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On weak solutions, stability and uniqueness in dynamics of dissipative bodies

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IN ORDER to include shocks in dynamic initial boundary-value problems weak solutions in the class of functions of bounded variation are investigated. The admissibility criterion for the weak solution is formulated. Stability and uniqueness results are obtained in this class of functions.

WE SHALL deal with a deformable dissipative body $\mathscr{B}$ for which constitutive relations of the theory with an internal state variable will be employed. The internal energy ε, the first Piola-Kirchhoff stress $\mathbf{T}$, the temperature ϑ are determined by the deformation gradient $\mathbf{F}$, the specific entropy η and the set of internal state variables $\mathbf{q} = (\mathbf{q}_{(i)})$, $i = 1, 2, \ldots, k$ by the constitutive equations

$$\varepsilon = \varepsilon^*(\mathbf{F}, \eta, \mathbf{q}), \quad \mathbf{T} = \mathbf{T}^*(\mathbf{F}, \eta, \mathbf{q}), \quad \vartheta = \vartheta^*(\mathbf{F}, \eta, \mathbf{q}). \tag{1}$$

These relations are accompanied with the so-called evolution equation for the internal state variable $\mathbf{q}$, namely

$$\dot{\mathbf{q}} = \mathbf{a}^*(\mathbf{F}, \eta, \mathbf{q}), \tag{2}$$

where the superposed dot denotes the material time derivative. The internal variables $\mathbf{q}_{(i)}$, $i = 1, 2, \ldots, k$ may be tensors, scalars or vectors; their geometrical character depends on the physical interpretation given for them. Moreover, they should have some invariance properties or satisfy transformation formulae whenever the spatial or material systems rotate (cf. VALANIS [4]). For the purpose of the present paper we assume that $\mathbf{q}$ behaves as a scalar in the spatial system [3].

The motion $\boldsymbol{\chi}(\mathbf{X}, t)$ determines the velocity field $\mathbf{v} = \dot{\boldsymbol{\chi}}$ and the deformation field $\mathbf{F} = \mathrm{Grad}\boldsymbol{\chi}$.

In what follows we assume that the material is a non-conductor of heat and ϱ is the reference positive mass density.

By a *thermodynamic process* with supply terms $(\mathbf{b}, r)(\mathbf{X}, t)$ we mean fields $(\boldsymbol{\chi}, \eta, \mathbf{q}) \times \times (\mathbf{X}, t)$, where $\mathbf{X} \in \mathscr{B}$, $t \in [0, t_0]$ such that $(\mathbf{v}, \mathbf{F}, \eta, \mathbf{q})(\mathbf{X}, t)$ are functions of locally bounded variation [5], in the sense of Tonelli-Cesari, satisfying the balance laws of momentum and energy in the sense of measures (or distributions)

$$\begin{gathered} \varrho\dot{\mathbf{v}} = \mathrm{Div}\,\mathbf{T} + \varrho\mathbf{b}, \\ \varrho\,\overline{\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)}^{\boldsymbol{\cdot}} = \mathrm{Div}(\mathbf{v}\mathbf{T}) + \varrho\mathbf{b}\cdot\mathbf{v} + \varrho r. \end{gathered} \tag{3}$$

The thermodynamic process is *proper* if it satisfies the Clausius-Duhem inequality in the sense of measures:

$$\dot{\eta}-\frac{r}{\vartheta}\geqslant 0. \tag{4}$$

A theromodynamic process $(\bar{\chi}, \bar{\eta}, \bar{q})(\mathbf{X}, t)$ with supply terms $(\bar{\mathbf{b}}, \bar{r})(\mathbf{X}, t)$ will be called *smooth* ([1]) if the functions $(\bar{\mathbf{v}}, \bar{\mathbf{F}}, \bar{\eta}, \mathbf{q})(\mathbf{X}, t)$ are Lipschitz uniformly continuous on bounded subsets of their domain. For smooth processes one may write the balance laws in reduced form (locally almost everywhere)

$$\begin{aligned} \varrho\dot{\bar{\mathbf{v}}} &= \operatorname{Div}\bar{\mathbf{T}}+\varrho\bar{\mathbf{b}},\\ \varrho\dot{\bar{\varepsilon}} &= \bar{\mathbf{T}}\cdot\dot{\mathbf{F}}+\varrho\bar{r}. \end{aligned} \tag{5}$$

