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Bending of thin plates in the linear theory of elastic mixtures

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AN APPROXIMATE theory of bending of thin plates of elastic mixtures in developed, based on simplifying assumptions of Kirchhoff's type.

Przedstawiono przybliżoną teorię zginania płyt cienkich wykonanych z mieszanin sprężystych na podstawie założeń upraszczających typu Kirchhoffa.

Представлена приближенная теория изгиба тонких плит, изготовленных из упругих смесей, опираясь на упрощающие предположения типа Кирхгофа.

Notation

σ_{kl}, π_{kl}	partial stresses,
π_l	diffusive force,
e_{kl}, g_{kl}, h_{kl}	characteristics of the deformation,
ω_l, η_l	displacements,
α_2, λ_1, etc.	elastic constants,
ϱ_1, ϱ_2	initial (constant) densities,
δ_{kl}	Kronecker's delta,
$(\ldots)_{,k}$	partial differentation with respect to x_k.

Other symbols are defined as they appear in the text.

1. Introduction

IN THE LAST few years many elasticity problems have been considered in the context of mixture theory. GREEN and NAGHDI [1] proposed a theory for a mixture of two interacting continua, GREEN and STEEL [2] and CROCHET and NAGHDI [3] then derived the constitutive equations for certain types of constituents, and STEEL [4] obtained the linearized equations for an isotropic mixture of two elastic solids.

Starting from these equations and assuming a number of simplifying hypotheses of Kirchhoff's type, we attempt to develop an approximate bending theory for thin plates. This theory reduces upon specialization to the classical one. In order to obtain the equilibrium equations and the boundary conditions of the theory, we use Lagrange's variational principle.

2. Basic formulae

Throughout the paper Greek suffixes take the values 1, 2, Latin suffixes take the values 1, 2, 3 and the convention of summation over repeated indices is understood.

Let B be a Cartesian three-dimensional domain occupied by a mixture of two homogeneous elastic solids, and let S_1, S_2 be two parts of its boundary S, such that $S_1 \cap S_2 = \phi$, $S_1 \cup S_2 = S$. The equations of the linear theory of the mixture are as follows [4]:

1) equilibrium equations (in the absence of body forces):

$$\sigma_{kl,k} - \pi_l = 0, \quad \pi_{kl,k} + \pi_l = 0; \tag{2.1}$$

2) constitutive equations:

$$\begin{aligned} \sigma_{(kl)} &= -\alpha_2 \delta_{kl} + \lambda_1 e_{pp} \delta_{kl} + 2\mu_1 e_{kl} - \lambda_3 g_{pp} \delta_{kl} + 2\mu_3 g_{kl}, \\ \pi_{(kl)} &= \alpha_2 \delta_{kl} + \lambda_2 g_{pp} \delta_{kl} + 2\mu_2 g_{kl} + \lambda_4 e_{pp} \delta_{kl} + 2\mu_3 e_{kl}, \\ \sigma_{[kl]} &= -\pi_{[kl]} = -2\lambda_5 h_{[kl]}, \\ \pi_l &= \frac{\alpha_2}{\varrho} (\varrho_1 g_{pp} + \varrho_2 e_{pp})_{,l}; \end{aligned} \tag{2.2}$$

3) kinematic relations:

$$e_{kl} = \omega_{(k,l)}, \quad g_{kl} = \eta_{(k,l)}, \quad h_{kl} = \eta_{k,l} + \omega_{l,k}; \tag{2.3}$$

4) boundary conditions [5]:

$$\begin{aligned} \omega_l &= \tilde{\omega}_l, \quad \eta_l = \tilde{\eta}_l \quad \text{on } S_1, \\ (\sigma_{kl} + \pi_{kl}) n_k &= \tilde{q}_l, \quad \omega_l - \eta_l = \tilde{u}_l \quad \text{on } S_2, \end{aligned} \tag{2.4}$$

where n_k are the components of the unit outward normal to S.

