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Plasticity and variable heredity

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VARIABLE heredity is defined as a characteristic of advanced systems prevalently found in the living worlds that can permanently change their internal make-up due to the action of inputs and/or environment. Materials are special examples of such systems; "plasticity" results if mechanical inputs alone change the material properties. For the operational definition of these characteristics a real system is idealized as a black box that receives inputs and emits outputs. Based on a comparison of suitable input-output pairs, a definition of variable and invariable heredity is given. We distinguish between environment-induced variable heredity (example: aging) and that induced by inputs (example: "plasticity"). The operational definition of variable heredity is compared with that of rate-dependence and they are shown to be unrelated. The problems of characterizing materials with "plasticity" effects are discussed. Finally, the operational definition for "plasticity" obtained herein is applied as a necessary condition to various constitutive equations previously proposed for "plasticity". Some of these are shown to be unsuited for "plasticity". Owing to the special nature of "plasticity", a functional constitutive equation showing history dependence in a mathematical sense may fail to reproduce "plasticity".

Zmienną dziedziczność definiuje się jako charakterystykę rozwiniętych układów spotykanych przeważnie w świecie ożywionym, które mogą stale zmieniać swą strukturę wewnętrzną pod wpływem wymuszeń zewnętrznych oraz/lub wpływów środowiska. Materiały są szczególnymi przypadkami takich układów, a "plastyczność" pojawia się wtedy, gdy same wymuszenia mechaniczne prowadzą do zmian własności tych materiałów. Dla skonstruowania operacyjnej definicji tych charakterystyk układ rzeczywisty idealizuje się jako „czarną skrzynkę", która otrzymuje sygnały wejściowe i emituje sygnały wyjściowe. Opierając się na porównaniu odpowiednich par „wejścia-wyjścia" podano definicję zmiennej i niezmiennej dziedziczności. Rozróżnia się zmienną dziedziczność wprowadzoną przez środowisko (np. starzenie) oraz przez wymuszenia (np. „plastyczność"). Definicję zmiennej dziedziczności porównano z definicją wrażliwości na prędkość i wykazano, że są one niezwiązane ze sobą. Omówiono problemy charakterystyki materiałów za pomocą efektów „plastyczności". Operacyjną definicję „plastyczności" wykorzystano jako warunek konieczny do różnych równań konstytutywnych zaproponowanych uprzednio dla „plastyczności". Niektóre z nich okazują się niestosowne dla „plastyczności". Wobec szczególnej natury „plastyczności", równanie konstytutywne wykazujące zależność od historii w sensie matematycznym może zawodzić przy próbie reprodukcji „plastyczności".

Переменная наследственность определяется как характеристика развитых систем, встречающихся прежде всего в живом мире, которые способны постоянно менять свою внутренную структуру под влиянием внещних стимулов или окружающей среды. Материалы являются частными случаями таких систем и „пластичность" появляется тогда, когда одни лищь механические стимулы вызывают изменение материальных свойств. Для получения операционного определения этих характеристик реальная система мыслится в виде „черного ящика" принимающего сигналы на входе и эмитирующего сигналы на выходе. Дается определение переменной и постоянной наследственности на основе сравнения соответствующих сигналов на входе и выходе. Различаем переменную наследственность, вызванную окружающей средой — пример: старение — и вызванную сигналами на входе (внещними стимулами) — пример: „пластичность". Операционное определение переменной наследственности сравнивается с определением на основе зависимости от скорости деформации; показывается, что они независимы. Обсуждаются вопросы характеристик материалов с „пластическими" эффектами. В заключение полученное операционное определение „пластичности" применяется как необходимое условие различных ранее предполагаемых определяющих уравнений для случая „пластичности". Оказывается, что некоторые из них не соответствуют „пластичности" Благодаря специфической природе „пластичности" функциональное определяющее уравнение, описывающее зависимость от истории процесса в математическом смысле может оказаться несостоятельным для описания „пластичности".

1. Introduction

FOR MANY years the theory of plasticity has occupied a special status in solid mechanics. It was almost always assumed that rate-independence, the yield surface and hardening laws are the only valid representation of plastic behavior. Indeed these notions were taken to be the distinguishing features of metal deformation behavior itself. At the same time this theory was not thought to be part of nonlinear continuum mechanics as it has evolved during the last thirty years.

During the last ten years other approaches to plasticity have been proposed which differ from the classical notions. Once one admits that the yield surface and other parts of the classical plasticity theory are not the only mathematical expressions through which the phenomenon plasticity can be modelled, one is immediately confronted with fundamental questions. First, it is necessary to define the unique features of the body of knowledge known as plasticity. Once this is accomplished, the class of mathematical models must be found which conform to this definition and which are suitable for the description of plasticity phenomena. (For a related discussion see [1].)

To many, dislocations are essential to plasticity. In a continuum theory state variables and separately postulated evolution laws are thought to be appropriate models. In other approaches an intrinsic time scale appears important. For a long time the modelling of a yield point appeared to be essential and hypoelasticity theories were proposed. These are just a few of the possible approaches.

In this paper we propose to identify essential phenomena of "plasticity" by suitable thought experiments which are also easily performed in the laboratory. The basis of our identification rests with the well-established fact by which every experienced technician can distinguish an annealed from a cold-worked metal. He compares the stress-strain diagrams of the two samples. The one with the highest stress-strain diagram is the one pertaining to the cold-worked metal.

Although our proposed identification has its origin in this simple observation in a tensile test, it is not limited to this test. Rather we can use other mechanical input (stimuli)–output (response) pairs for the identification of the system.

