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Source flow betweene two non-parallel rotating disks (*)

P.Å. JANSSON (GÖTEBORG)

LAMINAR source flow of an incompressible viscous fluid between two non-parallel disks has been analyzed. The disks are rotating with arbitrary angular velocities about axes perpendicular to the disks. The equations of motion are solved by a perturbation expansion about the creeping-flow solution for source flow between parallel rotating disks. A solution which is valid in an annular region is obtained. The combination of inclination and rotation is found to influence the pressure distribution and the flow pattern remarkably in some cases. The corresponding effects on the disks are discussed.

Rozważono laminarny przepływ źródłowy między dwiema nierównoległymi tarczami wirującymi. Tarcze wirują z dowolnymi prędkościami kątowymi wokół osi prostopadłych do ich płaszczyzn. Równania ruchu rozwiązano metodą rozwinięć perturbacyjnych względem rozwiązania dla przepływu pełzającego dla wirujących tarcz równoległych. Otrzymano rozwiązanie zachowujące swą ważność w obszarze pierścieniowym. Stwierdzono, że w pewnych przypadkach kombinacja wzajemnego nachylenia tarcz i ich prędkości obrotowych wpływa w istotny sposób na rozkład ciśnień i charakter przepływu. Przedyskutowano także wpływ tych czynników na tarcze.

Рассматривается ламинарное источниковое течение между двумя вращающимися непараллельными дисками. Диски вращаются с произвольными угловыми скоростями вокруг осей перпедикулярных к их плоскостям. Уравнения движения решены методом пертурбационных разложений по отношению к решению для ползающего течения для вращающихся параллельных дисков. Получено решение сохраняющее свою правильность в кольцевой области. Констатировано, что в некоторых случаях комбинация взаимного наклона дисков и их вращательных скоростей влияет существенным образом на распределение давлений и характер течения. Обсуждено также влияние этих факторов на диски.

Nomenclature

- d distance between the disks at the centre,
- F_x, F_y radial force components,
- F_n functions defined by Eq. (2.9),
- G_n functions defined by Eq. (2.8),
- H_n functions defined by Eq. (2.7),
- k ratio of inlet and outlet radii,
- M_x, M_y bending couple components,
- P_n functions defined by Eq. (2.10),
- p pressure,
- Q volumetric flow through an arbitrary surface $r =$ constant,
- q dimensionless volumetric flow rate $(Q/dr_0^2\omega)$,
- R distance defined by Fig. 1,

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R_β $R+r\cos\beta$,
Re Reynolds number ($d^2\omega/\nu$),
r radial coordinate,
r_0 radius of the disks,
s ratio of angular velocities,
u, v, w velocity components in axial, circumferential and radial directions, respectively,
X, Y, Z coordinates defined by Fig. 1,
x, y coordinates defined by Fig. 8,
α axial coordinate defined by Fig. 1,
α_0 angle between the disks,
β circumferential coordinate,
ν kinematic viscosity,
ξ α/α_0,
ϱ density,
ω angular velocity of the disk at $\alpha = 0$,

1. Introduction

LAMINAR source flow between two closely spaced parallel disks, stationary or rotating, is a problem of great interest because of its fundamental character and because of its applications in a number of practical cases, e.g. centrifugal pumps, face seals, air bearings, radial diffusers and rotating heat exchangers.

During the last twenty years several workers have investigated this problem theoretically and experimentally. In the case of stationary disks the works of MOLLER [1], PEUBE [2] and SAVAGE [3] may be mentioned. Flow between disks rotating with the same velocity has been studied by BREITER and POHLHAUSEN [4] and by PEUBE and KREITH [5], while KREITH and VIVIAND [6] treated the case of disks rotating with different speeds. PELECH and SHAPIRO [7] obtained a solution of the flow in the narrow gap between a flexible disk and a rigid wall while examining the mechanics of the disk. PÉCHEUX [8] discussed source flow between a fixed porous disk and a rotating impermeable one. More recently GOSWAMI and NANDA [9] investigated the problem of oscillating radial flow between rotating disks.

The influence of geometric deviations from the ideal case of flat, aligned surfaces has been studied by SNECK [10]–[12] using the "short bearing" approximation of the lubrication theory, modified to include inertial effects. The radial velocity is assumed to be small in this solution. An important example of geometric deviations is misalignment, i.e. the case when the disks are not strictly parallel. This problem was first studied by TAYLOR and SAFFMAN [13] in a attempt to explain the experimentally observed excess pressure at the centre of the airspace between two closely spaced parallel disks, one of them rotating. In their paper TAYLOR and SAFFMAN considered compressible as well as incompressible flow. The analysis, however, is restricted to zero radial volumetric flow rate and, furthermore, the tangential and radial velocity components are replaced by their mean values over the thickness of the fluid layer. Recently ETSION [14]–[16] has studied this problem using the "short bearing" approximation and creeping-flow conditions.

In the present paper the problem is solved by a perturbation expansion about the known creeping-flow solution for source flow between closely spaced parallel disks rotating with different velocities. A solution which is valid in an annular region is obtained.

2. Analysis

The coordinates (α, β, r) shown as in Fig. 1 and the corresponding velocity components (u, v, w) are used. The surfaces of the two disks are placed at $\alpha = 0$ and $\alpha = \alpha_0$. The disks are rotating about axes perpendicular to the disks at $r = 0$ with the angular velocities $s\omega(-1 \leqslant s \leqslant 1)$ and ω, respectively. The spacing between the disks at the centre of the disks is $d = R\alpha_0$. We consider an annular region with inlet radius kr_0 $(k < 1)$ and outlet radius r_0.

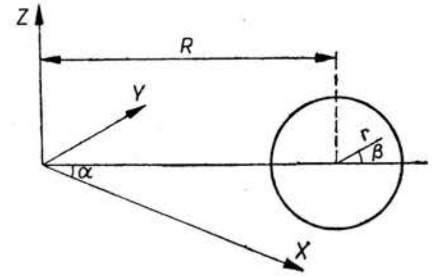

FIG. 1. Coordinate system.

