+1\right]^{\frac{1}{3}} a_0 \frac{\partial t}{\partial \xi}, \\ \frac{\partial x}{\partial \eta} &= \left[\frac{3E_1}{a_0 E_0}(\eta-\xi)+1\right]^{\frac{1}{3}} a_0 \frac{\partial t}{\partial \eta}. \end{aligned} \tag{3.10}$$

It is the canonical system of equations of nonlinear elasticity for a one-dimensional plane motion with constant entropy. It should be observed that, in contrast to the set of quasi-linear equations (2.1) or (2.4), the system (3.10) derived here is linear and has variable coefficients.

The system may be reduced to a single second order equation. To this end let us differentiate the first equation of the set (3.10) with respect to $\eta$, and the second one — with respect to $\xi$. As a result we obtain

$$\begin{aligned} \frac{\partial^2 x}{\partial \xi \partial \eta} &= -\left[\frac{3E_1}{a_0 E_0}(\eta-\xi)+1\right]^{\frac{1}{3}} a_0 \frac{\partial^2 t}{\partial \xi \partial \eta} - \frac{E_1}{E_0}\left[\frac{3E_1}{a_0 E_0}(\eta-\xi)+1\right]^{-\frac{2}{3}} \frac{\partial t}{\partial \xi}, \\ \frac{\partial^2 x}{\partial \eta \partial \xi} &= \left[\frac{3E_1}{a_0 E_0}(\eta-\xi)+1\right]^{\frac{1}{3}} a_0 \frac{\partial^2 t}{\partial \eta \partial \xi} - \frac{E_1}{E_0}\left[\frac{3E_1}{a_0 E_0}(\eta-\xi)+1\right]^{-\frac{2}{3}} \frac{\partial t}{\partial \eta}. \end{aligned} \tag{3.11}$$

Subtraction of both sides of Eqs. (3.11) yields

$$\frac{\partial^2 t}{\partial \xi \partial \eta} = \frac{\dfrac{E_1}{a_0 E_0}}{2\left[\dfrac{3E_1}{a_0 E_0}(\xi-\eta)-1\right]}\left(\frac{\partial t}{\partial \xi} - \frac{\partial t}{\partial \eta}\right). \tag{3.12}$$

Further simplifications are introduced by means of the following change of variables:

$$\begin{aligned} \alpha &= \frac{3E_1}{a_0 E_0}\xi - 1, \\ \beta &= \frac{3E_1}{a_0 E_0}\eta. \end{aligned} \tag{3.13}$$

Equation (3.12) expressed in terms of the variables (3.13) assumes (in passing from the variables $\xi$ and $\eta$ to $\alpha$ and $\beta$, the form of the notation for $t$ and $x$ remains unchanged)

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} = \frac{1}{6(\alpha-\beta)}\left(\frac{\partial t}{\partial \alpha} - \frac{\partial t}{\partial \beta}\right). \tag{3.14}$$

It is the linear partial second-order differential equation of the Euler-Darboux type [4]. Its general solution may be written in the form [4]

$$t(\alpha, \beta) = (\beta-\alpha)^{\frac{2}{3}} \int_0^1 \Phi[\alpha+(\beta-\alpha)\tau]\tau^{-\frac{1}{6}}(1-\tau)^{-\frac{1}{6}}d\tau + \int_0^1 \Psi[\alpha+(\beta-\alpha)\tau]\tau^{-\frac{5}{6}}(1-\tau)^{-\frac{5}{6}}d\tau, \tag{3.15}$$

where $\Phi$ and $\Psi$ are arbitrary functions of one variable. Their form is determined by the suitable boundary and initial conditions.

Once the function $t(\alpha, \beta)$ is determined, Eqs. (3.10) yield the function $x(\alpha, \beta)$ which assumes the form

$$x(\alpha, \beta) = x(\alpha_0, \beta_0) + \int_{\alpha_0}^{\alpha} P(\alpha_1, \beta)d\alpha_1 + \int_{\beta_0}^{\beta} Q(\alpha_0, \beta_1)d\beta_1. \tag{3.16}$$

Here

$$\begin{aligned} P(\alpha, \beta) &= a_0(\alpha-\beta)^{\frac{1}{3}} \frac{\partial t}{\partial \alpha}, \\ Q(\alpha, \beta) &= -a_0(\alpha-\beta)^{\frac{1}{3}} \frac{\partial t}{\partial \beta} \end{aligned} \tag{3.17}$$

and

$$\frac{\partial P(\alpha, \beta)}{\partial \beta} \| \frac{\partial Q(\alpha, \beta)}{\partial \alpha}. \tag{3.18}$$

The general solution represented by Eqs. (3.15)–(3.18) is constructed for the particular form of $W(m)$ expressed by Eqs. (2.3) in which the terms $0(m^4)$ have been disregarded. The question arises: what is the class of $W(m)$ which enables the construction of a closed-form solution of the equations of a one-dimensional plane motiôn of geometrically nonlinear isotropic elastic media with constant entropy? The question will be and swered in the following section.