In Eqs. (2.1)–(2.4) $\varrho = \varrho_1 + \varrho_2$, $\lambda_3 - \lambda_1 = \alpha_2$, and $\phi_{(kl)}$, $\phi_{[kl]}$ are the symmetric and skew-symmetric parts of ϕ_{kl}. The quantities $\tilde{w}_l$, $\tilde{\eta}_l$ and $\tilde{q}_l$, $\tilde{u}_l$ are prescribed on S_1 and S_2, respectively. We assume that all the functions involved in the subsequent calculations have the required degree of smoothness.

We now consider the potential of the diffusive force

$$\pi = \frac{\alpha_2}{\varrho} (\varrho_1 g_{pp} + \varrho_2 e_{pp}) + \varkappa, \quad (\varkappa = \text{arbitrary constant}), \tag{2.5}$$

and observe that the diffusive force may be replaced in Eq. (2.1) by a system of supplementary stresses $-\pi\delta_{kl}$, $\pi\delta_{kl}$. We define the generalized partial stresses t_{kl}, s_{kl} by

$$t_{kl} = \sigma_{kl} - \pi\delta_{kl}, \quad s_{kl} = \pi_{kl} + \pi\delta_{kl}, \tag{2.6}$$

the generalized surface tractions by

$$t_l = t_{kl} n_k, \quad s_l = s_{kl} n_k, \tag{2.7}$$

and the generalized internal energy per unit volume by

$$E = \frac{1}{2} (t_{(kl)} e_{kl} + s_{(kl)} g_{kl} + t_{[kl]} h_{[kl]}). \tag{2.8}$$

REMARK 2.1. The total generalized stresses and the total stresses are equal:

$$t_{kl} + s_{kl} = \sigma_{kl} + \pi_{kl},$$

and Eqs. (2.1) and $(2.2)_4$ remain unaltered by the choice of $\varkappa$.

REMARK 2.2. Taking into account the equilibrium equations and the physical meaning of the diffusive force [1], we may consider that both the total stresses and the supplementary stresses generated by the diffusive force contribute to the stress state of each constituent. If we assume that in the initial state under no applied forces not only the mixture but also each constituent is in equilibrium, it will appear natural to suppose that the generalized stresses are zero. Hence from Eqs. (2.2), (2.5) and (2.6) we obtain $\varkappa = -\alpha_2$.

REMARK 2.3. Operating with the generalized partial stresses and the generalized energy density, some theorems from classical elasticity can easily be extended to the linearized mixture theory [6]. It is to be noted that in this case t_{ki}, s_{ki} are also infinitesimal quantities.

Using Eqs. (2.5)–(2.7), we now re-write Eqs. (2.1), (2.2) and (2.4) as follows:

$$t_{kl,k} = 0, \quad s_{kl,k} = 0; \tag{2.9}$$

$$\begin{aligned} t_{(kl)} &= \left(\lambda_1 - \frac{\varrho_2}{\varrho}\alpha_2\right) e_{pp}\delta_{kl} + 2\mu_1 e_{kl} + \left(\lambda_3 - \frac{\varrho_1}{\varrho}\alpha_2\right) g_{pp}\delta_{kl} + 2\mu_3 g_{kl}, \\ s_{(kl)} &= \left(\lambda_2 + \frac{\varrho_1}{\varrho}\alpha_2\right) g_{pp}\delta_{kl} + 2\mu_2 g_{kl} + \left(\lambda_4 + \frac{\varrho_2}{\varrho}\alpha_2\right) e_{pp}\delta_{kl} + 2\mu_3 e_{kl}, \\ t_{[kl]} &= s_{[kl]} = -2\lambda_5 h_{[kl]}; \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\omega_i = \tilde{\omega}_i, \quad \eta_i = \tilde{\eta}_i \quad \text{on } S_1, \\ &t_i + s_i = \tilde{q}_i, \quad \omega_i - \eta_i = \tilde{u}_i \quad \text{on } S_2. \end{aligned} \tag{2.11}$$