The continuity equation and the Navier-Stokes equations for the incompressible steady flow of constant viscosity can easily be derived from the general equations in curvilinear orthogonal coordinates, see e.g. ROUSE [17]. In the present case

$$X = (R+r\cos\beta)\cos\alpha,$$
$$Y = (R+r\cos\beta)\sin\alpha,$$
$$Z = r\sin\beta.$$

The corresponding scale factors are

$$h_1 = R+r\cos\beta,$$
$$h_2 = r,$$
$$h_3 = 1.$$

The resulting equations are as follows:

$$(2.1)\qquad \frac{r}{\alpha_0}\frac{\partial u}{\partial \xi} + R_\beta \frac{\partial v}{\partial \beta} - r\sin\beta v + rR_\beta \frac{\partial w}{\partial r} + (R+2r\cos\beta)w = 0,$$

$$(2.2)\qquad \frac{1}{\alpha_0 R_\beta} u\frac{\partial u}{\partial \xi} + \frac{1}{r} v\frac{\partial u}{\partial \beta} + w\frac{\partial u}{\partial r} - \frac{\sin\beta}{R_\beta} uv + \frac{\cos\beta}{R_\beta} uw$$

$$= -\frac{1}{\alpha_0 R_\beta \varrho}\frac{\partial p}{\partial \xi} + \nu\left[\frac{1}{\alpha_0^2 R_\beta^2}\frac{\partial^2 u}{\partial \xi^2} - \frac{\sin\beta}{rR_\beta}\frac{\partial u}{\partial \beta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \beta^2}\right.$$

$$+\frac{R+2r\cos\beta}{rR_\beta}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\beta}{\alpha_0 R_\beta^2}\frac{\partial v}{\partial \xi}+\frac{2\cos\beta}{\alpha_0 R_\beta^2}\frac{\partial w}{\partial \xi}-\frac{1}{R_\beta^2}u\Bigg],$$

(2.3)
$$\frac{1}{\alpha_0 R_\beta}u\frac{\partial v}{\partial \xi}+\frac{1}{r}v\frac{\partial v}{\partial \beta}+w\frac{\partial v}{\partial r}+\frac{1}{r}vw+\frac{\sin\beta}{R_\beta}u^2$$
$$=-\frac{1}{r\varrho}\frac{\partial p}{\partial \beta}+\nu\Bigg[\frac{1}{\alpha_0^2 R_\beta^2}\frac{\partial^2 v}{\partial \xi^2}-\frac{\sin\beta}{rR_\beta}\frac{\partial v}{\partial \beta}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \beta^2}+\frac{R+2r\cos\beta}{rR_\beta}\frac{\partial v}{\partial r}$$
$$+\frac{\partial^2 v}{\partial r^2}+\frac{2}{r^2}\frac{\partial w}{\partial \beta}+\frac{2\sin\beta}{\alpha_0 R_\beta^2}\frac{\partial u}{\partial \xi}-\frac{R^2+2Rr\cos\beta+r^2}{r^2R_\beta^2}v-\frac{R\sin\beta}{rR_\beta^2}w\Bigg],$$

(2.4)
$$\frac{1}{\alpha_0 R_\beta}u\frac{\partial w}{\partial \xi}+\frac{1}{r}v\frac{\partial w}{\partial \beta}+w\frac{\partial w}{\partial r}-\frac{1}{r}v^2-\frac{\cos\beta}{R_\beta}u^2=-\frac{1}{\varrho}\frac{\partial p}{\partial r}+\nu\Bigg[\frac{1}{\alpha_0^2 R_\beta^2}\frac{\partial^2 w}{\partial \xi^2}$$
$$-\frac{\sin\beta}{rR_\beta}\frac{\partial w}{\partial \beta}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \beta^2}+\frac{R+2r\cos\beta}{rR_g}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \beta}+\frac{\sin\beta\cos\beta}{R_\beta^2}v$$
$$-\frac{2\cos\beta}{\alpha_0 R_\beta^2}\frac{\partial u}{\partial \xi}-\frac{R^2+2Rr\cos\beta+2r^2\cos^2\beta}{r^2R_\beta^2}w\Bigg],$$

where

$$\xi=\frac{\alpha}{\alpha_0},\quad R_\beta=R+r\cos\beta.$$

If the flow rate, i.e. the strength of the source, is Q, the boundary conditions are

$$u=w=0 \quad \text{at } \xi=0,\quad \xi=1,$$
$$v=sr\omega \quad \text{at } \xi=0,$$
$$v=r\omega \quad \text{at } \xi=1,$$
$$\int_0^1\int_{-\pi}^{\pi} w(R+r\cos\beta)\alpha_0 r\,d\beta\,d\xi=Q.$$

It is further assumed that the pressure is independent of the tangential coordinate β at the boundaries $r=kr_0$ and $r=r_0$.

The gap width d is assumed to be small compared with the disk radius r_0, which in its turn is small compared with R:

$$d\ll r_0\ll R$$

or

(2.5)
$$\alpha_0\ll\frac{r_0}{R}\ll 1.$$

The creeping-flow solution is approached when the Reynolds number

(2.6)
$$\mathrm{Re}=\frac{d^2\omega}{\nu}\ll 1.$$

These assumptions mean that the problem contains three mutually independent dimensionless parameters that are small compared with unity, namely Re, r_0/R and d/r_0. Thus the solution is expressed as a perturbation expansion in powers of these parameters:

$$u = d\omega\left\{H_0(\xi, r)+\mathrm{Re}\,H_1(\xi, r)+\frac{r_0}{R}H_2(\xi, \beta, r)+\frac{d}{r_0}H_3(\xi, r)+ \ldots\right\}, \tag{2.7}$$

$$v = r_0\omega\left\{G_3(\xi, r)+\mathrm{Re}\,G_1(\xi, r)+\frac{r_0}{R}G_2(\xi, \beta, r)+\frac{d}{r_0}G_3(\xi, r)+ \ldots\right\}, \tag{2.8}$$

$$w = r_0\omega\left\{F_0(\xi, r)+\mathrm{Re}\,F_1(\xi, r)+\frac{r_0}{R}F_2(\xi, \beta, r)+\frac{d}{r_0}F_3(\xi, r)+ \ldots\right\}, \tag{2.9}$$

$$p = \frac{\varrho\nu\omega r_0^2}{d^2}\left\{P_0(\xi, r)+\mathrm{Re}\,P_1(\xi, r)+\frac{r_0}{R}P_2(\xi, \beta, r)+\frac{d}{r_0}P_3(\xi, r)+ \ldots\right\}. \tag{2.10}$$