## 4. Closed solutions for a definite class of functions $W(m)$

From the expressions (2.2), (2.8), (3.5)–(3.7) it follows that

$$\begin{aligned} \frac{\partial x}{\partial \eta} &= \sqrt{\frac{1}{\varrho_0} W''(m)} \frac{\partial t}{\partial \eta}, \\ \frac{\partial x}{\partial \xi} &= -\sqrt{\frac{1}{\varrho_0} W''(m)} \frac{\partial t}{\partial \xi} \end{aligned} \tag{4.1}$$

and

$$\int_0^m \sqrt{\frac{1}{\varrho_0} W''(m_1)}\, dm_1 = \eta - \xi. \tag{4.2}$$

Let us assume that

$$\sqrt{\frac{1}{\varrho_0} W''(m)} = \varphi(\eta-\xi), \tag{4.3}$$

where $\varphi(\eta-\xi)$ is a positive definite differentiable function of the argument $\eta-\xi$. Then

$$\begin{aligned} \frac{\partial x}{\partial \eta} &= \varphi(\eta-\xi)\frac{\partial t}{\partial \eta}, \\ \frac{\partial x}{\partial \xi} &= -\varphi(\eta-\xi)\frac{\partial t}{\partial \xi}. \end{aligned} \tag{4.4}$$

Differentiation of the first equation of the set (4.4) with respect to $\xi$, and the second — with respect to $\eta$, yields

$$\begin{aligned} \frac{\partial^2 x}{\partial \eta\, \partial \xi} &= \varphi(\eta-\xi)\frac{\partial^2 t}{\partial \eta\, \partial \xi} - \varphi'(\eta-\xi)\frac{\partial t}{\partial \eta}, \\ \frac{\partial^2 x}{\partial \xi\, \partial \eta} &= -\varphi(\eta-\xi)\frac{\partial^2 t}{\partial \xi\, \partial \eta} - \varphi'(\eta-\xi)\frac{\partial t}{\partial \xi}. \end{aligned} \tag{4.5}$$

By eliminating the derivative $\partial^2 x/\partial\xi\partial\eta$ from Eqs. (4.5) we obtain the final equation of the problem under consideration

$$\frac{\partial^2 t}{\partial \xi\, \partial \eta} + \frac{1}{2}\frac{\varphi'(\eta-\xi)}{\varphi(\eta-\xi)}\left(\frac{\partial t}{\partial \xi} - \frac{\partial t}{\partial \eta}\right) = 0, \tag{4.6}$$

where

$$\varphi(\eta-\xi) \neq 0. \tag{4.7}$$

Let the function $\nu(\eta-\xi)$ be such that the equation

$$\frac{\partial^2 t}{\partial \xi\, \partial \eta} + \nu(\eta-\xi)\left(\frac{\partial t}{\partial \xi} - \frac{\partial t}{\partial \eta}\right) = 0$$

has a closed-form solution. Then, denoting for the sake of simplicity

$$\eta-\xi = z, \tag{4.8}$$

we may easily determine the character of $\varphi(z)$ from the following equation:

$$\frac{1}{2}\frac{\varphi'(z)}{\varphi(z)} = \nu(z)$$

or

$$\varphi(z) = C_1 e^{2\int \nu(z)dz}, \tag{4.9}$$

where $C_1$ is an arbitrary constant.

Introducing the notation (4.8) into the expressions (4.2) and (4.3), we obtain

$$\int_0^m \sqrt{\frac{1}{\varrho_0} W''(m_1)}\, dm_1 = z, \tag{4.10}$$

$$\sqrt{\frac{1}{\varrho_0} W''(m)} = \varphi(z), \tag{4.11}$$

what leads to the following relation between $m$ and $z$:

$$dm = \frac{dz}{\varphi(z)}$$

or

$$m = \int \frac{dz}{\varphi(z)} + C_2. \tag{4.12}$$

Here $C_2$ is an arbitrary constant.

The function $z(m)$ and $\varphi[z(m)]$ are determined from Eq. (4.12). The function $\varphi[z(m)]$ is then substituted into the formula (4.11) which, after double integration, yields

$$\frac{1}{\varrho_0} W(m) = \int \left\{ \int \varphi^2[z(m)]dm + C_3 \right\} dm + C_4, \tag{4.13}$$

$C_3$, $C_4$ are constants of arbitrary value.