As a particular case of the variational theorem given in [6], Lagrange's variational principle states that for arbitrary small variations of the displacements the change in the generalized energy is equal to the work of the generalized surface forces, i.e.

$$\delta \int_B E dv = \delta \int_S (t_i\omega_i + s_i\eta_i) da,$$

or, which is the same,

$$\delta \int E dv = \frac{1}{2}\delta \int [(t_i + s_i)(\omega_i + \eta_i) + (t_i - s_i)(\omega_i - \eta_i)] da. \tag{2.12}$$

3. Approximate theory

Let us consider a thin plate as defined in [7], and let C be the middle section, c is boundary (closed) curve in the (x_1, x_2)-plane and h the constant thickness of the plate. We assume that C is a regular domain (i.e. permitting the application of the divergence theorem), and that on the faces are prescribed the quantities

$$\tilde{q}_i\left(x_\alpha, \frac{h}{2}\right) = 2p(x_\alpha)\delta_{i3}, \quad \tilde{q}_i\left(x_\alpha, -\frac{h}{2}\right) = 0. \tag{3.1}$$

In order to construct a simplified bending theory, we make the following assumptions:

(i) There is no deformation in the middle plane of the plate.

(ii) Any linear element of the plate, initially normal to the middle plane, remains ormal to the middle surface after bending and its length is unaltered.

(iii) The generalized partial stresses t_{33}, s_{33} in the plate can be neglected with respect to the other components of the generalized stresses.

(iv) The difference $\omega_3-\eta_3$ can be neglected on the faces with respect to $\omega_\alpha-\eta_\alpha$.

(v) There is no shearing effect in either constituent on the faces.

REMARK 3.1. Assumptions (i) and (ii) are the same as in Kirchhoff's theory, and so is (iii) when considering independently the constituents of the mixture.

REMARK 3.2. Assumption (iv) has been introduced on account of the thinness of the plate and will permit us to get boundary conditions of a direct physical significance [5] and to determine the expressions of all the generalized stresses. Assumption (v) is based on mechanical considerations: from Eq. (3.1) we have $\sigma_{3\alpha}+\pi_{3\alpha}=0$ at $x_3=\pm h/2$ and it would appear implausible to assume that $\sigma_{3\alpha}=-\pi_{3\alpha}\neq 0$ on the faces in all situations.

REMARK 3.3. The theory constructed on the basis of (i)–(v) yields the same problems of mathematical rigour as Kirchhoff's, but as in the classical case it is simple and easily applicable.

Using classical arguments, from (i) and (ii) we obtain

$$\omega_3(x_i)=w(x_\alpha),\qquad \eta_3(x_i)=v(x_\alpha), \tag{3.2}$$

$$\omega_\alpha(x_i)=-x_3w_{,\alpha},\qquad \eta_\alpha(x_i)=-x_3v_{,\alpha}. \tag{3.3}$$

Then from Eq. (2.3) and Eqs. (3.2), (3.3)

$$\begin{gathered} e_{\alpha\beta}=-x_3w_{,\alpha\beta},\quad g_{\alpha\beta}=-x_3v_{,\alpha\beta},\quad e_{\alpha3}=g_{\alpha3}=0,\\ h_{[\alpha\beta]}=0,\quad h_{[\alpha3]}=w_{,\alpha}-v_{,\alpha}. \end{gathered} \tag{3.4}$$

As in Kirchhoff's theory (see for instance [8], p. 166) we use (iii) to eliminate e_{33}, g_{33}. Taking $t_{33}=s_{33}=0$ in Eq. (2.10), we obtain

$$e_{33}=c_1e_{\gamma\gamma}+c_2g_{\gamma\gamma},\qquad g_{33}=c_3e_{\gamma\gamma}+c_4g_{\gamma\gamma}, \tag{3.5}$$

where

(3.6)

$$c_0=\left(\lambda_1+2\mu_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_2+2\mu_2+\frac{\varrho_1}{\varrho}\alpha_2\right)-\left(\lambda_3+2\mu_3-\frac{\varrho_1}{\varrho}\alpha_2\right)\left(\lambda_4+2\mu_3+\frac{\varrho_2}{\varrho}\alpha_2\right),$$

