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Direct continuum model of an elastically-deformable polarizable and magnetizable body
I. Lagrangian function (*)

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THE NONLOCAL theory of an elastically-deformable, polarizable and magnetizable body under electromagnetic fields is introduced. We adopt in our considerations Tiersten's model of a polarizable continuum and Zorski's method based on the interactions between particles. The macroscopic state of the body is described by using the concept of generalized two-point densities which obey deterministic divergence equations very similar to those used for one-point densities in classical continuum mechanics.

Rozpatruje się nielokalną teorię sprężyście deformowalnego, polaryzowalnego i magnetyzowalnego ośrodka ciągłego znajdującego się pod wpływem pola elektromagnetycznego. Korzysta się z modelu Tierstena polaryzowalnego kontinuum oraz z metody Zorskiego opartej na oddziaływaniach międzycząsteczkowych. Makroskopowy stan ciała opisywany jest przy użyciu koncepcji uogólnionych dwupunktowych gęstości, które spełniają deterministyczne, dywergencyjne równania podobne do tych, których używa się w klasycznej teorii kontinuum dla jednopunktowych gęstości.

Рассматривается нелокальная теория упруго деформируемой, поляризующейся и намагничивающейся сплошной среды, находящейся под влиянием электромагнитного поля. Используется модель Тирстена поляризуемого континуума и метод Зорского, опирающийся на межмолекулярные взаимодействия. Макроскопическое состояние тела описывается при использовании концепции обобщенных двухточечных плотностей, которые удовлетворяют детерминистическим, дивергентным уравнениям, аналогичных тем, которые используются в классической теории континуума для одноточечных плотностей.

1. Introduction

THE PROBLEMS concerning interactions of electromagnetic fields with elastically-deformable, polarizable, and magnetizable bodies, in which there appear couplings between the different material fields, have been extensively considered in recent years by many research workers, beginning with the works of TOUPIN [1] and ERINGEN [2]. On the other hand, the ways of investigating these problems are very often similar in this sense that some global equations of motion and fields are postulated [3]. The problem of the local statements of these equations is still open at this moment, mainly in the nonlocal formulation [4], but after that the special material is proposed by means of the constitutive principles. There exist, of course, many different, sometimes very interesting methods (a variational approach [5] and the so-called Principle of Virtual Power used by PENFIELD and HAUS [6], MAUGIN and COLLET [7, 8]) but these formulations seem to shift the postulates to another place of essentially equivalent theories and then, confirming the global approach cannot be paradoxically a serious verification of the latter. It is to be noticed

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that the questions associated with a more direct and integral inclusion of different kinds of interactions to the theory appear very seldom or never in these studies. We believe that a simple, direct, particular, because of special kinds of interactions described in the notable books on the subject [9, 10], and reversible formulation will be necessary if it allows one to discuss the fundamental mechanical assumptions or results of the phenomenological theories. Of interest are then the studies of the form of the antisymmetric portion of the mechanical stress tensor, the concept of the local magnetic field vector, the exchange forces permitted to be of the contact type, the idea of the higher order surface couplings. We shall try to adopt in our considerations TIERSTEN's model of a polarizable continuum [11] and ZORSKI's method [12, 13] based on the interactions between particles. For the sake of simplicity, we restrict our considerations to two point potentials and include the central two point interactions, the spatially nonlocal spin-spin interactions, and the nonlocal interactions between the spin continuum and the lattice continuum (the one-ionic anisotropy effect). The polarization follows from the existence of two strictly interpenetrating ionic continua which are needed to allow the body to respond to actions of external electromagnetic fields. We refer the reader to the paper of TIERSTEN and TSAI [14], in particular to their model of a polarizable and magnetizable continuum and to ZORSKI's papers [12, 13] for an explanation of the basic principles of construction of the direct and nonlocal model of a continuum. In accordance with these remarks, our model consists of two strictly interpenetrating ionic continua and in a certain sense it may be understood as a generalization of the theories mentioned above. The microscopic particles, which build two continua, interact by means of two point potentials of different kinds. The transition from the microscopic level to the macroscopic level is possible by using the concept of generalized two-point densities [15, 16, 17], which obey deterministic divergence equations very similar to those used for one-point densities in classical continuum mechanics. When this is done, the model becomes simultaneously nonlocal and purely phenomenological but with a new concept of generalized densities. We are now in a position to construct a variational principle in order to obtain the equations of motion and fields, equations of conservation of energy, momentum, and moment of momentum and also equations of conservation connected with the internal symmetry of the body. It must be possible, of course, to obtain the local version of the theory. The way of taking into account the interactions of a magnetizable and polarizable continuum with electromagnetic fields should become clear while reading this paper. The resulting description is not Lorentz invariant. No boundary conditions are considered due to the strictly nonlocal formulation of the problem; the Lagrangian function does not depend on the material gradients of the material fields under consideration. Moreover, we assume that the body contains no surfaces of discontinuity.

2. Generalized densities

We assume that the body under consideration consists of two strictly interpenetrating ionic continua numbered $\alpha = a, b$. Initially, both continua occupy the same region of space and, hence, have the same material coordinates X in a fixed Cartesian rectangu-

lar coordinate system which is used throughout the paper; we shall frequently write X rather than $\mathbf{X}$ if no confusion results. The microscopic characteristics in such a body are the molecular masses m_a and m_b of the a-component and the b-component, respectively, and their molecular charges q_a and q_b. The macroscopic state of both continua at any given time (t) is described by the following set of generalized densities:

$$(2.1)\qquad (\alpha, \beta = a, b)\qquad \begin{aligned} &\bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t),\\ &\bar{\varrho}_{\alpha\beta}(x_\alpha, y_\beta, t) \equiv m_\alpha m_\beta \bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t),\\ &\bar{\sigma}_{\alpha\beta}(x_\alpha, y_\beta, t) \equiv q_\alpha q_\beta \bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t), \end{aligned}$$

where the quantity $\bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t)\, dx_\alpha dy_\beta$ expresses the probability of finding a molecule of the α-species in the volume element dx_α, and a molecule of the β-species in the volume element dy_β, at time t. As a rule we use either the usual tensor notations $\mathbf{x}_\alpha$, $\mathbf{y}_\alpha$, $d\mathbf{x}_\alpha$, $d\mathbf{y}_\alpha$ ($\alpha = a, b$) or the simplified notations x_α, y_α, dx_α, dy_α ($\alpha = a, b$) if there is no ambiguity. In view of the physical interpretation of the generalized densities, we have

$$(2.2)\qquad \bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t) = \bar{n}_{\beta\alpha}(y_\beta, x_\alpha, t).$$

It is not difficult to obtain the following relations between the usual one-point densities of classical mechanics and the generalized two-point densities introduced above:

$$\bar{n}_\alpha(x_\alpha, t) = \int_\infty dy_\beta \bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t) \Big/ \int_\infty dy^\beta \bar{n}_\beta(y_\beta, t),$$

$$(2.3)\qquad (\alpha, \beta = a, b)\qquad \begin{aligned} &\bar{\varrho}_\alpha(x_\alpha, t) = m_\alpha \bar{n}_\alpha(x_\alpha, t),\\ &\bar{\sigma}_\alpha(x_\alpha, t) = q_\alpha \bar{n}_\alpha(x_\alpha, t), \end{aligned}$$

where the quantity $\bar{n}_\alpha(x_\alpha, t)\, dx_\alpha$ expresses the probability that at time t the volume element dx_α is occupied by a molecule of the α-species, and the quantities $\bar{\varrho}_\alpha$ and $\bar{\sigma}_\alpha$ can be interpreted as a mass density and a charge density, respectively. The body contains no surfaces of discontinuity. The generalized two-point densities are continuous functions of their arguments. Moreover, we assume that the mass density $\varrho(x, t)$ vanishes when x is large enough. First of all we are interested in processes occurring during the time interval which is much larger than the time interval of the establishment of the quasi-equilibrium distribution of magnetization in the whole body [10]. It is reasonable then to assume that the generalized two-point densities describe the spatial distribution of particles under the strictly given distribution of the macroscopic magnetic moment per unit mass. To derive the equations governing the behaviour of $\bar{n}_{\alpha\beta}$ observe that if we follow the trajectories of the particles, the probabilities do not change. Thus