Our aim is to identify the evolution of our system in time as it is subjected to a given constant environment and suitable mechanical inputs. After some technical preliminaries we come to the main result of the paper. We can precisely identify two basic classes of systems. The first class consists of systems with invariable heredity [2, 3], i.e. systems that are always unchanged regardless of the environment and the stimuli. (Elasticity and viscoelasticity are members of this class.) The second class of systems has variable heredity. In these systems the environment and/or the mechanical inputs can change the *system properties permanently*. If these changes are caused by the environment alone (mechanical stimuli are absent), then we speak of aging. If the system properties are unaffected by the environment but may be changed by suitable mechanical stimuli, we speak of stimulus-induced variable heredity. The history dependence in the sense of plasticity [3] or simply "plasticity" is an example of a special kind of stimulus-induced variable heredity. Aside from these hereditary properties, rate-dependence and the aftereffect are identified for metrials by suitable input-output experiments distinct from those used to identify the hereditary properties.

With these identifications we can establish necessary conditions which a mathematical model must fulfill if it is to represent any one of these phenomena. These conditions are very general but nevertheless permit the clear statement that some of the constitutive equations proposed previously for plasticity do not exhibit variable heredity and are therefore, in our opinion, not a suitable model for plasticity.

2. Preliminaries

Our system of interest is idealized as a black box that receives mechanical inputs or stimuli or forcings $\boldsymbol{\varphi}(\tau)$ on $[0, t]$. The inputs produce outputs or responses or response functions $\boldsymbol{\rho}(t)$ at the present time t which depend on $\boldsymbol{\varphi}(\tau)$ on $[0, t]$. For generality they are introduced as vector (tensor) quantities. The comparison of input-output pairs on some time interval is used to identify the phenomenon of interest. We must be able to compare outputs and to recognize differences between outputs. Therefore, we postulate the existence of a "primitive observer" [4] who will be able to discern whether two outputs for a given stimulus are identical or different. This assumes that our inputs and outputs have a defined zero value and that excursions from zero can be determined. Because of these simple requirements we can only identify very general properties.

We assume that there is a time $\tau = 0$ at which our process starts(1). At this time we have as many *identical* samples of our system as we need. All the samples are in the same condition at $\tau = 0$, i.e. they have had the same method of preparation [5] and are subjected to a constant environment for $\tau \geqslant 0$. The only variables are time and the inputs which we select. The method is not limited to the identification of material properties, it can be applied to living systems as well (2) provided we have conditions which permit the identification of true system properties from the responses.

In the application of our methods to materials we must restrict ourselves to accelerationless, homogeneous motions and mechanical inputs since we want to identify properties of constitutive equations [6]. In [2] we have shown that constant rate tensile testing, creep, relaxation and low-cycle fatigue loading constitute the best experimentally obtainable homogeneous motions in solids and therefore we should use inputs from this set of tests. They can be based on the stress traction or displacement vector. The output is then the displacement or the stress traction vector. By using $\boldsymbol{\varphi}$ and $\boldsymbol{\rho}$ we want to emphasize that kinematics (finite or infinitesimal motions) and the role of stress and strain are immaterial for our identification.

(1) By stipulating this condition we deviate from treatments in continuum mechanics where it is generally assumed that we know the entire history from the distant past to the present time.

(2) An example would be a stem of bacteria in an incubator to which a chemical agent is administered and its effect on the bacteria is observed. Our black box would be identified with the stem of bacteria, the application of the chemical agent starting at $\tau = 0$ would be the stimulus. The increase (decrease) of the number of bacteria for $\tau > 0$ would be the response. We will return to this example later.

3. Forcing function histories

For the identification of our phenomena we need at least three samples, a reference stimulus designated as φ^b and a prior loading history φ^a which starts and ends at zero.

The sequence of histories is given in Fig. 1 and is described mathematically below. We are following a simple sheme in which the same forcing φ^b is applied after various prior histories. The quantities φ^a, φ^b and b are parameters of the test sequence. They are constants for a given test but can be varied from test to test. They must be selected from the set of suitable test histories mentioned above and from the set of time intervals b that are of interest.

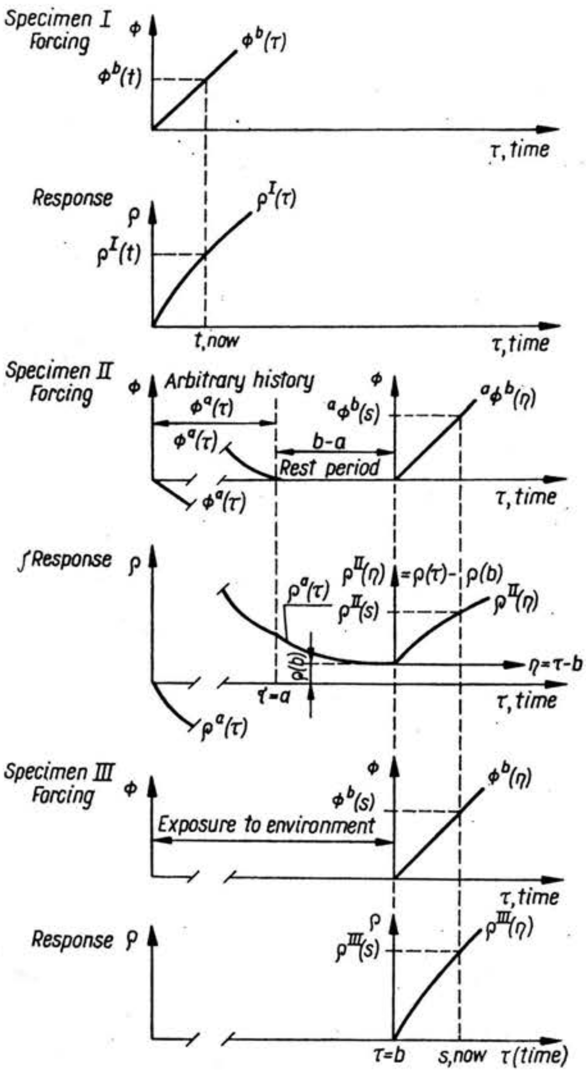

Fig. 1.