It is a standard result that every smooth process is proper if and only if

$$\mathbf{T}^{*}=\varrho\frac{\partial\varepsilon^{*}}{\partial\mathbf{F}},\quad \vartheta^{*}=\frac{\partial\varepsilon^{*}}{\partial\eta},\quad \mathbf{A}^{*}\cdot\dot{\bar{\mathbf{q}}}\leqslant 0, \tag{6}$$

where

$$\mathbf{A}^{*}(\mathbf{F},\eta,\mathbf{q})=\frac{\partial\varepsilon^{*}}{\partial\mathbf{q}}(\mathbf{F},\eta,\mathbf{q}). \tag{7}$$

NOTE that in a smooth process with the supply terms $(\bar{\mathbf{b}}, \bar{r})$ the energy balance law (5) reduces, under Eq. (6), to

$$-\dot{\bar{\eta}}+\frac{\bar{r}}{\bar{\vartheta}}=\frac{1}{\bar{\vartheta}}\bar{\mathbf{A}}\cdot\dot{\bar{\mathbf{q}}}, \tag{8}$$

where

$$\bar{\mathbf{A}}=\mathbf{A}^{*}(\bar{\mathbf{F}},\bar{\eta},\bar{\mathbf{q}}).$$

Now the main definition arises:

A proper thermodynamic process $(\chi, \eta, \mathbf{q})$ with supply terms is called *admissible* if

$$-\dot{\eta}+\frac{r}{\vartheta}\leqslant\frac{1}{\vartheta}\mathbf{A}^{*}\cdot\mathbf{a}. \tag{9}$$

NOTE(¹) that every proper and smooth process is admissible.

Adopting DI PERNA's idea from [2] and generalizing the definition of DAFERMOS [1] for any two processes: an admissible process $(\chi, \eta, \mathbf{q})$ with the supply terms $(\mathbf{b}, r)$ and a smooth one $(\bar{\chi}, \bar{\eta}, \bar{\mathbf{q}})$ with $(\bar{\mathbf{b}}, \bar{r})$, we introduce two functions H and $\mathbf{G}$ by the following formulae:

$$\begin{aligned} H(\mathbf{X},t) &= \frac{\varrho}{2}(\mathbf{v}-\bar{\mathbf{v}})\cdot(\mathbf{v}-\bar{\mathbf{v}})+\varrho\varepsilon^{*}-\varrho\bar{\varepsilon}^{*}-\bar{\mathbf{T}}\cdot(\mathbf{F}-\bar{\mathbf{F}})-\varrho\bar{\vartheta}^{*}\cdot(\eta-\bar{\eta})\\ &\qquad -\varrho\bar{\mathbf{A}}^{*}\cdot(\mathbf{q}-\bar{\mathbf{q}}),\\ G(\mathbf{X},t) &= -(\mathbf{v}-\bar{\mathbf{v}})(\mathbf{T}-\bar{\mathbf{T}}). \end{aligned} \tag{10}$$

(¹) In the present note the concepts of proper and admissible processes differ from that given by DAFERMOS in [1].

We shall calculate the four-dimensional divergence of the field $(H, \mathbf{G})$ in order to find the evolution of the "distance" between two solutions. We calculate $\dot{H}+\mathrm{Div}\mathbf{G}$ bearing in mind that the usual product differentiation rule applies (in the sense of measures) to the product of a Lipschitz continuous function with a function of bounded variation. Here we find

$$
(11) \qquad \dot{H}+\mathrm{Div}\,\mathbf{G} = \varrho\left(\overline{\varepsilon+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}}\right)^{\cdot}-\varrho\dot{\mathbf{v}}\cdot\bar{\mathbf{v}}+\varrho\bar{\mathbf{v}}\cdot\dot{\bar{\mathbf{v}}}-\varrho\dot{\bar{\varepsilon}}-\dot{\mathbf{T}}\cdot(\mathbf{F}-\bar{\mathbf{F}})
$$

$$
-\bar{\mathbf{T}}\cdot\dot{\mathbf{F}}-\varrho\dot{\bar{\vartheta}}(\eta-\bar{\eta})-\varrho\bar{\vartheta}(\eta-\dot{\bar{\eta}})-\mathrm{Div}(\mathbf{v}\mathbf{T})+(\mathbf{T}-\bar{\mathbf{T}})\cdot\dot{\bar{\mathbf{F}}}+\bar{\mathbf{T}}\cdot\dot{\mathbf{F}}-\varrho\mathbf{v}\cdot\dot{\bar{\mathbf{v}}}
$$