$$(2.4)\qquad (\alpha, \beta = a, b)\qquad \begin{aligned} &\frac{d}{dt}\left[\bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t) dx_\alpha dy_\beta\right] = 0,\\ &\frac{\partial}{\partial t}\bar{n}_{\beta} + \frac{\partial}{\partial x_{\alpha p}}\left(\bar{n}_{\alpha\beta} v_\alpha^p(x_\alpha, t)\right) + \frac{\partial}{\partial y_{\beta p}}\left(\bar{n}_{\alpha\beta} v_\beta^p(y_\beta, t)\right) = 0, \end{aligned}$$

where the usual summation convention over repeated indices is applied. d/dt, $\partial/\partial t$ are the convective derivative with respect to time following x_α and/or y_β, and the partial derivative with respect to time, respectively. $\mathbf{v}_\alpha(x_\alpha, t) = \dot{\mathbf{x}}_\alpha(x_\alpha, t) = d\mathbf{x}_\alpha(x_\alpha, t)/dt$ de-

notes the macroscopic velocity of the α-subcontinuum measured at the instant t at the point x_α. The solutions of the above equations have the form

$$\bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t) = n^0_{\alpha\beta}(X, Y) J_\alpha(x_\alpha, t) J_\beta(y_\beta, t), \tag{2.5}$$

where $J_\alpha(x_\alpha, t) \equiv \left| \frac{\partial X}{\partial x_\alpha} \right|$ $(\alpha = a, b)$ is the Jacobian determinant of motion

$$X = X_\alpha(x_\alpha, t). \tag{2.6}$$

The quantities $n^0_{\alpha\beta}(X, Y)$ $(\alpha, \beta = a, b)$ define the macroscopic state of the body at the initial instant $t = 0$. They can be determined theoretically or better by the experimental technique of X-ray scattering [15, 18, 19]. The continuity equations for the one-point densities result directly from Eqs. (2.3) and (2.4). To finish this point, we assume that at the initial instant $t = 0$ it is permitted to write

$$\sum_{(\alpha)} \sigma^0_\alpha(X) = 0, \tag{2.7}$$

where we used the following abbreviations:

$$\sum_{(\alpha)} \equiv \sum_{\alpha=a}^{b},$$

$$\sigma^0_\alpha(X) = q_\alpha n^0_\alpha(X) \quad (\alpha, \beta = a, b),$$

$$n^0_\alpha(X) = \int_\infty dY n^0_{\alpha\beta}(X, Y) \Big/ \int_\infty dY n^0_\beta(Y).$$

3. Polarization

The motion of our complex system is described completely by two sets of equations:

$$x_\alpha = x_\alpha(X, t) \quad (\alpha = a, b), \tag{3.1}$$

$$\mu_\alpha = \mu_\alpha(X, t) \quad (\alpha = a, b), \tag{3.2}$$

where it is supposed that $x_a(X, t)$ and $x_b(X, t)$ possess unique inverses, $X = X_a(x_a, t)$ and $X = X_b(x_b, t)$, at any given time t at all points of the a-subcontinuum and the b-subcontinuum, respectively. In addition, we assume that $x_a(X, t)$ and $x_b(X, t)$ are differentiable as many times as required [20]. The quantity $\mu_\alpha(X, t) = \mu_\alpha[X_\alpha(x_\alpha, t), t] = \mu_\alpha(x_\alpha, t)$ denotes the magnetic moment per unit mass in the α-subcontinuum. For the sake of simplicity, we used the same notation to express that the magnetic moment per unit mass depends on x_α rather than X. At this point our attention will be directed at the transformation of the motions (3.1) into the motion of a point of the resultant continuum, which occurs at the center of mass of the ionic continua, and into the ionic polarization per unit volume or per unit mass, respectively. It should be noticed that the differences between the motions (3.1) in the two continua are caused by the different microscopic characteristics of molecules of the first and the second species, and also by the macroscopic states of the two continua described by means of the generalized densities. The

macroscopic states of the two continua appear to differ in many respects at the initial instant $t = 0$. Thus, even if the reversible processes are under consideration, the external agencies (electromagnetic fields) and the internal agencies (the internal volume forces) influence the behaviour of the a-subcontinuum and the b-subcontinuum in a different way. The motion of the center of mass of the ionic continua is given by the equation

$$x(X, t) \equiv A_a x_a(X, t) + A_b x_b(X, t) \tag{3.3}$$

at all points of the body, where

$$A_\alpha \equiv \frac{\varrho_\alpha^0(X)}{\varrho_a^0(X) + \varrho_b^0(X)}. \tag{3.4}$$

For the sake of simplicity, we confine ourselves to a homogeneous body at the initial instant $t = 0$; this means that the quantities A_a and A_b are constant coefficients. At the same time the case of sufficiently short range forces, which is supported by the experimental investigations [21], allows one to assume that the influence of the boundary on the behaviour of the complex continuum can be neglected at all points of the body where the homogeneity assumption is accepted.

Generally, the mapping (3.3) is not one-to-one. The condition

$$\bigwedge_t \{(X \neq Y) \Rightarrow ((x_{ak} - y_{ak})(x_b^k - y_b^k) \geqslant 0)\} \tag{3.5}$$

assures the existence, but in the algebraic sense only, of the unique inverse $X = X(x, t)$ of the mapping (3.3) at all points of the body. It will be convenient to introduce the following set of quantities:

$$\begin{aligned} &z_{ab} = x_b - x_a, \quad j_a \equiv |x_a^k, K|, \quad j_b \equiv |x_b^k, K|, \\ &j_{ab} \equiv |z_{ab}^k, K|, \quad j \equiv |x^k, K|, \end{aligned} \tag{3.6}$$

where the commas denote partial differentiation with respect to X or x (here with respect to X), j_α is the strictly positive Jacobian determinant of the α-subcontinuum for all t, and j is the Jacobian determinant of the resultant continuum. After a simple calculation, we obtain

$$\begin{aligned} j = \sum_{(\alpha)} A_\alpha j_\alpha - \frac{1}{6} x^k, K \sum_{(\alpha,\beta)} A_\alpha A_\beta (1 - \delta_{\alpha\beta}) \frac{\partial j_{\alpha\beta}}{\partial z_{\alpha\beta}^k, K} \\ - \frac{1}{6} \sum_{(\alpha,\beta)} A_\alpha A_\beta z_{\alpha\beta}^k, K \left(\frac{\partial j_\beta}{\partial x_\beta^k, K} - \frac{\partial j_\alpha}{\partial x_\alpha^k, K} \right). \end{aligned} \tag{3.7}$$

The usual properties of the motion of the center of mass will be satisfied if we write

$$j > 0 \tag{3.8}$$

at all points of the body.