$$c_1=\frac{1}{c_0}\left[\left(\lambda_4+\frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_3+2\mu_3-\frac{\varrho_1}{\varrho}\alpha_2\right)-\left(\lambda_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_2+2\mu_2+\frac{\varrho_1}{\varrho}\alpha_2\right)\right],$$

$$c_2=\frac{1}{c_0}\left[\left(\lambda_2+\frac{\varrho_1}{\varrho}\alpha_2\right)\left(\lambda_3+2\mu_3-\frac{\varrho_1}{\varrho}\alpha_2\right)-\left(\lambda_3-\frac{\varrho_1}{\varrho}\alpha_2\right)\left(\lambda_2+2\mu_2+\frac{\varrho_1}{\varrho}\alpha_2\right)\right],$$

$$c_3=\frac{1}{c_0}\left[\left(\lambda_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_4+2\mu_3+\frac{\varrho_2}{\varrho}\alpha_2\right)-\left(\lambda_4+\frac{\varrho_2}{\varrho}\alpha_2\right)\left(\lambda_1+2\mu_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\right],$$

$$c_4=\frac{1}{c_0}\left[\left(\lambda_3-\frac{\varrho_1}{\varrho}\alpha_2\right)\left(\lambda_4+2\mu_3+\frac{\varrho_2}{\varrho}\alpha_2\right)-\left(\lambda_2+\frac{\varrho_1}{\varrho}\alpha_2\right)\left(\lambda_1+2\mu_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\right].$$

From Eqs. (2.10), (3.4), (3.5) it then follows

$$
\begin{aligned}
t_{(\alpha\beta)} &= -x_3[\Delta(d_1 w+d_3 v)\delta_{\alpha\beta}+2(\mu_1 w+\mu_3 v)_{,\alpha\beta}],\\
s_{(\alpha\beta)} &= -x_3[\Delta(d_3 w+d_2 v)\delta_{\alpha\beta}+2(\mu_3 w+\mu_2 v)_{,\alpha\beta}],\\
t_{[\alpha\beta]} &= s_{[\alpha\beta]} = 0,\\
t_{[\alpha 3]} &= -s_{[\alpha 3]} = -2\lambda_5(w-v)_{,\alpha}, \quad \Delta(\dots) = (\dots)_{,\alpha\alpha},
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
d_1 &= (1+c_1)\left(\lambda_1-\frac{\varrho_2}{\varrho}\alpha_2\right)+c_3\left(\lambda_3-\frac{\varrho_1}{\varrho}\alpha_2\right),\\
d_2 &= (1+c_4)\left(\lambda_2+\frac{\varrho_2}{\varrho}\alpha_2\right)+c_2\left(\lambda_4+\frac{\varrho_2}{\varrho}\alpha_2\right),\\
d_3 &= (1+c_4)\left(\lambda_3-\frac{\varrho_1}{\varrho}\alpha_2\right)+c_2\left(\lambda_1-\frac{\varrho_2}{\varrho}\alpha_2\right)\\
&= (1+c_1)\left(\lambda_4+\frac{\varrho_2}{\varrho}\alpha_2\right)+c_3\left(\lambda_2+\frac{\varrho_1}{\varrho}\alpha_2\right).
\end{aligned}
\tag{3.8}
$$

If we take $i=\alpha$ in Eq. (2.1), according to (iv), Eqs. $(3.1)_2$, (3.4) and (3.7), we obtain

$$
\begin{aligned}
t_{(\alpha 3)} &= \frac{4x_3^2-h^2}{8}\Delta[(d_1+2\mu_1)w+(d_3+2\mu_3)v]_{,\alpha}-2\lambda_5(w-v)_{,\alpha},\\
s_{(\alpha 3)} &= \frac{4x_3^2-h^2}{8}\Delta[(d_3+2\mu_3)w+(d_2+2\mu_2)v]_{,\alpha}+2\lambda_5(w-v)_{,\alpha}.
\end{aligned}
\tag{3.9}
$$