Specifically we use the following tests:

Specimen I (Reference)

Input

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^b(\tau), \quad 0 \leqslant \tau \leqslant t, \quad \boldsymbol{\varphi}^b(0) = \mathbf{0}. \tag{3.1}$$

Output

$$\boldsymbol{\rho}^{\mathrm{I}} = \boldsymbol{\rho}^{\mathrm{I}}(\tau), \quad 0 \leqslant \tau \leqslant t, \tag{3.2}$$

where t is the present time.

Specimen II (Stimulus-induced variable heredity)

$$\boldsymbol{\varphi} = \begin{cases} \boldsymbol{\varphi}^a(\tau), & 0 \leqslant \tau \leqslant a; \quad \boldsymbol{\varphi}^a(0) = \boldsymbol{\varphi}^a(\tau \geqslant a) = \mathbf{0}, \\ \mathbf{0}, & a \leqslant \tau \leqslant b, \\ {}^a\boldsymbol{\varphi}^b(\tau - b), & b \leqslant \tau \leqslant b+s, \ (^3) \end{cases} \tag{3.3}$$

$$\boldsymbol{\rho} = \begin{cases} \boldsymbol{\rho}^a(\tau) & 0 \leqslant \tau \leqslant b, \\ {}^a\boldsymbol{\rho}^b(\tau) = \boldsymbol{\rho}^a(b) + \boldsymbol{\rho}^{\mathrm{II}}(\tau - b), & b \leqslant \tau \leqslant b+s. \end{cases} \tag{3.4}$$

Specimen III (Environment-induced variable heredity)

$$\boldsymbol{\varphi} = \begin{cases} \mathbf{0}, & 0 \leqslant \tau \leqslant b, \\ \boldsymbol{\varphi}^b(\tau - b), & b \leqslant \tau \leqslant b+s, \end{cases} \tag{3.5}$$

$$\boldsymbol{\rho} = \begin{cases} \mathbf{0}, & 0 \leqslant \tau \leqslant b, \\ \boldsymbol{\rho}^{\mathrm{III}}(\tau - b), & b \leqslant \tau \leqslant b+s. \end{cases} \tag{3.6}$$

We note that for specimens II and III a new time origin is introduced at $\tau = b$ with $\eta = \tau - b$, see also Fig. 1. Also $\boldsymbol{\rho}^{\mathrm{II}}(s)$ is measured form a new origin, see Fig. 1. The present time in this new system is designated as s. Setting $s = t$ ensures that all forcing functions $\boldsymbol{\varphi}^b$ have the same duration.

3.1. System representation

For the representation of our system we use a functional which is thought to represent our system or material. We require that a zero input on $[0, t]$ produces a zero output at t. Formally we have

$$\boldsymbol{\rho}(t) = \mathbf{K}\big(\boldsymbol{\varphi}(\overset{t}{\underset{0}{\tau}})\big) \ (^4) \tag{3.7}$$

with

$$\mathbf{K}(\mathbf{0}(\overset{t}{\underset{0}{\tau}})) = \mathbf{0},$$

(3) By writing ${}^a\boldsymbol{\varphi}^b$ we want to make clear that this input is preceded by $\boldsymbol{\varphi}^a$; except for the shift on the time axis ${}^a\boldsymbol{\varphi}^b$ is identical to $\boldsymbol{\varphi}^b$.

(4) We mean by this symbolism simply that the present value of $\boldsymbol{\rho} = \boldsymbol{\rho}(\tau = t)$, $t \geqslant 0$ is determined by the function $\boldsymbol{\varphi}(\tau)$ defined on $[0, t]$. From the information given in some of the examples an observer could also conclude $\boldsymbol{\rho}(t) = \mathbf{H}(\boldsymbol{\varphi}(t))$, i.e., the present response $\boldsymbol{\rho}$ is determined only by the present value of $\boldsymbol{\varphi}$. The first conclusion is more general than the second and is therefore retained.

where $\mathbf{0}$ denotes the zero input or output. We do not assign any further properties to $\mathbf{K}$ except it must be such that the conclusions valid for real systems can also be derived using its representation $\mathbf{K}$.

4. Invariable heredity

DEFINITION. Intuitively invariable heredity implies that the system does not change no matter what the stimulus or the environment. Formally we define in reference to Fig. 1 and Eqs. (3.1)–(3.6).

A system is said to have *invariable heredity* if *for all* $t = s$

$$\boldsymbol{\rho}(t)^{\mathrm{I}} = \boldsymbol{\rho}(s)^{\mathrm{II}} = \boldsymbol{\rho}(s)^{\mathrm{III}} \tag{4.1}$$

for all $\boldsymbol{\varphi}^b$, $\boldsymbol{\varphi}^a(\tau)$ and b.

We spak of *invariable hereditary response* if Eq. (4.1) is true for all $\boldsymbol{\varphi}^b$ and at least one b and at least one $\boldsymbol{\varphi}^a(\tau)$.

4.1. Conditions on constitutive equations

Using Eq. (3.7) the responses are computed to be

$$\boldsymbol{\rho}^{\mathrm{I}} = \mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{t}{\underset{0}{\tau}})\big), \tag{4.2}$$

$$\boldsymbol{\rho}^{\mathrm{II}} = \mathbf{K}\big(\boldsymbol{\varphi}^a(\overset{b}{\underset{0}{\tau}}) + {}^a\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau - b}})\big) - \mathbf{K}\big(\boldsymbol{\varphi}^a(\overset{b}{\underset{0}{\tau}})\big), \tag{4.3}$$

and

$$\boldsymbol{\rho}^{\mathrm{III}} = \mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau - b}})\big). \tag{4.4}$$