The conditions (3.5) and (3.8) will be guaranteed if we assume that the displacements of the a-subcontinuum and the b-subcontinuum with respect to the resultant continuum associated with the center of mass of the ionic continua are small enough. We are now in a position to write

$$
\begin{aligned}
&x_\alpha(X,t) = x_\alpha(X(x,t),t) \equiv \bar{x}_\alpha(x,t),\\
&w_\alpha(x,t) \equiv \bar{x}_\alpha(x,t) - x \qquad (\alpha = a,b).
\end{aligned} \tag{3.9}
$$

Following Tiersten, we assume

$$
w^k_{\alpha,k} = 0 \qquad (\alpha = a,b) \tag{3.10}
$$

in order to assure that elements of the different continua with the same material coordinates have equal volumes at all times. Using Eq. (3.10) we obtain directly

$$
\begin{aligned}
n_{\alpha\beta}(x,y,t)dxdy &\equiv \bar{n}_{\alpha\beta}[\bar{x}_\alpha(x,t),\bar{y}_\beta(y,t),t]dxdy\\
&= \bar{n}_{\alpha\beta}(x_\alpha,y_\beta,t)dx_\alpha dy_\beta \qquad (\alpha,\beta = a,b).
\end{aligned} \tag{3.11}
$$

It results easily from Eqs. (2.4) and (3.11) that

$$
\begin{aligned}
&\frac{d}{dt}\left[n_{\alpha\beta}(x,y,t)dxdy\right] = 0 \qquad (\alpha,\beta = a,b),\\
&\frac{\partial}{\partial t}n_{\alpha\beta} + \frac{\partial}{\partial x_p}\left(n_{\alpha\beta}v^p(x,t)\right) + \frac{\partial}{\partial y_p}\left(n_{\alpha\beta}v^p(y,t)\right) = 0,
\end{aligned} \tag{3.12}
$$

where d/dt denotes the material time derivative following x and/or y and $\mathbf{v}(x,t)$ is the macroscopic velocity of the resultant continuum measured at the instant t at the point x. The classical velocity field is given by the equation

$$
\mathbf{v}(x,t) = \dot{\mathbf{x}}(x,t) = \frac{d\mathbf{x}(x,t)}{dt} = \frac{\partial}{\partial t}\mathbf{x} + v_k\frac{\partial}{\partial x_k}\mathbf{x}, \tag{3.13}
$$

where we adopted both dyadic and Cartesian tensor notations. The solutions of Eqs. (3.12) have the form

$$
n_{\alpha\beta}(x,y,t) = n^0_{\alpha\beta}(X,Y)J(x,t)J(y,t), \tag{3.14}
$$

where $J(x,t) = \left|\frac{\partial X}{\partial x}\right|$ is the Jacobian determinant of the unique inverse of the mapping (3.3). The quantity $J(x,t)$ is strictly positive for all x and t, in accordance with the inequality (3.8). We shall introduce the following set of generalized densities, which is more convenient to describe the behaviour of the body in terms of the motion of the center of mass:

$$
\begin{aligned}
&n_{\alpha\beta}(x,y,t) \equiv \bar{n}_{\alpha\beta}(\bar{x}_\alpha(x,t),\bar{y}_\beta(y,t),t),\\
&\varrho_{\alpha\beta}(x,y,t) \equiv m_\alpha m_\beta n_{\alpha\beta}(x,y,t),\\
&\sigma_{\alpha\beta}(x,y,t) \equiv q_\alpha q_\beta n_{\alpha\beta}(x,y,t),\\
&\sigma_2(x,y,t) \equiv \sum_{(\alpha,\beta)}\sigma_{\alpha\beta}(x,y,t),\\
&n_\alpha(x,t) = \int_\infty dy\, n_{\alpha\beta}(x,y,t)\Big/\int_\infty dy\, n_\beta(y,t),\\
&\varrho_\alpha(x,t) \equiv m_\alpha n_\alpha(x,t),\\
&\sigma_\alpha(x,t) \equiv q_\alpha n_\alpha(x,t),
\end{aligned} \tag{3.15}
$$

(3.15) [cont.]
$$\varrho(x, t) \equiv \sum_{(\alpha)} \varrho_\alpha(x, t),$$
$$\sigma(x, t) \equiv \sum_{(\alpha)} \sigma_\alpha(x, t),$$

where we used the useful abbreviation

(3.16)
$$\sum_{(\alpha, \beta)} \equiv \sum_{\alpha=a}^{b} \sum_{\beta=a}^{b}.$$

It is very easy to see that the following equation is satisfied:

(3.17)
$$\sigma(x, t) = 0$$

as a result of the assumption (2.7), the assumption (3.10), the deterministic divergence equation for one-point densities, and also as a result of Eqs. (3.15).

When the two volume elements are sufficiently far apart to make it impossible for conditions in the neighbourhood of one of them to influence the region in which the other is situated, the occupational probabilities will satisfy the condition of independence, and $n_{\alpha\beta}$ will reduce to the product $n_\alpha n_\beta$. In this case the following equation is also satisfied

(3.18)
$$\sigma_2(x, y, t) = 0.$$

We are now in a position to introduce the concept of polarization. Following TIERSTEN [11] and TIERSTEN and TSAI [14], we propose the following definition:

(3.19)
$$\varrho(x, t)\mathbf{\Pi}(x, t) = \mathbf{P}(x, t) = \sum_{(\alpha)} \sigma_\alpha(x, t)\mathbf{w}_\alpha(x, t),$$

where $\mathbf{\Pi}$ and $\mathbf{P}$ are the ionic polarizations per unit mass and per unit volume, respectively. We can write

(3.20)
$$\mathbf{w}_\alpha(x, t) = \bar{\mathbf{x}}_\alpha(x, t) - \mathbf{x} = \bar{x}_\alpha(x, t) - x = w_\alpha(x, t) \quad (\alpha = a, b)$$

consistently with our simplifying convention.

It follows directly from Eq. (3.19) that the ionic polarization is a consequence of displacements of the two continua with respect to each other. There exists a relation between the ionic polarization per unit mass or per unit volume on one hand, and infinitesimal displacements $\mathbf{w}_a$ and $\mathbf{w}_b$ of the a-subcontinuum and the b-subcontinuum with respect to the resultant continuum associated with the center of mass on the other hand. Tiersten's formulae have the form

(3.21)
$$\mathbf{w}_\alpha(x, t) = \frac{1}{[1 + C_{(\alpha \neq \varepsilon)}]\sigma_\alpha(x, t)} \mathbf{P}(x, t) \quad (\alpha = a, b),$$

where

$$C_{(\alpha \neq \varepsilon)} = \frac{\varrho_\alpha(x, t)}{\varrho_\varepsilon(x, t)} = \frac{\varrho_\alpha^0}{\varrho_\varepsilon^0} = \text{const} \quad (\alpha \neq \varepsilon).$$

We refer the reader to TIERSTEN's paper [11] and to that of TIERSTEN and TSAI [14] for more details on the subject.

4. Interactions

For two infinitesimal volumes dx_α and dy_β, we take the potential energies to be

$$\bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t)\,{}^{(1)}V_{\alpha\beta}(|x_\alpha - y_\beta|)dx_\alpha dy_\beta = n^0_{\alpha\beta}(X, Y)\,{}^{(1)}V_{\alpha\beta}(|x_\alpha(X,t) - y_\beta(Y,t)|)dX\,dY, \tag{4.1}$$

$$\begin{aligned}\bar{\varrho}_{\alpha\beta}(x_\alpha, y_\beta, t)\overset{pq}{J}_{\alpha\beta}(x_\alpha - y_\beta)\mu_{\alpha p}(x_\alpha, t)\mu_{\beta q}(y_\beta, t)dx_\alpha dy_\beta \\ = \varrho^0_{\alpha\beta}(X, Y)\overset{pq}{J}_{\alpha\beta}\big(x_\alpha(X,t) - y_\beta(Y,t)\big)\mu_{\alpha p}(X,t)\mu_{\beta q}(Y,t)dX\,dY \quad (\alpha, \beta = a, b),\end{aligned} \tag{4.2}$$