$$
+\bar{\mathbf{T}}\cdot\dot{\bar{\mathbf{F}}}+\mathbf{v}\cdot\mathrm{Div}\,\bar{\mathbf{T}}+\bar{\mathbf{v}}\cdot\mathrm{Div}\,\mathbf{T}-\bar{\mathbf{v}}\cdot\mathrm{Div}\,\bar{\mathbf{T}}-\varrho\dot{\bar{\mathbf{A}}}\cdot(\mathbf{q}-\bar{\mathbf{q}})-\varrho\bar{\mathbf{A}}\cdot(\dot{\mathbf{q}}-\dot{\bar{\mathbf{q}}}).
$$

Observing that $\mathrm{Grad}\,\mathbf{v}=\dot{\mathbf{F}}$, $\mathrm{Grad}\,\bar{\mathbf{v}}=\dot{\bar{\mathbf{F}}}$ and $\mathbf{a}^*(\mathbf{F},\eta,\mathbf{q})=\dot{\mathbf{q}}$, $\bar{\mathbf{a}}=\mathbf{a}^*(\bar{\mathbf{F}},\bar{\eta},\bar{\mathbf{q}})=\dot{\bar{\mathbf{q}}}$ and using the balance laws, we may rewrite Eq. (11) in the form

$$
(12) \qquad \dot{H}+\mathrm{Div}\,\mathbf{G} = \varrho(\mathbf{v}-\bar{\mathbf{v}})\cdot(\mathbf{b}-\bar{\mathbf{b}})
$$

$$
+\left\{\mathbf{T}-\bar{\mathbf{T}}-\frac{\partial\bar{\mathbf{T}}^*}{\partial\mathbf{F}}[\mathbf{F}-\bar{\mathbf{F}}]-\frac{\partial\bar{\mathbf{T}}^*}{\partial\eta}(\eta-\bar{\eta})-\frac{\partial\bar{\mathbf{T}}}{\partial\mathbf{q}}[\mathbf{q}-\bar{\mathbf{q}}]\right\}\cdot\dot{\bar{\mathbf{F}}}+\varrho\left\{\vartheta-\bar{\vartheta}\right.
$$

$$
\left.-\frac{\partial\bar{\vartheta}^*}{\partial\mathbf{F}}\cdot(\mathbf{F}-\bar{\mathbf{F}})-\frac{\partial\bar{\vartheta}^*}{\partial\eta}(\eta-\bar{\eta})-\frac{\partial\bar{\vartheta}^*}{\partial\mathbf{q}}\cdot(\mathbf{q}-\bar{\mathbf{q}})\right\}\cdot\dot{\bar{\eta}}+\varrho\left\{\mathbf{A}-\bar{\mathbf{A}}-\frac{\partial\bar{\mathbf{A}}^*}{\partial\mathbf{F}}[\mathbf{F}-\bar{\mathbf{F}}]\right.
$$

$$
\left.-\frac{\partial\bar{\mathbf{A}}^*}{\partial\eta}(\eta-\bar{\eta})-\frac{\partial\bar{\mathbf{A}}^*}{\partial\mathbf{q}}[\mathbf{q}-\bar{\mathbf{q}}]\right\}\cdot\dot{\bar{\mathbf{q}}}+\{\varrho(r-\bar{r})-\varrho(\vartheta-\bar{\vartheta})\dot{\bar{\eta}}-\varrho\bar{\vartheta}(\dot{\eta}-\dot{\bar{\eta}})
$$

$$
-\varrho(\mathbf{A}-\bar{\mathbf{A}})\cdot\bar{\mathbf{a}}-\varrho\bar{\mathbf{A}}(\mathbf{a}-\bar{\mathbf{a}})\},
$$

where the following identities have been used:

$$
\varrho\frac{\partial\bar{\vartheta}^*}{\partial\mathbf{F}}=\frac{\partial\bar{\mathbf{T}}^*}{\partial\eta}, \qquad \frac{\partial\bar{\mathbf{T}}^*}{\partial\mathbf{q}}\cdot((\mathbf{F}-\bar{\mathbf{F}})\otimes\dot{\bar{\mathbf{q}}})=\varrho\left(\frac{\partial\bar{\mathbf{A}}}{\partial\mathbf{F}}[\mathbf{F}-\bar{\mathbf{F}}]\right)\cdot\dot{\bar{\mathbf{q}}}.
$$