For Eq. (4.1) to be true *we must have the following properties of* $\mathbf{K}$ *for all* $s = t$:

$$\mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau - b}})\big) = \mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{s}{\underset{0}{\eta}})\big), \tag{4.5}$$

and

$$\mathbf{K}\big(\boldsymbol{\varphi}^a(\overset{a}{\underset{0}{\tau}}) + {}^a\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau - b}})\big) = \mathbf{K}\big(\boldsymbol{\varphi}^a(\overset{a}{\underset{0}{\tau}})\big) + \mathbf{K}\big({}^a\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau - b}})\big), \tag{4.6}$$

as well as

$$\mathbf{K}\big({}^a\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau - b}})\big) = \mathbf{K}\big({}^a\boldsymbol{\varphi}^b(\overset{s}{\underset{0}{\eta}})\big) = \mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{t}{\underset{0}{\tau}})\big). \tag{4.7}$$

If the condition (4.5) — usually referred to as time origin translation invariance — is fulfilled, then $\boldsymbol{\rho}^{\mathrm{I}} = \boldsymbol{\rho}^{\mathrm{III}}$. The condition (4.6) has been called additivity under disjoint support [7] and the principle of superposition. We also list the condition (4.7). It involves first Eq. (4.5) and then requires that the response to $\boldsymbol{\varphi}^b$ following $\boldsymbol{\varphi}^a$ be identical to the response to $\boldsymbol{\varphi}^b$ alone, Eq. (4.2). (Note that Eq. (4.7) is not identical to Eq. (4.5)).

A constitutive equation represents invariable heredity if Eqs. (4.5), (4.6) and (4.7) are true. In words these conditions represent: 1) invariance under time origin translation, 2) additivity under disjoint support, 3) no change in response due to prior forcings.

A constitutive equation represents *invariable hereditary response* if the above hold for all $\boldsymbol{\varphi}^b$ and at least one prior exposure interval b and at least one prior loading $\boldsymbol{\varphi}^a(\tau)$.

It should be noted that the response of specimen II is mesured relative to the new origin introduced at $\eta = 0$. We have, therefore, eliminated the "permanent set" and not attach any significance to this phenomenon[5] for purposes of identification.

5. Variable heredity. Environment and stimulus induced

Following the definition of invariable heredity we not proceed to identify two kinds of variable heredity. For the first kind we observe that only the exposure to the environment changes the response. We then speak of environment-induced variable heredity or simply aging. The other kind of variable heredity can be found in the absence of aging and is solely due to prior inputs. In this case we speak of stimulus-induced variable heredity. In the present context the interaction of these two effects is not considered.

5.1. Environment-induced variable heredity (aging)

DEFINITION. Our system is said to exhibit *aging* if for *some* $s = t$

$$\boldsymbol{\rho}^{\mathrm{I}}(t) \neq \boldsymbol{\rho}^{\mathrm{III}}(s) \tag{5.1}$$

for all b and at least one $\boldsymbol{\varphi}^b_s$.

It exhibits *aging response* if Eq. (5.1) is true for at least one b and *at least one* $\boldsymbol{\varphi}^b$[6].

5.2. Stimulus-induced variable heredity

DEFINITION. In the *absence* of *aging* we speak of stimulus-induced *variable heredity* if for some $s = t$

$$\boldsymbol{\rho}^{\mathrm{I}}(t) \neq \boldsymbol{\rho}^{\mathrm{II}}(s) \tag{5.2}$$

for all $\boldsymbol{\varphi}^a$ and *at least for one* ${}^a\boldsymbol{\varphi}^b$.

Our system exhibits a stimulus-induced *variable hereditary response* if Eq. (5.2) is true for at least one $\boldsymbol{\varphi}^a$ and at least one ${}^a\boldsymbol{\varphi}^b$[7].

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[5] Our opinion deviates therefore from the notions of classical plasticity.

[6] Definition of aging for the example of footnote 2). Following Fig. 1, specimen I, we add a chemical at a certain rate (the stimulus $\boldsymbol{\varphi}$) to the bacteria and observe the change in the number of bacteria with time (the response $\boldsymbol{\rho}$).

On an identical sample of bacteria we repeat the above experiment after b units of time have elapsed, specimen III of Fig. 1. If the change in the number of bacteria is identical to that of the first experiment, then we can conclude that aging does not occur in our sample. If the outcome is different, aging has occurred during b units of time. (To have a valid experiment, the size of the populations at $\tau = 0$ and $\tau = b$ should be identical.)

[7] Stimulus-induced variable heredity in the example of footnote 2). To simulate the conditions of the experiment with specimen II in Fig. 1, we administer a chemical $\boldsymbol{\varphi}^a(\tau)$ during $[0, a]$, then we wait until changes in the population have ceased before we administer the test chemical (${}^a\boldsymbol{\varphi}^b$) in the usual way. There are two possible outcomes: 1) The change in population is identical to that of the test with specimen I; 2) the change in population is different from that of the test with specimen II. If outcome 2) is observed,

5.3. Conditions on constitutive equations

Environment-induced variable heredity (aging). Comparison of Eqs. (4.2), (4.4) and (5.1) shows that the constitutive equation must be time origin translation variant for all b and that at least one $\boldsymbol{\varphi}^b$ represents *aging* so that for some $s = t$:

$$\mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau-b}})\big) \neq \mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{t}{\underset{0}{\tau}})\big). \tag{5.3}$$

To represent *aging response* we require time origin translation variance and therefore Eq. (5.3) to be true for at least one b and at least one $\boldsymbol{\varphi}^b$.