$$\begin{aligned}&[\bar{\varrho}_{\alpha\beta}(x_\alpha, y_\beta, t)\,{}^{(1)}\overset{pq}{C}_{\alpha\beta}(x_\alpha - y_\beta)\mu_{\alpha p}(x_\alpha, t)\mu_{\alpha q}(x_\alpha, t) \\ &\qquad + \bar{\varrho}_{\beta\alpha}(y_\beta, x_\alpha, t)\,{}^{(1)}\overset{pq}{C}_{\beta\alpha}(y_\beta - x_\alpha)\mu_{\beta p}(y_\beta, t)\mu_{\beta q}(y_\beta, t)]dx_\alpha dy_\beta \\ &= [\varrho^0_{\alpha\beta}(X, Y)\,{}^{(1)}\overset{pq}{C}_{\alpha\beta}\big(x_\alpha(X,t) - y_\beta(Y,t)\big)\mu_{\alpha p}(X,t)\mu_{\alpha q}(X,t) \\ &\qquad + \varrho^0_{\beta\alpha}(Y, X)\,{}^{(1)}\overset{pq}{C}_{\beta\alpha}\big(y_\beta(Y,t) - x_\alpha(X,t)\big)\mu_{\beta p}(Y,t)\mu_{\beta q}(Y,t)]dX\,dY,\end{aligned} \tag{4.3}$$

where the potential functions ${}^{(1)}V_{\alpha\beta}$ representing the mutual potential interactions of two molecules depend only on the distance between the particle of the α-species and the particle of the β-species. This is connected with the requirement that the potential energies (4.1) do not change after the rigid spatial translation and rotation of the particle systems under consideration. At the same time, the functions $\overset{pq}{J}_{\alpha\beta}$ describing the so-called isotropic exchange and superexchange integrals for interactions within the same subcontinuum and the cross-interactions between the different species and the one-ionic anisotropy functions ${}^{(1)}\overset{pq}{C}_{\alpha\beta}$ [22–24] including the anisotropic interactions between the magnetic particle of the α-species and the ionic particle of the β-species depend only on the difference between the two position vectors x_α and y_β on account of the homogeneity of space. In addition, we must write

$$\varepsilon_{ilk}\left\{\delta_{sl}\delta_{qr}\overset{kr}{J}_{\alpha\beta}(z) + \delta_{sr}\delta_{ql}\overset{rk}{J}_{\alpha\beta}(z) + z_l\frac{\partial\overset{sq}{J}_{\alpha\beta}(z)}{\partial z_k}\right\} = 0, \tag{4.4}$$

$$(\alpha, \beta = a, b)$$

$$\varepsilon_{ilk}\left\{\delta_{sl}\delta_{qr}\,{}^{(1)}\overset{kr}{C}_{\alpha\beta}(z) + \delta_{sr}\delta_{ql}\,{}^{(1)}\overset{rk}{C}_{\alpha\beta}(z) + z_l\frac{\partial\,{}^{(1)}\overset{sq}{C}_{\alpha\beta}(z)}{\partial z_k}\right\} = 0, \tag{4.5}$$

where ε_{ijk} denotes the permutation symbol and δ_{ij} is the Kronecker symbol. The conditions (4.4) and (4.5) are sufficient to ensure the independence of the potential energies (4.2) and (4.3) on the rigid spatial rotation of the particle system. If we assume that the potential energies (4.2) and (4.3) are symmetric with respect to the transposition of the magnetic moments per unit mass, the solutions of Eqs. (4.4) and (4.5) will have the form

$$\overset{kr}{J}_{\alpha\beta}(z) = \delta_{kr}\bar{J}_{\alpha\beta}(|z|) + z_k z_r\hat{J}_{\alpha\beta}(|z|) \quad (\alpha, \beta = a, b), \tag{4.6}$$

$${}^{(1)}\overset{kr}{C}_{\alpha\beta}(z) = \frac{1}{2}\delta_{kr}\bar{C}_{\alpha\beta}(|z|) + \frac{1}{2}z_k z_r\hat{C}_{\alpha\beta}(|z|), \tag{4.7}$$

where

$$\bar{J}_{\alpha\beta}(|z|) = \bar{J}_{\beta\alpha}(|z|), \quad \hat{J}_{\alpha\beta}(|z|) = \hat{J}_{\beta\alpha}(|z|) \quad (\alpha, \beta = a, b), \tag{4.8}$$

$$z \equiv x - y, \tag{4.9}$$

but generally

$$\bar{C}_{\alpha\beta}(|z|) \neq \bar{C}_{\beta\alpha}(|z|), \quad \hat{C}_{\alpha\beta}(|z|) \neq \hat{C}_{\beta\alpha}(|z|) \quad (\alpha, \beta = a, b). \tag{4.10}$$

The so-called anisotropic integrals $\hat{J}_{\alpha\beta}$ and $\hat{C}_{\alpha\beta}$ arise from the anisotropic exchange and superexchange interactions, and from the one-ionic anisotropic interactions, respectively. The nonlocal character of the anisotropic exchange and superexchange interactions associated with the spatial nonhomogeneity of magnetization has no importance in the purely phenomenological approaches. The form of the energy (4.3), although similar, differs in many respects from that of NEEL [25]. In contrast to the previous approaches, we introduce no deformation measure in order to localize our theory. Since $\mathbf{w}_a$ and $\mathbf{w}_b$ are all infinitesimal displacement fields, we expand ${}^{(1)}V_{\alpha\beta}(|x_\alpha - y_\beta|)$ in a Taylor series about $x-y$ and retain the first and the second terms only. Substituting Eq. (3.21) into these transformed expressions, we obtain directly

$$\begin{aligned}
(4.11)\quad & \bar{n}_{\alpha\beta}(x_\alpha, y_\beta, t)\,{}^{(1)}V_{\alpha\beta}(|x_\alpha - y_\beta|)\,dx_\alpha dy_\beta \\
&= n_{\alpha\beta}(x, y, t)\,{}^{(1)}V_{\alpha\beta}(|x-y|)\,dx dy + [\varrho_{\alpha\beta}(x, y, t)\,{}^{(1)}\overset{p}{R}_{\alpha\beta}(x-y)\Pi_p(x, t) \\
&\quad + \varrho_{\beta\alpha}(y, x, t)\,{}^{(1)}\overset{p}{R}_{\beta\alpha}(y-x)\Pi_p(y, t)]\,dx dy \\
&\quad + \varrho_{\alpha\beta}(x, y, t)\,\overset{pq}{I}_{\alpha\beta}(x-y)\Pi_p(x, t)\Pi_q(y, t)\,dx dy \\
&\quad + [\varrho_{\alpha\beta}(x, y, t)\,{}^{(1)}\overset{pq}{R}_{\alpha\beta}(x-y)\Pi_p(x, t)\Pi_q(x, t) \\
&\quad + \varrho_{\beta\alpha}(y, x, t)\,{}^{(1)}\overset{pq}{R}_{\beta\alpha}(y-x)\Pi_p(y, t)\Pi_q(y, t)]\,dx dy \quad (\alpha, \beta = a, b),
\end{aligned}$$

where we introduced the notations

$$\begin{aligned}
{}^{(1)}\overset{p}{R}_{\alpha\beta}(x-y) &\equiv \frac{\varrho^0}{[1 + C_{(\alpha\neq\varepsilon)}] m_\alpha m_\beta \sigma_\alpha^0}\,\frac{\partial {}^{(1)}V_{\alpha\beta}(|x-y|)}{\partial x_p}, \\
{}^{(1)}\overset{pq}{R}_{\alpha\beta}(x-y) &\equiv \frac{(\varrho^0)^2}{2[1 + C_{(\alpha\neq\varepsilon)}]^2 m_\alpha m_\beta (\sigma_\alpha^0)^2}\,\frac{\partial^2 {}^{(1)}V_{\alpha\beta}(|x-y|)}{\partial x_p \partial x_q}, \\
\overset{pq}{I}_{\alpha\beta}(x-y) &\equiv -\frac{(\varrho^0)^2}{[1 + C_{(\alpha\neq\varepsilon)}][1 + C_{(\beta\neq\varepsilon)}] m_\alpha m_\beta \sigma_\alpha^0 \sigma_\beta^0}\,\frac{\partial^2 {}^{(1)}V_{\alpha\beta}(|x-y|)}{\partial x_p \partial x_q}.
\end{aligned} \tag{4.12}$$