From the equality (8) and the inequality (9) we get for the last term in Eq. (12), denoted by $\mathcal{R}$, the following estimation:

$$
(13) \qquad \mathcal{R}\leqslant\frac{\varrho}{\vartheta}(\vartheta-\bar{\vartheta})(r-\bar{r})-\frac{\varrho\bar{r}}{\bar{\vartheta}\vartheta}(\vartheta-\bar{\vartheta})^2+\varrho(\mathbf{A}-\bar{\mathbf{A}})\cdot(\mathbf{a}-\bar{\mathbf{a}})
$$

$$
-\varrho(\vartheta-\bar{\vartheta})\left(\frac{1}{\vartheta}\mathbf{A}\cdot\mathbf{a}-\frac{1}{\bar{\vartheta}}\bar{\mathbf{A}}\cdot\bar{\mathbf{a}}\right).
$$

Hence Eq. (12) yields

$$
(14) \qquad \dot{H}+\mathrm{Div}\,\mathbf{G}\leqslant\varrho(\mathbf{v}-\bar{\mathbf{v}})\cdot(\mathbf{b}-\bar{\mathbf{b}})+\frac{\varrho}{\vartheta}(\vartheta-\bar{\vartheta})(r-\bar{r})-\frac{\varrho\bar{r}}{\vartheta\bar{\vartheta}}(\vartheta-\bar{\vartheta})^2
$$

$$
+\varrho(\mathbf{A}-\bar{\mathbf{A}})\cdot(\mathbf{a}-\bar{\mathbf{a}})-\varrho(\vartheta-\bar{\vartheta})\left(\frac{1}{\vartheta}\mathbf{A}\cdot\mathbf{a}-\frac{1}{\bar{\vartheta}}\bar{\mathbf{A}}\cdot\bar{\mathbf{a}}\right)
$$

$$
+\left\{\mathbf{T}-\bar{\mathbf{T}}-\frac{\partial\bar{\mathbf{T}}^*}{\partial\mathbf{F}}[\mathbf{F}-\bar{\mathbf{F}}]-\frac{\partial\bar{\mathbf{T}}^*}{\partial\eta}(\eta-\bar{\eta})-\frac{\partial\bar{\mathbf{T}}^*}{\partial\mathbf{q}}[\mathbf{q}-\bar{\mathbf{q}}]\right\}\cdot\dot{\bar{\mathbf{F}}}
$$

$$+\varrho\left\{\vartheta-\bar{\vartheta}-\frac{\partial\bar{\vartheta}^*}{\partial\mathbf{F}}\cdot[\mathbf{F}-\bar{\mathbf{F}}]-\frac{\partial\bar{\vartheta}^*}{\partial\eta}(\eta-\bar{\eta})-\frac{\partial\bar{\vartheta}^*}{\partial\mathbf{q}}\cdot(\mathbf{q}-\bar{\mathbf{q}})\right\}\cdot\dot{\bar{\eta}}$$

$$+\varrho\left\{\mathbf{A}-\bar{\mathbf{A}}-\frac{\partial\bar{\mathbf{A}}}{\partial\mathbf{F}}[\mathbf{F}-\bar{\mathbf{F}}]-\frac{\partial\bar{\mathbf{A}}^*}{\partial\eta}(\eta-\bar{\eta})-\frac{\partial\bar{\mathbf{A}}}{\partial\mathbf{q}}[\mathbf{q}-\bar{\mathbf{q}}]\right\}\cdot\dot{\bar{\mathbf{q}}}.$$

This inequality plays the same role in our analysis as the inequality (3.10) in the Dafermos paper [1] on the hyperelastic nonconductor. Assuming that the function $\mathbf{a}^*$ is Lipschitz continuous and ε^* is of the C^3-class, the right-hand side of Eq. (14) is of quadratic order in $(\mathbf{v}-\bar{\mathbf{v}}, \mathbf{F}-\bar{\mathbf{F}}, \eta-\bar{\eta}, \mathbf{q}-\bar{\mathbf{q}}, \mathbf{b}-\bar{\mathbf{b}}, r-\bar{r})$.