5.4. Stimulus-induced variable heredity

Assuming time origin translation invariance, i.e. that Eq. (4.5) holds always, the constitutive equation must be able to represent nonadditivity under disjoint support or change in response due to prior forcings. Therefore to represent *stimulus-induced variable heredity* we must have for some $s = t$

$$\mathbf{K}\big(\boldsymbol{\varphi}^a(\overset{a}{\underset{0}{\tau}})+{}^a\boldsymbol{\varphi}^b(\overset{s+b}{\underset{b}{\tau-b}})\big)-\mathbf{K}\big(\boldsymbol{\varphi}^a\,(\overset{a}{\underset{0}{\tau}})\big) \neq \mathbf{K}\big(\boldsymbol{\varphi}^b(\overset{t}{\underset{0}{\tau}})\big) \tag{5.4}$$

which implies that either Eq. (4.6) or Eq. (4.7) or both are not true for all $\boldsymbol{\varphi}^a$ and at least one ${}^a\boldsymbol{\varphi}^b$.

To represent *variable hereditary response* the above conditions must hold for at least one $\boldsymbol{\varphi}^a$ and at least one ${}^a\boldsymbol{\varphi}^b$.

The identifications and conditions were given very formally without regard to their physical implications. As an example, the definition of stimulus-induced variable heredity (test with specimens I and II in Fig. 1) says that a form change of the response to the same stimulus is observed. A short reflection will show that this can only be possible *physically* if the prior stimulus $\boldsymbol{\varphi}^a$ has chaged the internal make-up of our system. The systems ordinarily considered in mechanics (gases modelled by elastic balls, solids and fluids represented by springs and dashpots and combinations of them)([8]) do not normally exhibit variable heredity. Indeed only Ref. [8] mentions variable heredity in connection with aging.

It appears that variable heredity is a property of advanced systems that can be encountered in materials and in the living world. Examples from the latter area are the immune reaction cited in the footnotes, the improving effects of exercise in athletics (the muscles strengthen due to prior stimuli), and relations between persons or groups of persons (attitude changes due to prior experience).

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then we must necessarily conclude that the dose $\boldsymbol{\varphi}^a$ given on $[0, a]$ has changed the constitution of the bacteria, e.g. they may have developed an immune reaction (aging is assumed to be absent); we can now speak of stimulus-induced variable heredity. [A valid test requires the same population size at $\tau = 0$ and $\tau = b$.]

([8]) A mere rearrangement of the constituents from one to another random orientation is not sufficient to cause variable heredity.

Before returning to the subject of plasticity it is important to mention that *history dependence in a mathematical sense does not automatically represent variable heredity*. The former is given by Eq. (3.7). *Variable heredity*, however, requires that either Eq. (5.3) or Eq. (5.4) or both hold. Not every functional can satisfy these conditions.

6. Deformation phenomena in materials

6.1. Variable heredity

The previous definitions of course apply to the deformation behavior of *materials* which can show only *variable hereditary response* as we may always find a $\boldsymbol{\varphi}^a$ small enough (interpreted componentwise) such that $\boldsymbol{\rho}^{\mathrm{I}}$ is indistinguishable from $\boldsymbol{\rho}^{\mathrm{II}}$. Also the environment may or may not change the material.

Specifically we consider the definition of variable hereditary response (5.2) as appropriate for the plasticity phenomenon, provided that the *response* are *only different* in *degree* and not in *kind* as described in [3], p. 64 ([9]). This type of variable hereditary response has been called history dependence in the sense of plasticity in [2, 3]. We have given numerous examples which show that metals exhibit this phenomenon [2, 3].

"*Plasticity is a special type of variable hereditary response. A constitutive equation suitable for "plasticity" must as a minimum satisfy* (5.4) *for at least one* $\boldsymbol{\varphi}^a$ *and one* ${}^a\boldsymbol{\varphi}^b$. *Equation* (5.4) *represents a necessary condition for the modelling of "plasticity"*.

A constitutive equation *showing history dependence in a mathematical sense does not necessarily represent history dependence in the sense of plasticity or simply "plasticity"*. Also we note that rate-dependence or rate independence does not enter into this definition.

The above definition excludes the model of an elastic perfectly plastic material from "plasticity" [9]. This fact is not disturbing since the special nature of this model has long been recognized (no growth law for the yield surface is necessary in this case).

It is impossible to infer "plasticity" from one test alone. Only the comparison of two tests (specimens I and II) permits the identification. A given constitutive equation may very well match a stress-strain diagram, or a set of creep curves perfectly without reproducing "plasticity". *The critical test for* a constitutive equation intended for "plasticity" is therefore its behavior during *loading*, *unloading* and *subsequent reloading*.

We know that the physical reason for the observed history dependence in the sense of plasticity rests with the possible permanent microstructural changes induced in materials by deformation. (The material at $\tau = a$ can be different from the material at $\tau = 0$, specimen II in Fig. 1; in metals the dislocation density at the two points may be different by several orders of magnitude.) These changes proceed during deformation. Macroscopically we can only note the difference between $\boldsymbol{\rho}^{\mathrm{I}}$ and $\boldsymbol{\rho}^{\mathrm{II}}$ and we can only use this difference as a criterion for "plasticity". Consequently, we must accept a constitutive equation for "plasticity" as long as it can represent such a difference.

([9]) Living systems may exhibit responses which are different in kind.

The definition of "plasticity" represents a necessary condition. We are presently not able to give necessary and sufficient conditions for the representation of "plasticity".

6.2. Rate-dependence, rate-independence

It is easiest to consider a forcing $\boldsymbol{\varphi}(\tau)$ and an accelerated (retarded) forcing $\boldsymbol{\varphi}(\alpha\tau)$ with $\alpha > 0$. The forcing $\boldsymbol{\varphi}(\tau)$ and the accelerated forcing $\boldsymbol{\varphi}(\alpha\tau)$ reach the same value at $\tau = t$ and $\tau = t/\alpha$, respectively. We speak of rate-independence if

$$\mathbf{K}\big(\boldsymbol{\varphi}(\overset{t}{\underset{0}{\tau}})\big) = \mathbf{K}\big(\boldsymbol{\varphi}(\overset{t/\alpha}{\underset{0}{\alpha\tau}})\big)\ (^{10}) \tag{6.1}$$

is true for all t and all α (see also the definition in [10]). If Eq. (6.1) is not true for some t or some α, we speak of rate-dependence.