It must be remarked that the second-order terms with respect to the ionic polarization field in the expression (4.11) have the same form as the magnetic terms in Eqs. (4.2) and (4.3). These similarities become less surprising if we take into account simple symmetry considerations. There is no doubt that the analogous procedure may be adopted in the functions $\overset{pq}{J}_{\alpha\beta}$ and ${}^{(1)}\overset{pq}{C}_{\alpha\beta}$ in order to obtain the couplings between the different material fields. On the other hand, the goal of this paper is to establish the general idea only and couplings between the magnetization and polarization fields will not be introduced here. From the mathematical point of view, we confine ourselves to terms of the first and the

second order with respect to the ionic polarization field and the magnetic moment field respectively, and neglect all terms of higher than the second degree in the $\mathbf{\Pi}$ and $\boldsymbol{\mu}$. In addition, the effect of time reversal [26] allows one to exclude from our study terms which are proportional to $\boldsymbol{\mu}$ and $\mathbf{\Pi}\boldsymbol{\mu}$. The question regarding types of interactions which should be described by the functions ${}^{(1)}V_{\alpha\beta}$, $J_{\alpha\beta}$, ${}^{(1)}C_{\alpha\beta}$ is very difficult to be answered. Our suggestion is that, even if the arbitrary electromagnetic fields are under consideration, we have to add to Eq. (4.11) in the way described below the interactions of Coulomb's type, too. Indeed, in the usual macroscopic electrodynamics the concept of correlation between two particles is unknown [27]; it means that the following condition of independence is desired in order to obtain the classical electrodynamics

$$n_{\alpha\beta}(x, y, t) = n_\alpha(x, t) n_\beta(y, t) \quad (\alpha, \beta = a, b). \tag{4.13}$$

If the quasi-static electric approximation is sufficient, we are in a position to write the electric part of interactions in the following form [28]:

$$\begin{aligned}\sum_{(\alpha,\beta)} \int\int_{\infty} \bar{\sigma}_{\alpha\beta}(x_\alpha, y_\beta, t) \frac{1}{|x_\alpha - y_\beta|} dx_\alpha dy_\beta \\ = \sum_{(\alpha,\beta)} \int\int_{\infty} \bar{\sigma}_\alpha(x_\alpha, t)\bar{\sigma}_\beta(y_\beta, t) \frac{1}{|x_\alpha - y_\beta|} dx_\alpha dy_\beta \\ + \sum_{(\alpha,\beta)} \int\int_{\infty} [\bar{\sigma}_{\alpha\beta}(x_\alpha, y_\beta, t) - \bar{\sigma}_\alpha(x_\alpha, t)\bar{\sigma}_\beta(y_\beta, t)] \frac{1}{|x_\alpha - y_\beta|} dx_\alpha dy_\beta \\ = \frac{1}{8\pi} \int_{\infty} {}^{(w)}\mathbf{E}^2(x, t) dx \\ + \sum_{(\alpha,\beta)} \int\int_{\infty} [\bar{\sigma}_{\alpha\beta}(x_\alpha, y_\beta, t) - \bar{\sigma}_\alpha(x_\alpha, t)\bar{\sigma}_\beta(y_\beta, t)] \frac{1}{|x_\alpha - y_\beta|} dx_\alpha dy_\beta,\end{aligned} \tag{4.14}$$

where

$${}^{(w)}\mathbf{E}(x, t) \equiv \frac{\partial}{\partial \mathbf{x}} \left\{ \int_{\infty} \mathbf{P}(y, t) \cdot \frac{\partial}{\partial \mathbf{x}} \frac{1}{|x - y|} dy \right\}.$$

The term $1/8\pi \int_{\infty} {}^{(w)}\mathbf{E}^2(x, t)\, dx$ is very general and exists in the case of arbitrary electromagnetic fields, too. The second term in the expression (4.14) will be adequate if we assume that the radius of correlation between two particles is small enough. It is evident that if this assumption is accepted, the effect of time retardation has no importance. In order to accomplish a transition from the concept of motions of two continua to the concepts of motion of the center of mass of the ionic continua and the ionic polarization field, respectively, we expand the functions $1/|x_\alpha - y_\beta|$; $(\alpha, \beta = a, b)$ in a Taylor series about $x - y$ and retain the first terms only. Using Eq. (3.21), we find

$$\sum_{(\alpha,\beta)} \int\int_{\infty} [\bar{\sigma}_{\alpha\beta}(x_\alpha, y_\beta, t) - \bar{\sigma}_\alpha(x_\alpha, t)\bar{\sigma}_\beta(y_\beta, t)] \frac{1}{|x_\alpha - y_\beta|} dx_\alpha dy_\beta \tag{4.15}$$

(4.15) [cont.]
$$= \int\int_{\infty} \sigma_2(x,y,t)\frac{1}{|x-y|}dxdy + \int\int_{\infty}\Big\{\Big[\sum_{(\alpha)}\sigma_\alpha(y,t)\mathbf{P}_\alpha(y,x,t) - \sum_{(\alpha)}\sigma_\alpha(x,t)\mathbf{P}_\alpha(x,y,t)\Big]\frac{\partial}{\partial \mathbf{x}}\frac{1}{|x-y|}\Big\}dxdy = \sum_{(\alpha,\beta)}\int\int_{\infty} n_{\alpha\beta}(x,y,t)^{(2)}V_{\alpha\beta}(|x-y|)dxdy + \sum_{(\alpha,\beta)}\int\int_{\infty}[\varrho_{\alpha\beta}(x,y,t)^{(2)}\overset{p}{R}_{\alpha\beta}(x-y)\Pi_p(x,t) + \varrho_{\beta\alpha}(y,x,t)^{(2)}\overset{p}{R}_{\beta\alpha}(y-x)\Pi_p(y,t)]dxdy,$$

where

$$^{(2)}V_{\alpha\beta}(|x-y|) \equiv q_\alpha q_\beta \frac{1}{|x-y|},$$

(4.16)
$$^{(2)}\overset{p}{R}_{\alpha\beta}(x-y) \equiv \frac{\varrho^0}{[1+C_{(\alpha\neq\varepsilon)}]m_\alpha m_\beta \sigma_\alpha^0}\frac{\partial^{(2)}V_{\alpha\beta}(|x-y|)}{\partial x_p}.$$

$\mathbf{P}_\alpha(y, x, t)$ denotes the ionic polarization per unit volume measured at the point x at the instant t when it is known that a molecule of the α-species is situated at the same instant t at the point y. Of course, if the condition of independence is taken into account, the correlation energy (4.15) vanishes. Our hypothesis is that the term

(4.17)
$$n_{\alpha\beta}(x,y,t)^{(2)}V_{\alpha\beta}(|x-y|)dxdy + [\varrho_{\alpha\beta}(x,y,t)^{(2)}\overset{p}{R}_{\alpha\beta}(x-y)\Pi_p(x,t) + \varrho_{\beta\alpha}(y,x,t)^{(2)}\overset{p}{R}_{\beta\alpha}(y-x)\Pi_p(y,t)]dxdy$$

must be included in the mechanical part of interactions.

5. Lagrangian function

This section is devoted to the discussion of the structure of the Lagrangian function describing the two dynamical systems, the body and the electromagnetic field, respectively. It is very well known that if two arbitrary systems are isolated, the resultant action functional is the sum of the action functionals corresponding to each system. In order to describe the interactions between two systems, it is necessary to introduce some additional terms in the action functional, each of the terms containing dynamical variables corresponding to both systems [29]. In our special case the ionic polarization per unit mass or per unit volume and the magnetic moment per unit mass or per unit volume allow the body to respond to actions of electromagnetic fields. From the theoretical point of view the coupling between the body and the electromagnetic field as well as the principles governing the behaviour of the electromagnetic field can be treated nonlocally. In our ap-

proach, however, such a general case is not introduced; it means that the usual laws for the electromagnetic field are obeyed.