Finally, from Eqs. (2.8), (3.4) and (3.7) we have

$$
\begin{aligned}
U &= \int_B E dv = \int_C \int_{-h/2}^{+h/2} (t_{(\alpha\beta)}e_{\alpha\beta}+s_{(\alpha\beta)}g_{\alpha\beta}+2t_{[\alpha 3]}h_{[\alpha 3]})dx_3 da\\
&= \frac{1}{2}\int_C \left\{\frac{h^3}{12}[d_1(\Delta w)^2+2d_3\Delta\Delta v+d_2(\Delta v)^2+2\mu_1 w_{,\alpha\beta}w_{,\alpha\beta}\right.\\
&\quad \left.+4\mu_3 w_{,\alpha\beta}v_{,\alpha\beta}+2\mu_2 v_{,\alpha\beta}v_{,\alpha\beta}]-4h\lambda_5(w-v)_{,\alpha}(w-v)_{,\alpha}\right\} da.
\end{aligned}
\tag{3.10}
$$

We next define

$$
\begin{aligned}
M_{\alpha\beta}^{(1)} &= \int_{-h/2}^{+h/2} x_3 t_{\alpha\beta}dx_3, \quad Q_\alpha^{(1)} = \int_{-h/2}^{+h/2} t_{\alpha 3}dx_3,\\
M_{\alpha\beta}^{(2)} &= \int_{-h/2}^{+h/2} x_3 s_{\alpha\beta}dx_3, \quad Q_\alpha^{(2)} = \int_{-h/2}^{+h/2} s_{\alpha 3}dx_3,
\end{aligned}
\tag{3.11}
$$

and observe that

$$
M_{\alpha\beta} = M_{\alpha\beta}^{(1)}+M_{\alpha\beta}^{(2)}, \quad Q_\alpha = Q_\alpha^{(1)}+Q_\alpha^{(2)}
\tag{3.12}
$$

are the total bending and twisting moments and shearing stress resultants.

Further, we put

$$M_{nn}^{(\gamma)} = M_{\alpha\beta}^{(\gamma)} n_\alpha n_\beta, \quad M_{ns}^{(\gamma)} = \varepsilon_{\lambda\beta} M_{\alpha\beta}^{(\gamma)} n_\alpha n_\lambda, \quad Q_\eta^{(\gamma)} = Q_\alpha^{(\gamma)} n_\alpha, \tag{3.13}$$

where n_α are the components of the unit outward normal to c and $\varepsilon_{\lambda\beta}$ is the alternating symbol in the plane. Then

$$M_{nn} = M_{nn}^{(1)} + M_{nn}^{(2)}, \quad M_{ns} = M_{ns}^{(1)} + M_{ns}^{(2)}, \quad Q_n = Q_n^{(1)} + Q_n^{(2)} \tag{3.14}$$

are the total moments and resultants acting on an elementary section of c. (The direction of the tangent s to c is such that the system of axis (n, s) has the same orientation as (x_1, x_2)). From Eqs. (3.7), (3.11) and (3.13) we obtain

$$\begin{aligned}
M_{nn}^{(1)} &= \frac{h^3}{12}\left\{-\Delta[(d_1+2\mu_1)w+(d_3+2\mu_3)v]+2\left(\frac{1}{r}\frac{\partial}{\partial n}+\frac{\partial^2}{\partial s^2}\right)(\mu_1 w+\mu_3 v)\right\},\\
M_{nn}^{(2)} &= \frac{h^3}{12}\left\{-\Delta[(d_3+2\mu_3)w+(d_2+2\mu_2)v]+2\left(\frac{1}{r}\frac{\partial}{\partial n}+\frac{\partial^2}{\partial s^2}\right)(\mu_3 w+\mu_2 v)\right\},
\end{aligned}$$