It can be shown that rate-dependence implies the existence of an aftereffect and that the modelling of the aftereffect imposes a fading memory on the constitutive functionals. These restrictions are applicable for viscoplastic materials and are shown to be related [11] to the "fading memory hypothesis" imposed in the theory [6] of constitutive functionals.

7. Representation and characterization of materials

The rate-dependence, the aftereffect and fading memory are related to each other. They affect those properties of materials which were called viscous or rheological in [5, 12, 17]. Physically the details of the deformation mechanisms at the present time, i.e. dislocation bowing, are responsible for the presence or absence of viscosity. Mathematically, viscosity involves Eq. (6.1).

Variable heredity, on the other hand, involves the conditions (5.3) and (5.4) which are mathematically separate from Eq. (6.1). Physically variable heredity is caused by the accumulated effects of past exposure to environment or to inputs. They lead to a given microstructure at the present time through internal structural mechanisms [5, 12, 17].

For illustration we have listed physical phenomena and the corresponding necessary mathematical conditions in Table 1 to illustrate our point. The phenomena variable heredity and rate-dependence (aftereffect) are unrelated. The continuum mechanics theory has mostly concentrated on the fading memory aspects and has therefore almost exclusively dealt with the viscosity of deformation behavior. Variable heredity has not yet been generally recognized as an important phenomenon in the evolution in time of material systems. This may be the reason why "plasticity" is not yet included in these theories.

Based on the evidence presented so far it would seem natural to use separate repositories for variable heredity and viscosity (rate-dependence). This approach was followed in [3, 5, 12, 15, 17].

(10) Unfortunately this notation is not completely clear. It is not meant to imply that an equality can always be obtained by introducing a new time variable $\eta = \alpha\tau$. A notation like $\mathbf{K}(\boldsymbol{\varphi}(\overset{t}{\underset{0}{\tau}}), t-\tau)$ is less ambiguous but is less frequently used than $\mathbf{K}(\boldsymbol{\varphi}(\overset{t}{\underset{0}{\tau}}))$.

Table. 1. Various phenomena and necessary conditions on functionals intended for their representation

Name / Necessary condition	Invariable heredity⁽¹⁾	Stimulus induced variable heredity⁽¹⁾	Environment induced variable heredity⁽¹⁾	Rate-dependence	Rate-dependence
$\mathbf{K}\Big(\overset{a}{\underset{0}{\boldsymbol{\varphi}}}(\tau)+{}^{a}\overset{s+b}{\underset{b}{\boldsymbol{\varphi}^{b}}}(\tau-b)\Big)=\mathbf{K}\Big(\overset{a}{\underset{0}{\boldsymbol{\varphi}}}(\tau)\Big)+\mathbf{K}\Big({}^{a}\overset{s+b}{\underset{b}{\boldsymbol{\varphi}^{b}}}(\tau-b)\Big)$ $\mathbf{K}\Big({}^{a}\overset{s+b}{\underset{b}{\boldsymbol{\varphi}^{b}}}(\tau-b)\Big)=\mathbf{K}\Big(\overset{t}{\underset{0}{\boldsymbol{\varphi}^{b}}}(\tau)\Big)$	true for all $s=t$	not true for some $s=t$	true for all $s=t$	unrelated	unrelated
$\mathbf{K}\Big(\overset{s+b}{\underset{b}{\boldsymbol{\varphi}^{b}}}(\tau-b)\Big)=\mathbf{K}\Big(\overset{t}{\underset{0}{\boldsymbol{\varphi}^{b}}}(\eta)\Big)$	true for all $s=t$	true for all $s=t$	not true for some $s=t$	unrelated	unrelated
$\mathbf{K}\Big(\overset{t}{\underset{0}{\boldsymbol{\varphi}}}(\tau)\Big)=\mathbf{K}\Big(\overset{t/\alpha}{\underset{0}{\boldsymbol{\varphi}}}(\alpha\tau)\Big)$	unrelated	unrelated	unrelated	not always true	always true

(1) For simplicity no distinction is made between property and response, Also interactions between stimuli and environment induced effects are excluded,

Since we have defined two kinds of variable heredity it is natural to ask how these two phenomena can be characterized (i.e. what kind of tests are necessary to obtain the material properties) and how one might represent them in constitutive equations.

To characterize an aging system one could vary b for a fixed $\boldsymbol{\varphi}^b$ and then repeat the process for a different $\boldsymbol{\varphi}^b$ until one knows how b and $\boldsymbol{\varphi}^b$ modify the response. A different specimen is necessary for each test.

For the representation of aging an explicit dependence on time of the constitutive equation is frequently used, i.e.

$$\boldsymbol{\rho}(t)=\mathbf{K}\Big(\overset{t}{\underset{0}{\boldsymbol{\varphi}}}(\tau),t\Big). \tag{7.1}$$

The response of specimen I for a material represented by Eq. (7.1) is

$$\boldsymbol{\rho}^{\mathrm{I}}(t)=\mathbf{K}\Big(\overset{t}{\underset{0}{\boldsymbol{\varphi}^{b}}}(\tau),t\Big) \tag{7.2}$$

and of specimen III

$$\boldsymbol{\rho}^{\mathrm{III}}(s)=\mathbf{K}\Big(\overset{s}{\underset{0}{\boldsymbol{\varphi}^{b}}}(\eta),s+b\Big) \tag{7.3}$$

and we see that Eq. (5.1) is satisfied, for all $s=t$.

The representation (7.1) is said to violate the principle of material indifference [13, footnote p. 45]. There are at least two ways of reconciling this potential conflict. One way is to adopt the derivation in [14], Eqs. (2.11) or (2.12). Another way is to consider that the present response is a functional of both the mechanical and the environmental input. Since the latter is constant here, its functional dependence can be "integrated out" leaving only a function of time and a functional of the mechanical input, i.e. Eq. (7.1), [15]. If we therefore interpret Eq. (7.1) not as a fundamental form but rather a specific representation valid for constant environment only, no conflict arises.