Let us assume that our complex system (the body+the electromagnetic field) is described by the following nonlocal action functional:

$$\underset{(t_1,t_2)}{\overset{\Omega}{W}} = \int_{t_1}^{t_2} dt \int_{\Omega} dx L, \tag{5.1}$$

where (t_1, t_2) denotes the arbitrary time interval, Ω is the arbitrary spatial region in which the homogeneity assumption with respect to the generalized densities at the initial instant $t = 0$ is accepted, and L is the nonlocal Lagrangian function described below,

$$L(x,t) \equiv \varrho(x,t)\mathscr{L}^B(x,t)+L^{EM}(x,t),$$

$$\mathscr{L}^B(x,t) \equiv \mathscr{L}^{(1)B}(x,t)+\int_{\infty} dy\,\varrho(y,t)\mathscr{L}^{(2)B}(x,y,t),$$

$$\mathscr{L}^{(2)B} \equiv \sum_{\alpha,\beta)} \mathscr{L}^{(2)B}_{\alpha\beta},$$

$$\mathscr{L}^{(1)B}(\mathbf{v}, \mathbf{\Pi}, \mathbf{\Pi}, \mu_a, \mu_b, \varphi_{,k}, \mathbf{A}, \mathbf{A}_{,k})$$

$$= \frac{1}{2}\mathbf{v}^2+\frac{m_a m_b}{2|q_a q_b|}\dot{\mathbf{\Pi}}^2+\frac{1}{c}\dot{\Pi}_k A^k+\frac{v_k}{c}\Pi_r A^k_{,r}$$

$$+\varepsilon_{krs}\sum_{(\alpha)}\frac{C_{(\alpha\neq\varepsilon)}}{\{1+C_{(\alpha\neq\varepsilon)}\}}\mu_{\alpha k}A^s_{,r}-\Pi_k\varphi_{,k},$$

$$\mathscr{L}^{(2)B}_{\alpha\beta}(X, Y, x(X,t), y(Y,t), \mathbf{\Pi}(X,t), \mathbf{\Pi}(Y,t), \mu_\alpha(X,t), \mu_\beta(Y,t))$$

$$\equiv -\frac{1}{2\varrho^0(X)\varrho^0(Y)}\{n^0_{\alpha\beta}(X,Y)V_{\alpha\beta}(|x-y|)+[\varrho^0_{\alpha\beta}(X,Y)\overset{p}{{}^{(3)}R}_{\alpha\beta}(x-y)\Pi_p(X,t)$$

$$+\varrho^0_{\beta\alpha}(Y,X)\overset{p}{{}^{(3)}R}_{\beta\alpha}(y-x)\Pi_p(Y,t)]+\varrho^0_{\alpha\beta}(X,Y)\overset{pq}{I}_{\alpha\beta}(x-y)\Pi_p(X,t)\Pi_q(Y,t)$$

$$+[\varrho^0_{\alpha\beta}(X,Y)\overset{pq}{{}^{(1)}R}_{\alpha\beta}(x-y)\Pi_p(X,t)\Pi_q(X,t)+\varrho^0_{\beta\alpha}(Y,X)\overset{pq}{{}^{(1)}R}_{\beta\alpha}(y-x)\Pi_p(Y,t)\Pi_q(Y,t)]$$

$$+\varrho^0_{\alpha\beta}(X,Y)\overset{pq}{J}_{\alpha\beta}(x-y)\mu_{\alpha p}(X,t)\mu_{\beta q}(Y,t)$$

$$+[\varrho^0_{\alpha\beta}(X,Y)\overset{pq}{{}^{(1)}C}_{\alpha\beta}(x-y)\mu_{\alpha p}(X,t)\mu_{\alpha q}(X,t)$$

$$+\varrho^0_{\beta\alpha}(Y,X)\overset{pq}{{}^{(1)}C}_{\beta\alpha}(y-x)\mu_{\beta p}(Y,t)\mu_{\beta q}(Y,t)]\}, \tag{5.2}$$

$$L^{EM} \equiv \frac{1}{8\pi}\left\{\left(\varphi_{,k}+\frac{1}{c}\partial_t A_k\right)\left(\varphi_{,k}+\frac{1}{c}\partial_t A_k\right)\right\},$$

where a superimposed dot indicates the usual material time derivative, ∂_t denotes the partial time derivative, c is the velocity of light in vacuum, $(\varphi, \mathbf{A})$ are the so-called electromagnetic potentials. In accordance with our previous remarks, we introduced the new potential functions $V_{\alpha\beta}$ and $\overset{p}{{}^{(3)}R}_{\alpha\beta}$ given by the following formulae:

$$V_{\alpha\beta} \equiv {}^{(1)}V_{\alpha\beta}+{}^{(2)}V_{\alpha\beta},$$

$$\overset{p}{{}^{(3)}R}_{\alpha\beta} \equiv \overset{p}{{}^{(1)}R}_{\alpha\beta}+\overset{p}{{}^{(2)}R}_{\alpha\beta}. \tag{5.3}$$

The first thing we do is to state the principle of construction of the local Lagrangian function $\mathscr{L}^{(1)B}$ connected with the interactions between two systems. The local Lagrangian function $L_\alpha^{(1)B}$ for the α-subcontinuum has the following generally accepted form:

$$L_\alpha^{(1)B}(x_\alpha, t) \equiv \frac{1}{2}\bar{\varrho}_\alpha(x_\alpha, t)[\mathbf{v}_\alpha(x_\alpha, t)]^2 - \bar{\sigma}_\alpha(x_\alpha, t)\varphi(x_\alpha, t) + \frac{1}{c} j_\alpha^k(x_\alpha, t) A_k(x_\alpha, t), \tag{5.4}$$

where

$$\mathbf{j}_\alpha(x_\alpha, t) \equiv \bar{\sigma}_\alpha(x_\alpha, t)\mathbf{v}_\alpha(x_\alpha, t) + c\nabla \times \left(\bar{\varrho}_\alpha(x_\alpha, t)\boldsymbol{\mu}_\alpha(x_\alpha, t)\right).$$

$\mathbf{j}_\alpha$ denotes the total volume current density of the α-subcontinuum consisting of either the volume current density connected with charges belonging to the α-subcontinuum or the volume current density associated with the magnetic moment per unit volume in the α-subcontinuum. Following the same line of arguments as given in the paper of TIERSTEN and TSAI [14] and also in GROT's lecture [3], we add $L_a^{(1)B}(x_a, t)$ and $L_b^{(1)B}(x_b, t)$, expand the electromagnetic potentials φ and $\mathbf{A}$ in a Taylor series about x and neglect the higher order products of the magnitudes of the infinitesimal diplacement fields $\mathbf{w}_a$ and $\mathbf{w}_b$ with the magnitudes of the gradients of the electromagnetic potentials φ and $\mathbf{A}$; it means that we confine ourselves to the case of sufficiently smooth electromagnetic potentials φ and $\mathbf{A}$.