It has been assumed that the solution is axisymmetrical when the limit $r_0/R \to 0$ is taken, i.e. when the disks are parallel. However, no fundamental difficulty is avoided by this restriction. Nonaxisymmetrical boundary conditions can easily be treated, if the unknown functions are dependent on β. Substituting Eqs. (2.7)–(2.10) into Eqs. (2.1)–(2.4) and collecting terms of equal powers of the perturbation parameters yields the following equations:

System 1 (terms of order unity; the solution of this system describes the behaviour in the limit when $\mathrm{Re} = r_0/R = d/r_0 = 0$)

$$\frac{r}{r_0}\frac{\partial H_0}{\partial \xi}+r\frac{\partial F_0}{\partial r}+F_0 = 0,$$

$$\frac{\partial P_0}{\partial \xi} = 0,$$

$$\frac{\partial^2 G_0}{\partial \xi^2} = 0,$$

$$\frac{\partial^2 F_0}{\partial \xi^2} = r_0\frac{\partial P_0}{\partial r}$$

with the boundary conditions

$$F_0(0, r) = F_0(1, r) = 0,$$

$$G_0(0, r) = s\frac{r}{r_0},$$

$$G_0(1, r) = \frac{r}{r_0},$$

$$\int_0^1\int_{-\pi}^{\pi} r_0\omega F_0 R\alpha_0 r\, d\beta\, d\xi = Q.$$

The solution is simply the creeping-flow solution for the case of parallel disks:

$$H_0 = 0, \tag{2.11}$$

(2.12) $$G_0 = \frac{r}{r_0}[(1-s)\xi+s],$$

(2.13) $$F_0 = -\frac{3}{\pi}q\frac{r_0}{r}(\xi^2-\xi),$$

(2.14) $$P_0 = -\frac{6}{\pi}q\ln\frac{r}{r_0}+\text{const},$$

where

$$q = \frac{Q}{dr_0^2\omega}.$$

System 2 (terms of order Re; the solution of this system describes the corrections to system 1 due to inertial effects)

(2.15) $$\frac{r}{r_0}\frac{\partial H_1}{\partial \xi}+r\frac{\partial F_1}{\partial r}+F_1 = 0,$$

(2.16) $$\frac{\partial P_1}{\partial \xi} = 0,$$

(2.17) $$\frac{\partial^2 G_1}{\partial \xi^2} = H_0\frac{\partial G_0}{\partial \xi}+r_0F_0\frac{\partial G_0}{\partial r}+\frac{r_0}{r}F_0G_0,$$

(2.18) $$\frac{\partial^2 F_1}{\partial \xi^2} = r_0\frac{\partial P_1}{\partial r}+H_0\frac{\partial F_0}{\partial \xi}+r_0F_0\frac{\partial F_0}{\partial r}-\frac{r_0}{r}G_0^2$$

with

(2.19) $$\begin{aligned} H_1(0,r) &= H_1(1,r) = 0,\\ G_1(0,r) &= G_1(1,r) = 0,\\ F_1(0,r) &= F_1(1,r) = 0,\\ \int_0^1 F_1\,d\xi &= 0. \end{aligned}$$

Substituting Eqs. (2.11)–(2.13) and (2.16) into Eqs. (2.17)–(2.18) yields

$$\frac{\partial^2 G_1}{\partial \xi^2} = -\frac{6}{\pi}q\frac{r_0}{r}[(1-s)\xi^3+(2s-1)\xi^2-s\xi],$$

$$\frac{\partial^2 F_1}{\partial \xi^2} = r_0\frac{dP_1}{dr}-\frac{9}{\pi^2}q^2\frac{r_0^3}{r^3}(\xi^2-\xi)^2-\frac{r}{r_0}[(1-s)^2\xi^2+2s(1-s)\xi+s^2]$$

which may be integrated to give

(2.20) $$G_1 = -\frac{1}{10\pi}q\frac{r_0}{r}[3(1-s)\xi^5+5(2s-1)\xi^4-10s\xi^3+(3s+2)\xi],$$

(2.21) $$F_1 = \frac{1}{2}r_0\frac{dP_1}{dr}(\xi^2-\xi)-\frac{3}{20\pi^2}q^2\frac{r_0^3}{r^3}(2\xi^6-6\xi^5+5\xi^4-\xi)$$

$$-\frac{1}{12}\frac{r}{r_0}[(1-s)^2\xi^4+4s[(1-s)\xi^3+6s^2\xi^2-(3s^2+2s+1)\xi].$$

Substituting Eq. (2.21) into Eq. (2.15) yields

$$\frac{\partial H_1}{\partial \xi} = -\frac{1}{2}\frac{r_0^2}{r^2}\left(r^2\frac{d^2P_1}{dr^2}+r\frac{dP_1}{dr}\right)(\xi^2-\xi)-\frac{3}{10\pi^2}q^2\frac{r_0^4}{r^4}(2\xi^6-6\xi^5+5\xi^4-\xi)$$
$$+\frac{1}{6}[(1-s)^2\xi^4+4s(1-s)\xi^3+6s^2\xi^2-(3s^2+2s+1)\xi].$$

As there are two boundary conditions $(2.19)_1$ to this first order equation, not only an expression for H_1 but also a differential equation for P_1 are obtained:

$$H_1 = -\frac{3}{70\pi^2}q^2\frac{r_0^4}{r^4}(2\xi^7-7\xi^6+7\xi^5-3\xi^3+\xi^2)+\frac{1}{30}[(1-s)^2\xi^5$$
$$+5s(1-s)\xi^4+(7s^2-4s-3)\xi^3-(3s^2-s-2)\xi^2], \tag{2.22}$$

$$r^2\frac{d^2P_1}{dr^2}+r\frac{dP_1}{dr} = -\frac{27}{35\pi^2}q^2\frac{r_0^2}{r^2}+\frac{1}{5}(3s^2+4s+3)\frac{r^2}{r_0^2}. \tag{2.23}$$

The solution of Eq. (2.23) is

$$P_1 = A+B\ln\frac{r}{r_0}-\frac{27}{140\pi^2}q^2\frac{r_0^2}{r^2}+\frac{1}{20}(3s^2+4s+3)\frac{r^2}{r_0^2}. \tag{2.24}$$

The constant B is determined by substituting Eq. (2.24) into Eq. (2.21) and using the condition $\int\limits_0^1 F_1 d\xi = 0$ $(2.19)_4$. The result is simply $B = 0$.