$$\begin{aligned}
M_{ns}^{(1)} &= -\frac{h^3}{6}\left(\frac{1}{r}\frac{\partial}{\partial s}-\frac{\partial^2}{\partial s\,\partial n}\right)(\mu_1 w+\mu_3 v),\\
M_{ns}^{(2)} &= -\frac{h^3}{6}\left(\frac{1}{r}\frac{\partial}{\partial s}-\frac{\partial^2}{\partial s\,\partial n}\right)(\mu_3 w+\mu_2 v),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
Q_n^{(1)} &= -\frac{h^3}{12}\frac{\partial}{\partial n}\Delta[(d_1+2\mu_1)w+(d_3+2\mu_3)v]-4h\lambda_5\frac{\partial}{\partial n}(w-v),\\
Q_n^{(2)} &= -\frac{h^3}{12}\frac{\partial}{\partial n}\Delta[(d_3+2\mu_3)w+(d_2+2\mu_2)v]+4h\lambda_5\frac{\partial}{\partial n}(w-v),
\end{aligned}$$

where r is the radius of curvature of c.

On the lateral surface of the plate we denote

$$\begin{aligned}
\tilde{M}_{nn}^{(1)} &= \int_{-h/2}^{+h/2} x_3 t_\alpha n_\alpha dx_3, & \tilde{M}_{nn}^{(2)} &= \int_{-h/2}^{+h/2} x_3 s_\alpha n_\alpha dx_3,\\
\tilde{M}_{ns}^{(1)} &= \int_{-h/2}^{+h/2} x_3 \varepsilon_{\alpha\beta} t_\beta n_\alpha dx_3, & \tilde{M}_{ns}^{(2)} &= \int_{-h/2}^{+h/2} x_3 \varepsilon_{\alpha\beta} s_\beta n_\alpha dx_3,\\
\tilde{Q}_n^{(1)} &= \int_{-h/2}^{+h/2} t_3 dx_3, & \tilde{Q}_n^{(2)} &= \int_{-h/2}^{+h/2} s_3 dx_3,
\end{aligned} \tag{3.16}$$

where s_α are the components of the unit tangent vector to c. Then

$$\tilde{M}_{nn} = \tilde{M}_{nn}^{(1)} + \tilde{M}_{nn}^{(2)}, \quad \tilde{M}_{ns} = \tilde{M}_{ns}^{(1)} + \tilde{M}_{ns}^{(2)}, \quad \tilde{Q}_n = \tilde{Q}_n^{(1)} + \tilde{Q}_n^{(2)} \tag{3.17}$$

are the total moments and resultants acting on c.

The work of the forces acting on the lateral surface is

$$\frac{1}{2}\int_c \Big[-\tilde{M}_{nn}\frac{\partial}{\partial n}(w+v)+\Big(\tilde{Q}_n-\frac{\partial}{\partial s}\tilde{M}_{ns}\Big)(w+v) - \Big(\tilde{M}^{(1)}_{nn}-\tilde{M}^{(2)}_{nn}\frac{\partial}{\partial n}(w-v)+\Big(\tilde{Q}^{(1)}_n-\tilde{Q}^{(2)}_n-\frac{\partial}{\partial s}\tilde{M}^{(1)}_{ns}+\frac{\partial}{\partial s}\tilde{M}^{(2)}_{ns}(w-v)\Big]ds, \tag{3.18}$$

and, according to Eq. (3.1), (iv) and (v), the work of the surface forces acting on the faces is

$$\int_C P(w+v)da. \tag{3.19}$$

Using Eqs. (2.12) and (3.14)–(3.19), integrating by parts and using the divergence theorem, we now obtain