We obtain directly

$$L_a^{(1)B}(x_a, t) + L_b^{(1)B}(x_b, t) \cong \varrho(x, t)\mathscr{L}^{(1)B}(x, t). \tag{5.5}$$

We shall compute now, in accordance with the important work of ROGULA [30], the variation of the action functional $\overset{\Omega}{W}_{(t_1, t_2)}$ resulting from virtual variations of the time, the coordinates, the material fields and the electromagnetic fields, respectively

$$\begin{gathered} t^* = t + \delta t, \quad X^* = X + \delta X, \quad x^*(X^*, t^*) = x(X, t) + \delta x, \\ \boldsymbol{\Pi}^*(X^*, t^*) = \boldsymbol{\Pi}(X, t) + \delta\boldsymbol{\Pi}, \quad \boldsymbol{\mu}_\alpha^*(X^*, t^*) = \boldsymbol{\mu}_\alpha(X, t) + \delta\boldsymbol{\mu}_\alpha; \quad (\alpha = a, b), \\ \varphi^*(x^*, t^*) = \varphi(x, t) + \delta\varphi, \quad \mathbf{A}^*(x^*, t^*) = \mathbf{A}(x, t) + \delta\mathbf{A}. \end{gathered} \tag{5.6}$$

The above variations of fields should be understood as arbitrary functions of the undeformed coordinates X and/ or the deformed coordinates x. It is possible to change the independent variables of variations from the X to the x and inversely. The total variations (5.6) consist of course of the infinitesimal changes in the functional form of the fields under consideration and of the infinitesimal transformations generated by the change in independent variables of variations. On the basis of these remarks, it is convenient to introduce the following variations:

$$\delta_0 q = q^*(X, t) - q(X, t), \quad \delta_1 q = q^*(x, t) - q(x, t), \tag{5.7}$$

where q is an arbitrary function of their arguments. After very long and laborious calculations we obtain

$$\delta \overset{\Omega}{W}_{(t_1, t_2)} = \delta_0 \overset{\Omega}{W}_{(t_1, t_2)} + \delta_1 \overset{\Omega}{W}_{(t_1, t_2)}, \tag{5.8}$$

where

$$\delta_0 \overset{\Omega}{W}_{(t_1, t_2)} \equiv \int_{t_1}^{t_2} dt \int_\Omega dx \left\{ \varrho(x, t) \left[\frac{\partial \mathscr{L}^B}{\partial \varphi} \varphi_{,t} + \frac{\partial \mathscr{L}^B}{\partial A_k} A^k_{,i} + \frac{\partial \mathscr{L}^B}{\partial \varphi_{,r}} \varphi_{,ri} \right. \right. \tag{5.9}$$

(5.9) [cont.)

$$+\frac{\partial \mathscr{L}^B}{\partial A^k_{,r}} A^k_{,rt} - \frac{d}{dt}\frac{\partial \mathscr{L}^B}{\partial v_t} - \frac{1}{\varrho(x,t)} \sum_{(\alpha,\beta)} \int_\infty dy \left(n_{\alpha\beta}(x,y,t) \frac{\partial V_{\alpha\beta}}{\partial x_t} \right.$$

$$+ (\varrho_{\alpha\beta}\Pi_p)_A(x,y,t) \frac{\partial \overset{p}{R}_{\alpha\beta}}{\partial x_t} + (\varrho_{\alpha\beta}\Pi_p\Pi_q)(x,y,t) \frac{\partial \overset{pq}{I}_{\alpha\beta}}{\partial x_t}$$

$$+ (\varrho_{\alpha\beta}\Pi_p\Pi_q)_S(x,y,t) \frac{\partial \overset{pq}{R}_{\alpha\beta}}{\partial x_t} + (\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\beta q})(x,y,t) \frac{\partial \overset{pq}{J}_{\alpha\beta}}{\partial x_t}$$

$$\left. + (\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\alpha q})_S(x,y,t) \frac{\partial \overset{pq}{C}_{\alpha\beta}}{\partial x_t} \right) \Big] (\delta_0 x_t)(x,t)$$

$$+\varrho(x,t)\left[\frac{\partial \mathscr{L}^{(1)B}}{\partial \Pi_t} - \frac{d}{dt}\frac{\partial \mathscr{L}^{(1)B}}{\partial \dot{\Pi}_t} - \frac{1}{\varrho(x,t)} \sum_{(\alpha,\beta)} \int_\infty dy \left(\frac{1}{2} \varrho_{\alpha\beta}(x,y,t) \overset{t}{R}_{\alpha\beta}(x-y) \right.\right.$$

$$+ \varrho_{\alpha\beta}(x,y,t) \overset{tq}{I}_{\alpha\beta}(x-y)\Pi_q(y,t)$$

$$\left.\left. + \varrho_{\alpha\beta}(x,y,t) \overset{tq}{R}_{\alpha\beta}(x-y)\Pi_q(x,t) \right) \right] (\delta_0 \Pi_t)(x,t) + \sum_{(\alpha)} \varrho(x,t) \left[\frac{\partial \mathscr{L}^{(1)B}}{\partial \mu_{\alpha t}} \right.$$

$$- \frac{1}{\varrho(x,t)} \sum_{(\beta)} \int_\infty dy \big(\varrho_{\alpha\beta}(x,y,t) \overset{tq}{J}_{\alpha\beta}(x-y)\mu_{\beta q}(y,t)$$

$$\left. + \varrho_{\alpha\beta}(x,y,t) \overset{tq}{C}_{\alpha\beta}(x-y)\mu_{\alpha q}(x,t) \big) \right] (\delta_0 \mu_{\alpha t})(x,t)$$

$$+ \left[\frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial_t \varphi)} - \frac{\partial}{\partial x_k}\frac{\partial L}{\partial \varphi_{,k}} \right] (\delta_1 \varphi)(x,t)$$

$$\left. + \left[\frac{\partial L}{\partial A_t} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial_t A_t)} - \frac{\partial}{\partial x_k}\frac{\partial L}{\partial A^t_{,k}} \right] (\delta_1 A_t)(x,t) \right\};$$

(5.10)

$$\delta_1 \overset{\Omega}{W}_{(t_1,t_2)} \equiv \int_{t_1}^{t_2} dt \int_\Omega dx \left\{ \varrho(x,t) \frac{d}{dt} \left(\mathscr{L}^B \delta t + \frac{\partial \mathscr{L}^B}{\partial v_t} \delta_0 x_t + \frac{\partial \mathscr{L}^B}{\partial \dot{\Pi}_t} \delta_0 \Pi_t \right) \right.$$

$$+ \frac{\partial}{\partial t} \left(L^{EM} \delta t + \frac{\partial L}{\partial(\partial_t A_t)} \delta_1 A_t \right) + \frac{\partial}{\partial x_r} \Big[\varrho(x,t) \mathscr{L}^B \delta X_K x^r_{,K}$$

$$- \left(\frac{1}{2} \sum_{(\alpha,\beta)} \int_\infty dz \frac{z^r z^k}{|z|} V'_{\alpha\beta}(|z|) \int_0^1 d\xi n_{\alpha\beta}(x+\xi z, x+\xi z - z, t)(\delta_0 x_k)(x+\xi z, t) \right.$$

$$+ \frac{1}{2} \sum_{(\alpha,\beta)} \int_\infty dz z^r \frac{\partial \overset{p}{R}_{\alpha\beta}}{\partial z_k} \int_0^1 d\xi (\varrho_{\alpha\beta}\Pi_p)_A(x+\xi z, x+\xi z - z, t)(\delta_0 x_k)(x+\xi z, t)$$

$$\left. + \frac{1}{2} \sum_{(\alpha,\beta)} \int_\infty dz z^r \frac{\partial \overset{pq}{I}_{\alpha\beta}}{\partial z_k} \int_0^1 d\xi (\varrho_{\alpha\beta}\Pi_p\Pi_q)(x+\xi z, x+\xi z - z, t)(\delta_0 x_k)(x+\xi z, t) \right.$$

(5.10) [cont.]