Hence

$$P_1 = -\frac{27}{140\pi^2}q^2\frac{r_0^2}{r^2}+\frac{1}{20}(3s^2+4s+3)\frac{r^2}{r_0^2}+\text{const}, \tag{2.25}$$

$$F_1 = -\frac{3}{140\pi^2}q^2\frac{r_0^3}{r^3}(14\xi^6-42\xi^5+35\xi^4-9\xi^2+2\xi)$$
$$-\frac{1}{60}\frac{r}{r_0}[5(1-s)^2\xi^4+20s(1-s)\xi^3+3(7s^2-4s-3)\xi^2-2(3s^2-s-2)\xi]. \tag{2.26}$$

System 3 (terms proportional to r_0/R; the solution of this system describes the corrections to system 1 due to inclination of the disks)

$$\frac{r}{r_0}\frac{\partial H_2}{\partial \xi}+\frac{\partial G_2}{\partial \beta}-\frac{r}{r_0}\sin\beta G_0+r\frac{\partial F_2}{\partial r}+\frac{r^2}{r_0}\cos\beta\frac{\partial F_0}{\partial r}+F_2+2\frac{r}{r_0}\cos\beta F_0 = 0, \tag{2.27}$$

$$\frac{\partial P_2}{\partial \xi} = 0,$$

$$\frac{\partial^2 G_2}{\partial \xi^2} = \frac{r_0}{r}\frac{\partial P_2}{\partial \beta}, \tag{2.28}$$

$$\frac{\partial^2 F_2}{\partial \xi^2} = r_0\frac{\partial P_2}{\partial r}+2\frac{r}{r_0}\cos\beta\frac{\partial^2 F_0}{\partial \xi^2}, \tag{2.29}$$

with

$$(2.30)\quad \begin{aligned} &H_2(0,\beta,r) = H_2(1,\beta,r) = G_2(0,\beta,r) = G_2(1,\beta,r) = F_2(0,\beta,r) \\ &\qquad = F_2(1,\beta,r) = 0, \\ &\int\limits_0^1 \int\limits_{-\pi}^{\pi} \left(F_2 + F_0 \frac{r}{r_0}\cos\beta\right) d\beta d\xi = 0 \Rightarrow \int\limits_0^1 \int\limits_{-\pi}^{\pi} F_2 d\beta d\xi = 0. \end{aligned}$$

Substituting Eqs. (2.12) and (2.13) into Eqs. (2.27) and (2.29) yields

$$(2.31)\quad \frac{r}{r_0}\frac{\partial H_2}{\partial \xi} + \frac{\partial G_2}{\partial \beta} - \frac{r^2}{r_0^2}[(1-s)\xi + s]\sin\beta + r\frac{\partial F_2}{\partial r} - \frac{3}{\pi}q(\xi^2 - \xi)\cos\beta + F_2 = 0,$$

$$(2.32)\quad \frac{\partial^2 F_2}{\partial \xi^2} = r_0 \frac{\partial P_2}{\partial r} - \frac{12}{\pi} q\cos\beta.$$

The functions H_2, G_2, F_2 and P_2 are expanded as follows:

$$(2.33)\quad H_2(\xi,\beta,r) = \sum_{n=0}^{\infty} H_{cn}(\xi,r)\cos n\beta + \sum_{n=1}^{\infty} H_{sn}(\xi,r)\sin n\beta,$$

$$(2.34)\quad G_2(\xi,\beta,r) = \sum_{n=0}^{\infty} G_{cn}(\xi,r)\cos n\beta + \sum_{n=1}^{\infty} G_{sn}(\xi,r)\sin n\beta,$$

$$(2.35)\quad F_2(\xi,\beta,r) = \sum_{n=0}^{\infty} F_{cn}(\xi,r)\cos n\beta + \sum_{k=1}^{\infty} F_{sn}(\xi,r)\sin n\beta,$$

$$(2.36)\quad P_2(\beta,r) = \sum_{n=0}^{\infty} P_{cn}(r)\cos n\beta + \sum_{n=1}^{\infty} P_{sn}(r)\sin n\beta.$$

Substituting the expressions (2.34)–(2.36) into Eqs. (2.28) and (2.32) and collecting terms yields

$$\frac{\partial^2 G_{cn}}{\partial \xi^2} = n\frac{r_0}{r} P_{sn},$$

$$\frac{\partial^2 G_{sn}}{\partial \xi^2} = -n\frac{r_0}{r} P_{cn},$$

$$\frac{\partial^2 F_{c1}}{\partial \xi^2} = r_0 \frac{dP_{c1}}{dr} - \frac{12}{\pi} q,$$

$$\frac{\partial^2 F_{cn}}{\partial \xi^2} = r_0 \frac{dP_{cn}}{dr} \qquad (n \neq 1),$$

$$\frac{\partial^2 F_{sn}}{\partial \xi^2} = r_0 \frac{dP_{sn}}{dr}$$

which may be integrated to give

$$G_{cn} = \frac{n}{2}\frac{r_0}{r} P_{sn}(\xi^2-\xi),$$

$$G_{sn} = -\frac{n}{2}\frac{r_0}{r} P_{cn}(\xi^2-\xi),$$

$$F_1 = \frac{1}{2} r_0 \frac{dP_{c1}}{dr}(\xi^2-\xi) - \frac{6}{\pi} q(\xi^2-\xi), \tag{2.37}$$

$$F_{cn} = \frac{1}{2} r_0 \frac{dP_{cn}}{dr}(\xi^2-\xi) \quad (n \neq 1),$$

$$F_{sn} = \frac{1}{2} r_0 \frac{dP_{sn}}{dr}(\xi^2-\xi).$$

Substituting Eqs. (2.33)–(2.35) and (2.37) into Eq. (2.31) and collecting terms yields the differential equations:

$$\frac{\partial H_{c1}}{\partial \xi} = -\frac{1}{2}\frac{r_0^2}{r^2}\left(r^2\frac{d^2P_{c1}}{dr^2} + r\frac{dP_{c1}}{dr} - P_{c1}\right)(\xi^2-\xi) + \frac{9}{\pi} q \frac{r_0}{r}(\xi^2-\xi),$$