For the characterization of stimulus-induced variable heredity we must vary the $\boldsymbol{\varphi}^a$ for a given fixed ${}^a\boldsymbol{\varphi}^b$ and then repeat the process for a different ${}^a\boldsymbol{\varphi}^b$ until we know how the $\boldsymbol{\varphi}^a$ change the response. However, there are difficulties.

Strictly speaking this is a formidable, if not impossible task as one may have to run all conceivable combinations of $\boldsymbol{\varphi}^a$ and ${}^a\boldsymbol{\varphi}^b$. Elsewhere this difficulty has been recognized for metals as evidenced by the statement: "In a strict sense it is not possible to predict the strain components which will be found for a given stress history; the experiment itself must be run to get the answer" [16].

However, the situation is not quite as complex as it appears when we deal with history dependence in the sense of plasticity and with metals. Their responses retain certain characteristics which are invariant with respect to prior deformation [3, pp. 63–66]. Further there are certain stimuli, notably the "elastic ones" which do not appear to cause "plasticity" effects. It may therefore be possible to characterize the history dependence of metals with a limited number of tests and suitable interpolations (see the discussion by E. H. LEE and E. KRÖNER on p. 86 of [16]).

The above definition of history dependence was given for a continuum. Nothing in that definition suggests that a theory of simple materials (in the sense of [6]) would not be capable of reproducing history dependence in the sense of plasticity. Although metals may be "nonsimple bodies" on a microscopic level [16, p. 47] it appears that they can be represented macroscopically within the continuum theory of simple materials as defined in [6].

Suppose two primitive observers are told that they will have to compare the outcome of the tests on the same material with specimen I and II, respectively (see Fig. 1). Observer I witnesses the tests with specimens I and II, Observer II sees *only* the second part ($0 \leqslant \eta \leqslant s$) of the test on specimen II. We stipulate that Eq. (5.2) holds, i.e. we have stimulus-induced variable heredity.

Since they know that the same material is tested, both observers would use the same functional for representing the data. However, because of Eq. (5.2) (we use $\boldsymbol{\varphi}^b$ instead of ${}^a\boldsymbol{\varphi}^b$ since Observer II does not know that $\boldsymbol{\varphi}^a$ was applied)

$$\mathbf{K}\big(\boldsymbol{\varphi}^b \overset{s}{\underset{0}{(\eta)}}\big) \neq \mathbf{K}\big(\boldsymbol{\varphi}^b \overset{t}{\underset{0}{(\tau)}}\big) \tag{7.4}$$

for some $s = t$, a result which contradicts our initial stipulation that one functional can describe a material.

Observer I must necessarily conclude that $\boldsymbol{\varphi}^a$ on $0 \leqslant \tau \leqslant a$ must have changed the material from $\mathbf{K}$ to say $\hat{\mathbf{K}}$. However, there must be a way to obtain $\hat{\mathbf{K}}$ from $\mathbf{K}$. Indeed one can write

$$\mathbf{K}\big(\boldsymbol{\varphi}^a \overset{a}{\underset{0}{(\tau)}} + {}^a\boldsymbol{\varphi}^b \overset{s+b}{\underset{b}{(\tau - b)}}\big) - \mathbf{K}\big(\boldsymbol{\varphi}^a \overset{a}{\underset{0}{(\tau)}}\big) = \hat{\mathbf{K}}\big(\boldsymbol{\varphi}^b \overset{s}{\underset{0}{(\eta)}}\big), \tag{7.5}$$

where we have again used $\boldsymbol{\varphi}^b$ instead of ${}^a\boldsymbol{\varphi}^b$ on the right-hand side to demonstrate that we do not know the forcings for $\eta \leqslant 0$ if we use $\hat{\mathbf{K}}$ as a representation.

The above can be related to commonly accepted notions in materials science. "We regard it as self-evident, then, all current properties of a material are entirely determined by its current state" [18].

The "current state" at $\tau = 0$ and $\tau = b$ are represented in Eq. (7.5) by the material functions and constants in $\mathbf{K}$ and $\hat{\mathbf{K}}$, respectively. The material functions and constants in $\mathbf{K}$ and $\hat{\mathbf{K}}$ must be different to represent the different states implied by the different responses.

The task in materials science is to determine the current state knowing the state at some previous time and what happened to the materials between now and the previous time.

The task in continuum mechanics is similar. Equation (7.5) says that knowing $\mathbf{K}$, i.e. the material functions and constants, at $\tau = 0$ and the forcings on $[0, b]$ should enable us to determine $\hat{\mathbf{K}}$ (the material functions and constants at $\tau = b$).

It is interesting to note that similar ideas were expressed in [5, p. 125]: "To determine the actual thermomechanical state ... it is insufficient to have the actual deformation temperature configuration of a particle $\mathbf{X}$ but we additionally need the method of preparation of this configuration".

Once the need for information on the current state or the method of preparation is recognized one must immediately ask whether all states are equivalent or whether there is a preferred state.

From experience it appears that the annealed state (all the effects of prior mechanical loading have been removed by appropriate heat treatment) is a preferred state. A material can stay in this state if the inputs are small. Sufficiently large inputs will change this state and no mechanical input can return the material to this state.

If the annealed state is left behind by a suitable stimulus, then we can always define a new state relative to which we can characterize the material. The part of the stimulus leading from the annealed state to a new state that caused "plasticity" will together with the old state be absorbed in the new method of preparation or in the new state of the material. As such we can go from one state to the other as implied by Eq. (7.5) Then the initial annealed state does not appear anymore in an *explicit way*.