$$+\frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\frac{\partial \overset{pq}{R}_{\alpha\beta}}{\partial z_k}\int_0^1 d\xi(\varrho_{\alpha\beta}\Pi_p\Pi_q)_S(x+\xi z, x+\xi z-z, t)(\delta_0 x_k)(x+\xi z, t)$$

$$+\frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\frac{\partial \overset{pq}{J}_{\alpha\beta}}{\partial z_k}\int_0^1 d\xi(\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\beta q})(x+\xi z, x+\xi z-z, t)(\delta_0 x_k)(x+\xi z, t)$$

$$+\frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\frac{\partial \overset{pq}{C}_{\alpha\beta}}{\partial z_k}\int_0^1 d\xi(\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\alpha q})_S(x+\xi z, x+\xi z-z, t)(\delta_0 x_k)(x+\xi z, t)$$

$$+\frac{1}{4}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\overset{k}{R}_{\alpha\beta}(z)\int_0^1 d\xi\varrho_{\alpha\beta}(x+\xi z, x+\xi z-z, t)(\delta_0\Pi_k)(x+\xi z, t)$$

$$+\frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\overset{kp}{I}_{\alpha\beta}(z)\int_0^1 d\xi\varrho_{\alpha\beta}(x+\xi z, x+\xi z-z, t)\Pi_p(x+\xi z-z, t)(\delta_0\Pi_k)(x+\xi z, t)$$

$$+\frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\overset{kp}{R}_{\alpha\beta}(z)\int_0^1 d\xi\varrho_{\alpha\beta}(x+\xi z, x+\xi z-z, t)\Pi_p(x+\xi z, t)(\delta_0\Pi_k)(x+\xi z, t)$$

$$+\frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\overset{kp}{J}_{\alpha\beta}(z)\int_0^1 d\xi\varrho_{\alpha\beta}(x+\xi z, x+\xi z-z, t)\mu_{\beta p}(x+\xi z-z, t)(\delta_0\mu_{\alpha k})(x+\xi z, t)$$

$$+\frac{1}{2}\sum_{(\alpha,\beta)}\int_{\infty}dzz^{r}\overset{kp}{C}_{\alpha\beta}(z)\int_0^1 d\xi\varrho_{\alpha\beta}(x+\xi z, x+\xi z$$

$$-z, t)\mu_{\alpha p}(x+\xi z, t)(\delta_0\mu_{\alpha k})(x+\xi z, t)\Big)$$

$$+L^{EM}\delta x_r+\frac{\partial L}{\partial\varphi_{,r}}\delta_1\varphi+\frac{\partial L}{\partial A^k_{,r}}\delta_1 A^k\Big]\Big\}.$$

We introduced the following useful notations and abbreviations:

$$\overset{p}{R}_{\alpha\beta}(x-y)\equiv 2^{(3)}\overset{p}{R}_{\alpha\beta}(x-y),\quad \overset{pq}{R}_{\alpha\beta}(x-y)\equiv 2^{(1)}\overset{pq}{R}_{\alpha\beta}(x-y),$$

$$\overset{pq}{C}_{\alpha\beta}(x-y)\equiv 2^{(1)}\overset{pq}{C}_{\alpha\beta}(x-y),\quad V'_{\alpha\beta}(|x-y|)\equiv\left.\frac{dV_{\alpha\beta}}{d|z|}\right|_{|z|=|x-y|},$$

$$(\varrho_{\alpha\beta}\Pi_p)_A(x,y,t)\equiv\frac{1}{2}[\varrho_{\alpha\beta}(x,y,t)\Pi_p(x,t)-\varrho_{\alpha\beta}(y,x,t)\Pi_p(y,t)],$$

(5.11)

$$(\varrho_{\alpha\beta}\Pi_p\Pi_q)(x,y,t)\equiv\varrho_{\alpha\beta}(x,y,t)\Pi_p(x,t)\Pi_q(y,t),$$

$$(\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\beta q})(x,y,t)\equiv\varrho_{\alpha\beta}(x,y,t)\mu_{\alpha p}(x,t)\mu_{\beta q}(y,t),$$

(5.11)
[cont.]

$$(\varrho_{\alpha\beta}\Pi_p\Pi_q)_S(x,y,t) \equiv \frac{1}{2}[\varrho_{\alpha\beta}(x,y,t)\Pi_p(x,t)\Pi_q(x,t)+\varrho_{\alpha\beta}(y,x,t)\Pi_p(y,t)\Pi_q(y,t)],$$

$$(\varrho_{\alpha\beta}\mu_{\alpha p}\mu_{\alpha q})_S(x,y,t) \equiv \frac{1}{2}[\varrho_{\alpha\beta}(x,y,t)\mu_{\alpha p}(x,t)\mu_{\alpha q}(x,t)+\varrho_{\alpha\beta}(y,x,t)\mu_{\alpha p}(y,t)\mu_{\alpha q}(y,t)].$$

It must be remarked that the structure of the potential functions $\overset{pq}{R}_{\alpha\beta}$ and $\overset{pq}{I}_{\alpha\beta}$ is the same as the potential functions $\overset{pq}{C}_{\alpha\beta}$ and $\overset{pq}{J}_{\alpha\beta}$. Indeed, it follows directly from Eq. (4.12) that

$$\begin{aligned}\overset{pq}{R}_{\alpha\beta}(z) &= \delta_{pq}\bar{R}_{\alpha\beta}(|z|)+z^pz^q\hat{R}_{\alpha\beta}(|z|),\\ \overset{pq}{I}_{\alpha\beta}(z) &= \delta_{pq}\bar{I}_{\alpha\beta}(|z|)+z^pz^q\hat{I}_{\alpha\beta}(|z|) \quad (\alpha,\beta=a,b),\end{aligned} \tag{5.12}$$

where we do not give the expressions for $\bar{R}_{\alpha\beta}$, $\hat{R}_{\alpha\beta}$, $\bar{I}_{\alpha\beta}$, $\hat{I}_{\alpha\beta}$, respectively. The decomposition of the total variation of the action functional $\overset{\Omega}{\underset{(t_1,t_2)}{W}}$ into two structurally different parts is due entirely to the requirement that only the integrals including the variations $\delta_0 x, \delta_0\mathbf{\Pi}$, $\delta_0\mu_\alpha$, $\delta_1\varphi$, $\delta_1\mathbf{A}$ measured at x (see Eq. (5.9)) and not their spatial distributions around x (see Eq. (5.10)) allow one to find the equations of motion and fields by equating to zero all coefficients of the above variations. The integrals with the parameter ξ in Eq. (5.10) must be explained. Let us assume that $h(x,y)$ is the continuous antisymmetric function with respect to the transposition of the coordinates x and y. We obtain directly

$$\begin{aligned}\int_\infty dy h(x,y) &= -\frac{1}{2}\int_\infty dy[h(y,x)-h(x,y)] = -\frac{1}{2}\int_\infty dz[h(x+z,x)-h(x,x-z)]\\ &= -\frac{1}{2}\int_\infty dz\int_0^1 d\xi\frac{d}{d\xi}[h(x+\xi z,x+\xi z-z)] = -\frac{1}{2}\frac{\partial}{\partial x_r}\left\{\int_\infty dz z^r\int_0^1 d\xi h(x\right.\\ &\qquad \left.+\xi z,x+\xi z-z)\right\}.\end{aligned} \tag{5.13}$$

For the first time the integrals of this type were introduced to continuum mechanics by ZORSKI [12–13]. On the other hand, the suspicion that such expressions are important has already been noted in the literature on statistical mechanics [31]. It is worth noting that

$$\begin{aligned}\int_\infty dz z^r\int_0^1 d\xi h(x+\xi z,x+\xi z-z) &= \int_\infty dz z^r h(x,x-z)\\ &\quad +\frac{1}{2}\frac{\partial}{\partial x_s}\int_\infty dz z^r z^s h(x,x-z)+\ \ldots\ +.\end{aligned} \tag{5.14}$$

It will be possible to obtain the higher order surface couplings if we consider the terms of higher than the first degree in the z.

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