$$\frac{\partial H_{cn}}{\partial \xi} = -\frac{1}{2}\frac{r_0^2}{r^2}\left(r^2\frac{d^2P_{cn}}{dr^2} + r\frac{dP_{cn}}{dr} - n^2P_{cn}\right)(\xi^2-\xi) \quad (n \neq 1),$$

$$\frac{\partial H_{s1}}{\partial \xi} = -\frac{1}{2}\frac{r_0^2}{r^2}\left(r^2\frac{d^2P_{s1}}{dr^2} + r\frac{dP_{s1}}{dr} - P_{s1}\right)(\xi^2-\xi) + \frac{r}{r_0}[(1-s)\xi+s],$$

$$\frac{\partial H_{sn}}{\partial \xi} = -\frac{1}{2}\frac{r_0^2}{r^2}\left(r^2\frac{d^2P_{sn}}{dr} + r\frac{dP_{sn}}{dr} - n^2P_{sn}\right)(\xi^2-\xi) \quad (n \geq 2).$$

These equations may be integrated to give the axial velocity functions $H_{c0}, H_{c1}, \ldots$ and a set of differential equations for the pressure functions:

$$H_{cn} \equiv 0,$$

$$H_{s1} = \frac{r}{r_0}[(1+s)\xi^3 - (1+2s)\xi^2 + s\xi],$$

$$H_{sn} \equiv 0 \quad (n \geq 2)$$

and

$$r^2\frac{d^2P_{cn}}{dr^2} + r\frac{dP_{cn}}{dr} - n^2P_{cn} = 0 \quad (n \neq 1),$$

$$r^2\frac{d^2P_{c1}}{dr^2} + r\frac{dP_{c1}}{dr} - P_{c1} = \frac{18}{\pi} q \frac{r}{r_0},$$

$$r^2\frac{d^2P_{s1}}{dr^2} + r\frac{dP_{s1}}{dr} - P_{s1} = -6(1+s)\frac{r^3}{r_0^3},$$

$$r^2\frac{d^2P_{sn}}{dr^2} + r\frac{dP_{sn}}{dr} - n^2P_{sn} = 0 \quad (n \geq 2).$$

The solution is

$$P_{c0} = C_{c0}+D_{c0}\ln\frac{r}{r_0},$$

$$P_{c1} = C_{c1}\frac{r}{r_0}+D_{c1}\frac{r_0}{r}+\frac{9}{\pi}q\frac{r}{r_0}\ln\frac{r}{r_0},$$

$$P_{cn} = C_{cn}\left(\frac{r}{r_0}\right)^n+D_{cn}\left(\frac{r}{r_0}\right)^{-n} \quad (n \geqslant 2),$$

$$P_{s1} = C_{s1}\frac{r}{r^0}+D_{s1}\frac{r_0}{r}-\frac{3}{4}(1+s)\frac{r^3}{r_0^3},$$

$$P_{sn} = C_{sn}\left(\frac{r}{r_0}\right)^n+D_{sn}\left(\frac{r}{r_0}\right)^{-n} \quad (n \geqslant 2).$$

The condition that the pressure is independent of β at the boundaries $r = kr_0$ and $r = r_0$ yields

$$C_{c1} = -D_{c1} = \frac{9}{\pi}q\frac{k^2\ln k}{1-k^2},$$

$$C_{cn} = D_{cn} \equiv 0 \quad (n \geqslant 2),$$

$$C_{s1} = \frac{3}{4}(1+s)(1+k^2),$$

$$D_{s1} = -\frac{3}{4}(1+s)k^2,$$

$$C_{sn} = D_{sn} \equiv 0 \quad (n \geqslant 2).$$

Substituting the expression for P_{c0} into Eq. $(2.37)_4$ and using the condition $\int\limits_0^1\int\limits_{-\pi}^{\pi} F_2\, d\beta 3\xi = 0$ $(2.30)_2$ determines $D_{c0} = 0$.

Hence

$$(2.38) \qquad H_2 = \frac{r}{r_0}[(1+s)\xi^3-(1+2s)\xi^2+s\xi]\sin\beta,$$

$$(2.39) \qquad G_2 = -\frac{9}{2\pi}q\left[\ln\frac{r}{r_0}+\frac{k^2\ln k}{1-k^2}\left(1-\frac{r_0^2}{r^2}\right)\right](\xi^2-\xi)\sin\beta + \frac{3}{8}(1+s)\left(1+k^2-k^2\frac{r_0^2}{r^2}-\frac{r^2}{r_0^2}\right)(\xi^2-\xi)\cos\beta,$$

$$(2.40) \qquad F_2 = \frac{3}{8}(1+s)\left(1+k^2+k^2\frac{r_0^2}{r^2}-3\frac{r^2}{r_0^2}\right)(\xi^2-\xi)\sin\beta + \frac{3}{2\pi}q\left[3\ln\frac{r}{r_0}-1+3\frac{k^2\ln k}{1-k^2}\left(1+\frac{r_0^2}{r^2}\right)\right](\xi^2-\xi)\cos\beta,$$

(2.41) $$P_2 = \frac{3}{4}(1+s)\left[(1+k^2)\frac{r}{r_0} - k^2\frac{r_0}{r} - \frac{r^3}{r_0^3}\right]\sin\beta + \frac{9}{\pi}q\left[\frac{r}{r_0}\ln\frac{r}{r_0} + \frac{k^2\ln k}{1-k^2}\left(\frac{r}{r_0} - \frac{r_0}{r}\right)\right]\cos\beta.$$

System 4 (terms of order d/r_0)

The system is found to be identical to system 1 (with subscripts 3 instead of 0) but with homogeneous boundary conditions.