$$\begin{aligned}\int_C \{[\Delta\Delta(D_1 w+D_3 v)+4h\lambda_5\Delta(w-v)-p]\,\delta w \\ +[\Delta\Delta(D_3 w+D_2 v)-4h\lambda_5\Delta(w-v)-p]\,\delta v\}da \\ +\frac{1}{2}\int_c\Big\{(-M_{nn}+\tilde{M}_{nn})\,\delta\frac{\partial}{\partial n}(w+v)+\Big[\Big(Q_n-\frac{\partial}{\partial s}M_{ns}\Big)-\Big(\tilde{Q}_n-\frac{\partial}{\partial s}\tilde{M}_{ns}\Big)\Big]\delta(w+v) \\ +(-M^{(1)}_{nn}+M^{(2)}_{nn}+\tilde{M}^{(1)}_{nn}-\tilde{M}^{(2)}_{nn})\,\delta\frac{\partial}{\partial n}(w-v) \\ +\Big[\Big(Q^{(1)}_n-\frac{\partial}{\partial s}M^{(1)}_{ns}-Q^{(2)}_n+\frac{\partial}{\partial s}M^{(2)}_{ns}\Big)-\Big(\tilde{Q}^{(1)}_n-\frac{\partial}{\partial s}\tilde{M}^{(1)}_{ns}-\tilde{Q}^{(2)}_n+\frac{\partial}{\partial s}\tilde{M}^{(2)}_{ns}\Big)\Big]\delta(w-v)ds,\end{aligned} \tag{3.20}$$

where D_i are the partial rigidities

$$D_i = \frac{h^3}{12}(d_i+2\mu_i). \tag{3.21}$$

By standard arguments from Eq. (3.20) we can derive the equilibrium equations and boundary conditions of the theory. We will restrict our attention only to those of physical interest. Thus we obtain:

(a) equilibrium equations:

$$\begin{aligned}&\Delta\Delta(D_1 w+D_3 v)+4h\lambda_5\Delta(w-v)-p = 0,\\ &\Delta\Delta(D_3 w+D_2 v)-4h\lambda_5\Delta(w-v)-p = 0, \quad \text{in } C;\end{aligned} \tag{3.22}$$

(b) boundary conditions:

$$w = \tilde{w}, \quad v = \tilde{v}, \quad \frac{\partial w}{\partial n} = \tilde{\phi}, \quad \frac{\partial v}{\partial n} = \tilde{\psi} \quad \text{on } c_1; \tag{3.23}$$

$$\begin{aligned}&M_{nn} = \tilde{M}_{nn}, \quad Q_n-\frac{\partial}{\partial s}M_{ns} = \tilde{Q}_n-\frac{\partial}{\partial s}\tilde{M}_{ns},\\ &w-v = \tilde{\nu}, \quad \frac{\partial}{\partial n}(w-v) = \tilde{\chi} \quad \text{on } c_2,\end{aligned} \tag{3.24}$$

where c_1, c_2 are parts of c such that $c_1\cap c_2 = \phi$, $c_1\cup c_2 = c$.

REMARK 3.4. If instead of the mixture being initially isotropic as a whole each solid is initially isotropic, then $\lambda_5 = 0$ [9] and Eq. (3.22) reduces to

$$\Delta\Delta(D_1 w + D_3 v) - p = 0,$$
$$\Delta\Delta(D_3 w + D_2 v) - p = 0.$$

This system yields uncoupled equations of Sophie Germain's type for w and v when the conditions given in [5] for the positive definiteness of U are satisfied.

REMARK 3.5. If the two constituents coincide, we write $\varrho_1 = \varrho_2 = \varrho$, $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$, $\lambda_3 = \lambda_4 = \lambda_5 = \mu_3 = \alpha_2 = 0$, $w = v$ and from Eqs. (2.7), (3.1), (3.6)–(3.8) and (3.21) we obtain

$$D_1 = D_2 = D = \frac{h^3}{12}\frac{E}{1-\sigma^2}, \quad D_3 = 0, \quad \tilde{q}_i\left(x_\alpha, \frac{h}{2}\right) = p(x_\alpha)\delta_{i3},$$

where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and $\sigma = \dfrac{\lambda}{2(\lambda+\mu)}$ are Young's modulus and Poisson's ratio, respectively. Then Eq. (3.22) reduces to Sophie Germain's equation:

$$\Delta\Delta w = \frac{p}{D}$$

and Eqs. (3.23) and (3.24) become the boundary conditions of Kirchhoff's theory [8].

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