Parallel to Theorem 4.1 in [1] we may formulate

Theorem 1. Assume that $\mathscr{B}$ is bounded and has a finite perimeter ([5]). Let $(\bar{\boldsymbol{\chi}}, \bar{\eta}, \bar{\mathbf{q}})$ be a smooth process defined on $\mathscr{B}\times[0, t_0]$ residing in the convexity region of internal energy ε^*, with the supply terms $(\bar{\mathbf{b}}, r)(\cdot,\cdot)\in\mathscr{L}^2(\mathscr{B}\times[0, t_0])$. Then there exist positive constants δ, α, M, N with the following property:

If $(\boldsymbol{\chi}, \eta, \mathbf{q})$ is any admissible process defined on $\mathscr{B}\times[0, t_0]$ with the supply terms $(\mathbf{b}(\cdot,t), r(\cdot, t)\in\mathscr{L}^1([0, t_0], \mathscr{L}^2(\mathscr{B}))$ and such that

$$|\mathbf{F}(\mathbf{X}, t)-\bar{\mathbf{F}}(\mathbf{X}, t)|+|\eta(\mathbf{X}, t)-\bar{\eta}(\mathbf{X}, t)|+|\mathbf{q}(\mathbf{X}, t)-\bar{\mathbf{q}}(\mathbf{X}, t)|<\delta,$$

$$(\mathbf{X}, t)\in\mathscr{B}\times[0, t_0],$$

$$(\mathbf{v}-\bar{\mathbf{v}})\cdot(\mathbf{T}-\bar{\mathbf{T}})\mathbf{N}\leqslant 0 \text{ on } \partial\mathscr{B}, \quad \mathbf{N} \text{ normal to } \partial\mathscr{B},$$

then we have for any $s\in[0, t_0]$

$$\|(\mathbf{v}-\bar{\mathbf{v}}, \mathbf{F}-\bar{\mathbf{F}}, \eta-\bar{\eta}, \mathbf{q}-\bar{\mathbf{q}})(\cdot, s)\|_{\mathscr{L}^2(\mathscr{B})}\leqslant Me^{\alpha s}\|(\mathbf{v}-\bar{\mathbf{v}}, \mathbf{F}-\bar{\mathbf{F}}, \eta-\bar{\eta}, \mathbf{q}-\bar{\mathbf{q}})(\cdot, 0)\|_{\mathscr{L}^2(\mathscr{B})}+Ne^{\alpha s}\int_0^s\|(\mathbf{b}-\bar{\mathbf{b}}, r-\bar{r})(\cdot, t)\|_{\mathscr{L}^2(\mathscr{B})}dt.$$

Corollary. Let $\mathscr{B}$, $(\bar{\boldsymbol{\chi}}, \bar{\eta}, \bar{\mathbf{q}})(\cdot,\cdot)$ and $(\boldsymbol{\chi}, \eta, \mathbf{q})(\cdot,\cdot)$ be as in Theorem 1. Assume that the corresponding supply terms $(\bar{\mathbf{b}}, \bar{r})(\cdot,\cdot)$ and $(\mathbf{b}, r)(\cdot,\cdot)$ coincide on $\mathscr{B}\times[0, t_0]$ and that both processes originate from the same state, i.e.

$$\bar{\boldsymbol{\chi}}(\mathbf{X}, 0)=\bar{\boldsymbol{\chi}}(X, 0), \quad \mathbf{v}(X, 0)=\bar{\mathbf{v}}(\mathbf{X}, 0), \quad \eta(\mathbf{X}, 0)=\bar{\eta}(\mathbf{X}, 0),$$

$$\mathbf{q}(X, 0)=\bar{\mathbf{q}}(\mathbf{X}, 0), \quad \mathbf{X}\in\mathscr{B}$$

then $(\bar{\boldsymbol{\chi}}, \bar{\eta}, \bar{\mathbf{q}})(\cdot,\cdot)$ and $(\boldsymbol{\chi}, \eta, \mathbf{q})(\cdot,\cdot)$ coincide on $\mathscr{B}\times[0, t_0]$.

It seems not to be difficult to give the appropriate theorems under the assumption of strong ellipticity for the internal energy. It can be done in the way of that given by Dafermos in Sect. 5 of his paper [1].

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