Hence

(2.42) $$H_3 = G_3 = F_3 = P_3 = 0.$$

Substituting into Eqs. (2.7)–(2.10) yields the final solution:

(2.43) $$u = \frac{1}{30}d\omega\mathrm{Re}[(1-s)^2\xi^5 + 5s(1-s)\xi^4 + (7s^2-4s-3)\xi^3 - (3s^2-s-2)\xi^2] - \frac{3}{70\pi^2}d\omega\mathrm{Re}q^2\frac{r_0^4}{r^4}(2\xi^7 - 7\xi^6 + 7\xi^5 - 3\xi^3 + \xi^2) + d\omega\frac{r}{R}[(1+s)\xi^3 - (1+2s)\xi^2 + s\xi]\sin\beta,$$

(2.44) $$v = r\omega[(1-s)\xi + s] - \frac{1}{20\pi}r_0\omega\mathrm{Re}q\frac{r_0}{r}[6(1-s)\xi^5 + 10(2s-1)\xi^4 - 20s\xi^3 + 2(3s+2)\xi] - \frac{9}{2\pi}r_0\omega\frac{r_0}{R}q\left[\ln\frac{r}{r_0} + \frac{k^2\ln k}{1-k^2}\left(1-\frac{r_0^2}{r^2}\right)\right](\xi^2-\xi)\sin\beta + \frac{3}{8}(1+s)r_0\omega\frac{r_0}{R}\left(1+k^2-k^2\frac{r_0^2}{r^2} - \frac{r^2}{r_0^2}\right)(\xi^2-\xi)\cos\beta,$$

(2.45) $$w = -\frac{3}{\pi}r_0\omega q\frac{r_0}{r}(\xi^2-\xi) - \frac{3}{140\pi^2}r_0\omega\mathrm{Re}\,q^2\frac{r_0^2}{r^3}(14\xi - 42\xi^5 + 35\xi^4 - 9\xi^2 + 2\xi) - \frac{1}{60}r_0\omega\mathrm{Re}\frac{r}{r_0}[5(1-s)^2\xi^4 + 20[(1-s)\xi^3 + 3(7s^2-4s-3)\xi^2 - 2(3s^2-s-2)\xi] + \frac{3}{8}(1+s)r_0\omega\frac{r_0}{R}\left(1+k^2+k^2\frac{r_0^2}{r^2} - 3\frac{r^2}{r_0^2}\right)(\xi^2-\xi)\sin\beta + \frac{3}{2\pi}r_0\omega\frac{r_0}{R}q\left[3\ln\frac{r}{r_0} - 1 + 3\frac{k^2\ln k}{1-k^2}\left(1+\frac{r_0^2}{r^2}\right)\right](\xi^2-\xi)\cos\beta,$$

(2.46) $$\frac{p-p(r_0)}{\varrho\nu\omega r_0^2/d^2} = -\frac{6}{\pi}q\ln\frac{r}{r_0} + \frac{27}{140\pi^2}\mathrm{Re}q^2\left(1-\frac{r_0^2}{r^2}\right)$$

$$+\frac{1}{20}(3s^2+4s+3)\mathrm{Re}\left(\frac{r^2}{r_0^2}-1\right)+\frac{3}{4}(1+s)\frac{r_0}{R}\Bigg[(1+k^2)\frac{r}{r_0}$$

$$-k^2\frac{r_0}{r}-\frac{r^3}{r_0^3}\Bigg]\sin\beta+\frac{9}{\pi}\frac{r_0}{R}q\left[\frac{r}{r_0}\ln\frac{r}{r_0}+\frac{k^2\ln k}{1-k^2}\left(\frac{r}{r_0}-\frac{r_0}{r}\right)\right]\cos\beta.$$

3. Discussion

The solution (2.43)–(2.46) can be compared with the results obtained by SNECK [11]. The assumption $r_0 \ll R$ used in Sneck's pressure distribution yields after some manipulation

$$(3.1)\qquad \frac{p-p(r_0)}{\varrho\nu\omega r_0^2/d^2}=-\frac{6}{\pi}q\ln\frac{r}{r_0}+\frac{1}{20}(3s^2+4s+3)\mathrm{Re}\left(\frac{r^2}{r_0^2}-1\right)$$

$$+\frac{2}{3}(1+s)\frac{r_0}{R}\left[\frac{k^3-1}{\ln k}\ln\frac{r}{r_0}-\frac{r^3}{r_0^3}+1\right]\sin\beta-\frac{18}{\pi}\frac{r_0}{R}q\left[1-\frac{r}{r_0}-\frac{1-k}{\ln k}\ln\frac{r}{r_0}\right]\cos\beta.$$

However, Sneck's solution is valid only for a small volumetric flow rate under the "short bearing" approximation. It can be shown that Eqs. (2.46) and (3.1) coincide if $q \ll 1$, $r \approx r_0$ and $k \approx 1$. In the same way the pressure distribution obtained by ETSION [14], [16], which is valid under the same restrictions and creeping-flow conditions can be shown to agree with Eq. (2.46).

The analytical solution by TAYLOR and SAFFMAN [13] is valid for compressible flow (with $p \propto \varrho$). If the analysis is repeated for incompressible flow, the result is

$$\frac{p-p(r_0)}{\varrho\nu\omega r_0^2/d^2}=\frac{3}{4}\frac{r_0}{R}\left(\frac{r}{r_0}-\frac{r^3}{r_0^3}\right)\sin\beta$$

which is identical to Eq. (2.46) when $q = s = \mathrm{Re} = k = 0$. As was pointed out by Taylor and Saffman, it is obviously a good approximation to replace v and w by their mean values through the thickness of the fluid layer.

It should be noted that the validity of the solution is restricted not only by the assumptions (2.5) and (2.6). Some terms in Eqs. (2.2)–(2.4) become very large for small values of r. As r_0 has been considered to be a typical value of r, this means that terms that have been assumed to be small during the analysis cannot be neglected generally. However, it is possible for any given combination of d/r_0, Re, r_0/R and q to determine a value of k that justifies these assumptions. According to PELECH and SHAPIRO [7] $\mathrm{Re} = 10^{-2}$, $q = 10^{-2}$ and $d/r_0 = 10^{-3}$ are typical values in a practical case. If $r_0/R = 10^{-2}$, it can be shown that 0.1 is an acceptable value of k in this case.