From the relations obtained it follows that once a definite function $\nu(z)$ is selected, the internal energy $W(m)$ is determined by the functional equation (4.13) in which four arbitrary constants appear: $C_1, C_2, C_3, C_4$, apart from the constants which may be contained in the function $\nu(z)$.

Thus the question posed at the end of the preceding section has been answered, and we may formulate the answer in the following way: there exists a relatively broad class of functions $W = W(m)$ allowing for a general closed-form solution of a certain nonlinear one-dimensional motion of elastic bodies.

## 5. Example

Assume the function $\nu(z)$ to have the form

$$\nu(z) = \frac{n}{\eta - \xi} = \frac{n}{z}, \tag{5.1}$$

where $n$ is a real number. Then, according to Eqs. (4.9) and (4.13),

$$\begin{aligned} \varphi(z) &= C_1 z^{2n}, \\ m &= \frac{1}{C_1(1-2n)} z^{1-2n} + C_2, \quad n \neq \frac{1}{2}, \end{aligned} \tag{5.2}$$

whence

$$z = [C_1(1-2n)(m-C_2)]^{\frac{1}{1-2n}} \tag{5.3}$$

and

$$\varphi[z(m)] = C_1[C_1(1-2n)(m-C_2)]^{\frac{2n}{1-2n}}. \tag{5.4}$$

On substituting the expression (5.4) into the formula (4.13) we obtain, after integrations

$$\frac{1}{\varrho_0} W(m) = (1-2n)^{\frac{4n}{1-2n}} C_1^{\frac{2}{1-2n}} \left[ \frac{(1-2n)^2}{2(2n+1)} (m-C_2)^{\frac{2}{1-2n}} + C_3 m + C_4 \right]. \tag{5.5}$$

Hence, if the internal energy of the medium considered is expressed by the formula (5.5), the plane motion of that medium subject to one-dimensional finite deformation is described by the Euler-Darboux equation

$$\frac{\partial^2 t}{\partial\xi\,\partial\eta}-\frac{n}{\xi-\eta}\left(\frac{\partial t}{\partial\xi}-\frac{\partial t}{\partial\eta}\right)=0, \tag{5.6}$$

the general solution of which has the form

$$t(\xi,\eta)=\frac{\partial^{2n-2}}{\partial\xi^{n-1}\partial\eta^{n-1}}\left[\frac{\Phi(\xi)-\Psi(\eta)}{\xi-\eta}\right], \tag{5.7}$$

for positive integral $n$, the form

$$t(\xi,\eta)=(\xi-\eta)^{2n+1}\frac{\partial^{2n}}{\partial\xi^{n}\partial\eta^{n}}\left[\frac{\Phi(\xi)-\Psi(\eta)}{\xi-\eta}\right] \tag{5.8}$$

for negative integers $n$, and the form

$$t(\xi,\eta)=(\eta-\xi)^{1-2n}\int_0^1\Phi[\xi+(\eta-\xi)\tau]\tau^{-n}(1-\tau)^{-n}d\tau + \int_0^1\Psi[\xi+(\eta-\xi)\tau]\tau^{n-1}(1-\tau)^{n-1}d\tau \tag{5.9}$$

for fractional $n$ satisfying the conditions $0<n<1$ and $2n\neq 1$.

If $2n=1$, then

$$t(\xi,\eta)=\int_0^1\Phi[\xi+(\eta-\xi)\tau]\tau^{-n}(1-\tau)^{n-1}d\tau + \int_0^1\Psi[\xi+(\eta-\xi)\tau]\tau^{-n}(1-\tau)^{n-1}\ln[\tau(1-\tau)(\eta-\xi)]d\tau. \tag{5.10}$$

Here $\Phi$ and $\Psi$ are arbitrary functions of a single variable. Their form is determined by the boundary and initial conditions of the particular problem considered.

To conclude, let us present the solution for the case $n=0$. According to the derived formulae

$$t(\xi,\eta)=\Phi(\xi)+\Psi(\eta), \tag{5.11}$$

while the function $W(m)$ assumes the form

$$W(m)=\frac{1}{2}\varrho_0 C_1^2m^2+\varrho_0C_1^2(C_3-C_2)m+\varrho_0C_1^2\left(\frac{1}{2}C_2+C_4\right)$$

or

$$W(m)=k_0+k_1m+k_2m^2.$$

Here $k_0$, $k_1$, $k_2$ are arbitrary constants.

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