The pressure and the velocity components have been calculated for this case as functions of the tangential coordinate β for different values of s, r/r_0 and ξ. It can be seen that the combination of inclined disks and rotation has a remarkable influence on the

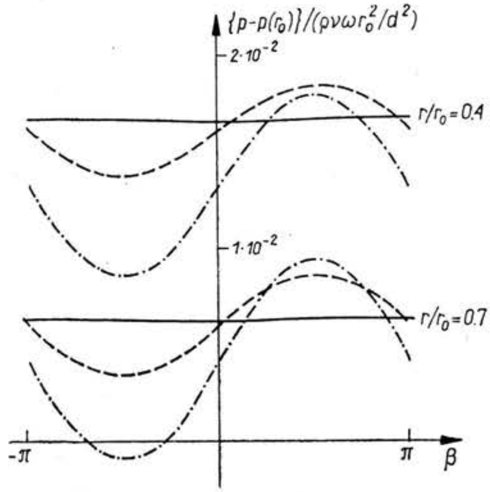

FIG. 2. Pressure at a small volumetric flow rate as a function of β for various values of r/r_0 and s(——, $s = -1$; — — —, $s = 0$; —·—, $s = 1$). $q = 0.01$, $\text{Re} = 0.01$, $r_0/R = 0.01$.

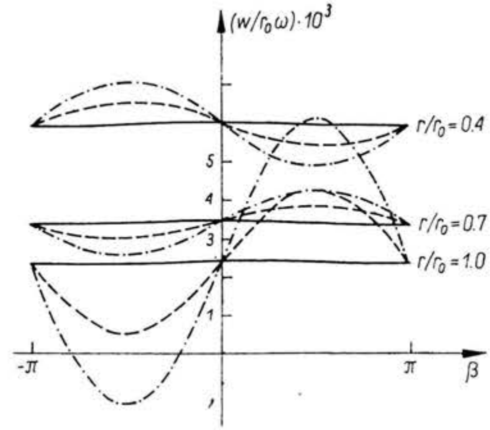

FIG. 3. Radial velocity at a small volumetric flow rate as a function of β for various values of r/r_0 and s (——, $s = -1$, — — —, $s = 0$; —·—, $s = 1$). $q = 0.01$, $\text{Re} = 0.01$, $r_0/R = 0.01$, $k = 0.1$, $\xi = 0.5$.

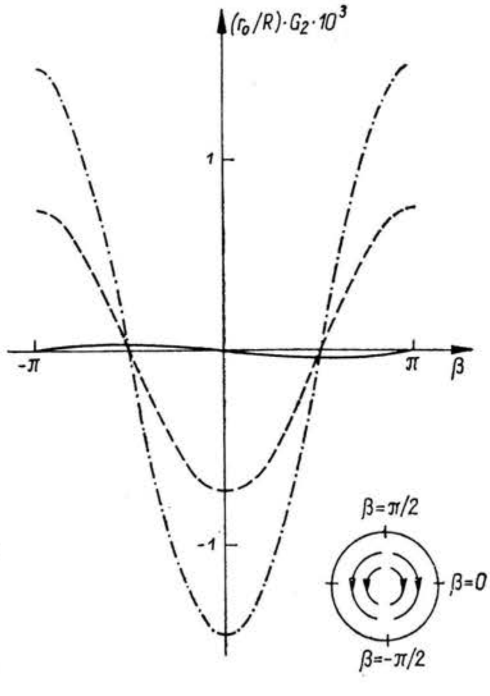

FIG. 4. Perturbation of circumferential velocity due to inclination at a small volumetric flow rate for various values of s (——, $s = -1$; — — —, $s = 0$; —·—, $s = 1$). $q = 0.01$, $\text{Re} = 0.01$, $r_0/R = 0.01$, $k = 0.1$, $\xi = 0.05$, $r/r_0 = 0.4$.

pressure and the radial velocity (Figs. 2–3) except in the case $s = -1$ (counterrotating disks at the same angular velocity). This effect is analogous to that of a journal bearing. If $s = 1$ (corotating disks), the calculated values will even correspond to negative pressure and negative radial velocity, i.e. backflow, in some cases. It should also be noted that

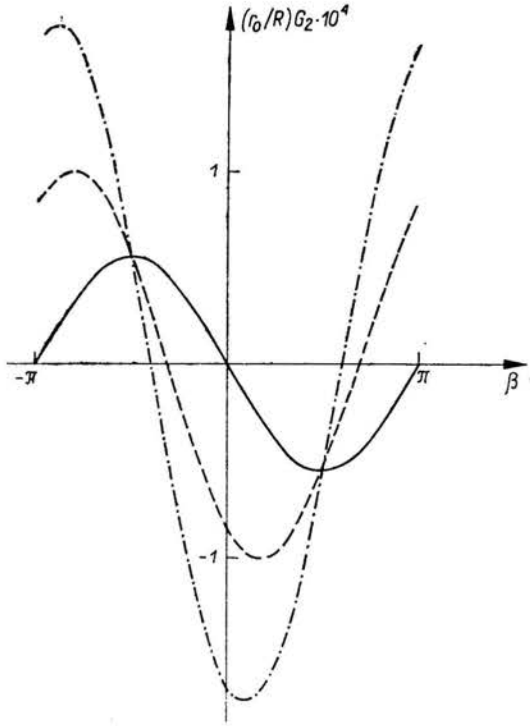

FIG. 5. Perturbation of circumferential velocity due to inclination at a higher volumetric flow rate for various values of s (——, $s = -1$; – – –, $s = 0$, —·—, $s = 1$). $q = 0.5$, Re $= 0.01$, $r_0/R = 0.01$, $k = 0.7$, $\xi = 0.5$, $r/r_0 = 0.85$.

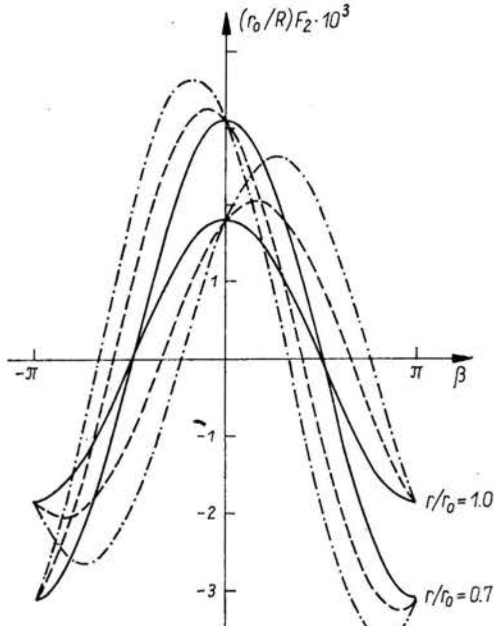

FIG. 6. Perturbation of radial velocity due to inclination at a higher volumetric flow rate for various values of r/r_0 and s (——, $s = -1$; – – –, $s = 0$; —·—, $s = 1$). $q = 0.5$, Re $= 0.01$, $r_0/R = 0.01$, $k = 0.7$, $\xi = 0.5$.

the sign of the angle-dependent velocity term depends on the value of the radial coordinate. The effect of the flow rate (q) is a small pressure increase and a small radial velocity decrease at that part of the region where the disks are closer to each other ($\cos\beta < 0$), as would be expected.

The combination of inclination and source flow will cause a tangential flow from regions of a smaller gap width towards regions of a higher one. If the angular velocity ω is small or even zero (i.e. fixed disks), this contribution will be dominant. The effect of the rotation (if $s \neq -1$) is a tangential flow from $\beta = \frac{\pi}{2}$ towards $\beta = -\frac{\pi}{2}$ (Fig. 4). This angular dependence of the tangential velocity is in complete agreement with the pressure variation, as could be seen from Eqs. (2.44) and (2.46).

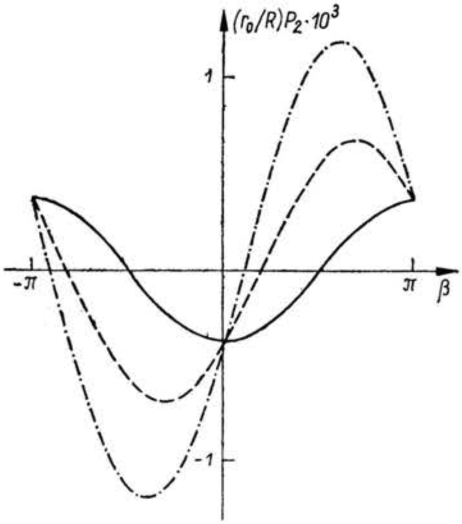

FIG. 7. Perturbation of pressure due to inclination at a higher volumetric flow rate for various of s (——, $s = -1$; – – –, $s = 0$; —·—, $s = 1$). $q = 0.5$, $\mathrm{Re} = 0.01$, $r_0/R = 0.01$, $k = 0.7$, $r/r_0 = 0.85$.

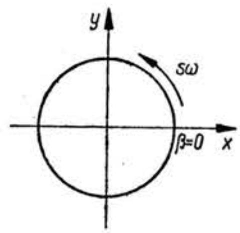

FIG. 8. The disk at $\alpha = 0$ viewed from the outside.

At higher values of q the terms that are independent of β will dominate. The angle-dependent terms of the pressure and the velocity components have been calculated for the case of $q = 0.5$. Some results are presented in Figs. 5–7.

It is now possible to calculate the bending couple exerted by the pressure forces on the disks. With notations according to Fig. 8 the components on the disk $\alpha = 0$ are

(3.2) $$M_x = \frac{\pi}{16}(1+s)\frac{r_0}{R}\frac{\varrho\nu\omega r_0^5}{d^2}(1-k^2)^3 \geqslant 0,$$

(3.3) $$M_y = \frac{9}{4}q\frac{r_0}{R}\frac{\varrho\nu\omega r_0^5}{d^2}\left(k^2\ln k + \frac{1-k^4}{4}\right) > 0.$$

As the pressure is independent of α, the couple on the disk $\alpha = \alpha_0$ is the same but oppositely directed. These results are in agreement with ETSION's results [14], [16] when $k \to 1$ and $s = 0$.

A simple examination shows that if the disks are corotating ($s > 0$), the component M_x tends to change the angular momenta in a way that corresponds to a decrease of the angle α_0. The rotation thus has a stabilizing effect, although the analysis of course assumes that the angle α_0 is fixed. If $s \leqslant 0$, it is not possible to deduce anything about stabilizing tendencies in general. However, if the moments of inertia about the axes of rotation are equal for both disks, it can be shown that the effect is destabilizing. If the disks are non-rotating, only the component M_y exists, which obviously has a restoring effect.

The inclination of the disks will also produce a radial force. The force acting on the disk $\alpha = 0$ has the components

$$F_x = \int_{-\pi}^{\pi}\int_{kr_0}^{r_0} (\tau_{r\alpha}\cos\beta - \tau_\beta \sin\beta) r dr d\beta,$$

$$F_y = \int_{-\pi}^{\pi}\int_{kr_0}^{r_0} (\tau_{r\alpha}\sin\beta + \tau_{\beta\alpha}\cos\beta) r dr d\beta,$$

where $\tau_{r\alpha}$ and $\tau_{\beta\alpha}$ are the shear stress components at $\alpha = 0$. The result is

$$F_x = 3q\frac{r_0}{R}\frac{\varrho\nu\omega r_0^3}{d}(1-k^2), \tag{3.4}$$

$$F_y = 0. \tag{3.5}$$

These results differ from those obtained by ETSION [15] probably because Etsion in part has neglected the circumferential pressure gradient. Thus the value of $\tau_{\beta\alpha}$ will be incorrect. Etsion's results (if $r_0 \ll R$) are

$$F_x = \frac{3}{2}q\frac{r_0}{R}\frac{\varrho\nu\omega r_0^3}{d}(1-k^2),$$

$$F_y = \frac{\pi}{8}\frac{r_0}{R}\frac{\varrho\nu\omega r_0^3}{d}(1+k)^3(1-k).$$

These results are claimed to be valid if $s = 0$ and $k \approx 1$.

The case of a precessing disk can be analyzed in the same way using a rotating coordinate system.

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DIVISION OF MECHANICS,
CHALMERS UNIVERSITY OF TECHNOLOGY, GÖTEBORG, SWEDEN.

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Corrigendum

Source flow between non-parallel rotating disks

P. Å. JANSSON

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Eqs. (3.4)—(3.5) should read:

$$F_x = \frac{3}{2} q \frac{r_0}{R} \frac{\varrho\nu\omega r_0^3}{d} (1-k^2), \tag{3.4}$$

$$F_y = -\frac{\pi}{4} (1-s) \frac{r_0}{R} \frac{\varrho\nu\omega r_0^3}{d} (1-k^4). \tag{3.5}$$

The component F_y still differs from the one obtained by Etsion. However, when the limit $k \to 1$ is taken, the results are in agreement.