8. Application of the necessary conditions developed earlier

Here we examine various constitutive equations which have been considered at one time or another as representations of plasticity. We check whether Eq. (5.4) holds. Only if the answer is affirmative do we consider it a valid representation for "plasticity". (Of course there may be other objections to models found to be valid by this procedure.)

8.1. Classical plasticity and classical viscoplasticity

Strains are split either additively, e.g. [19] and others or multiplicatively, e.g. [20] and others, and a growth law for the yield surface must be given such that the elastic range can change under the application of at least one $\boldsymbol{\varphi}^a$ in Fig. 1. The change in the elastic range alone is sufficient so that Eq. (5.4) can hold and therefore most classical theories can represent stimulus-induced variable heredity.

8.2. Linear and nonlinear viscoelasticity

Integral representations of the form

$$(8.1)_1 \qquad \boldsymbol{\rho}(t) = \int_0^t \mathbf{J}(t-\tau)\cdot\frac{d\boldsymbol{\varphi}}{d\tau} + \int_0^t\int_0^t \mathbf{G}(t-\tau_1, t-\tau_2)\cdot\frac{d\boldsymbol{\varphi}}{d\tau_1}\cdot\frac{d\boldsymbol{\varphi}}{d\tau_2}d\tau_1 d\tau_2 + \ldots$$

or

$$(8.1)_2 \qquad \boldsymbol{\rho}(t) = \int_0^t \mathbf{H}\big(t-\tau, \boldsymbol{\varphi}(\tau)\big)\cdot\frac{d\boldsymbol{\varphi}}{d\tau}d\tau$$

or

$$(8.1)_3 \qquad \boldsymbol{\rho}(t) = \boldsymbol{\varphi}(t) + \int_0^t \mathbf{K}(t-\tau)\cdot\mathbf{f}\big(\boldsymbol{\varphi}(\tau)\big)d\tau$$

with $\mathbf{f(0)} = \mathbf{0}$ have the property required in (3.1) and can be written in the form

$$(8.1) \qquad \boldsymbol{\rho}(t) = \mathbf{K}\Big(\overset{t}{\underset{0}{\boldsymbol{\varphi}(\tau)}}, t-\tau\Big).$$

They all show invariable heredity and are unsuitable to model "plasticity"([11]).

8.3. Intrinsic time, endochronic theory, arc length parametrization

These theories postulate the existence of an internal time ([21]–[30] and earlier papers quoted therein) which is mostly based on the second invariant of strain increments. The theories can be formulated for infinitesimal and finite strains and a stress-based time scale has also been proposed [29].

Most theories employ convolution integrals in the intrinsic time scale z and can be written symbolically in a form similar to

$$(8.2) \qquad \boldsymbol{\rho}\big(z(t)\big) = \mathbf{F}\Big(\overset{z(t)}{\boldsymbol{\varphi}(z')}, z-z'\Big).$$

Equation (8.2) exhibits additivity under disjoint support so that Eq. (4.6) holds. The modelling of variable heredity rests with violation of Eq. (4.5) or Eq. (4.7) depending on the relation between the intrinsic and real time, see Appendix I.

Appendix I clearly demonstrates that the introduction of convolution integrals in intrinsic time $z = \hat{z}$ with z defined as

$$(8.3) \qquad \hat{z}(t) = \int_0^t (d\boldsymbol{\varphi}\cdot\mathbf{P}\cdot d\boldsymbol{\varphi})^{1/2}d\tau$$

is not sufficient for the modelling of stimulus-induced variable heredity([12]). An additional

([11]) If the matching of metal stress-strain or creep curves were to be a criterion for "plasticity", then each of these equations could be a valid model.

([12]) Again, if only the matching of tensile and shear stress-strain diagrams is considered to be important to represent "plasticity", then an intrinsic time alone is sufficient.

mapping between $\hat{z}$ and the z must be employed in Eq. (8.2). This important ingredient is normally not considered a part of intrinsic time theories. It is, except for the condition $dz/d\hat{z} > 0$, unrestricted from a theoretical point of view.

If this additional mapping is not used and the theory is formulated only as a convolution integral in the $\hat{z}$-parameter, then it fails to reproduce "plasticity". An example is the theory presented in [24].

If the convolution form in Eq. (8.2) is not used, e.g. if the $z-z'$ dependence is replaced by a z, z' dependence, then $z = \hat{z}$ is sufficient to model plasticity. Then $f(\xi)$ in (A.6) and $f(\zeta)$ in (A. 11) must be constant and g in (A.5) and (A.11) has to be annulled to avoid aging in real time.

Postscript: Initial thoughts on this subject were given in [31]. Since the completion of the first draft of this paper (Spring 1976) the theory of material divagation on the basis of continuum mechanics was formulated [32]. The ideal material defined in [32] has properties similar to our material with invariable heredity.

Appendix I

We subject Eq. (8.2) to Eq. (3.1) and obtain, considering Eq. (3.3)

$$\boldsymbol{\rho}(z_{s+b}) = \mathbf{F}\left(\boldsymbol{\varphi}^a\overset{z_a}{\underset{0}{(z')}}, z-z'\right)+\mathbf{F}\left({}^a\boldsymbol{\varphi}^b\overset{z_{s+b}}{\underset{z_b}{(z'-z_b)}}, z-z'\right), \tag{A.1}$$

where $z_{s+b} = z(\tau = s+b)$, $z_a = z(\tau = a)$ and $z_b = z(\tau = b)$. The second term can be rewritten by introducing $z'' = z'-z_b$ to yield

$$\mathbf{F}\left({}^a\boldsymbol{\varphi}^b\overset{z_{s+b}}{\underset{z_b}{(z')}}, z-z'\right) = \mathbf{F}\left({}^a\boldsymbol{\varphi}^b\overset{z_{s+b}-z_b}{\underset{0}{(z'')}}, z_{s+b}-z_b-z''\right),$$

so that

$$\boldsymbol{\rho}^{\mathrm{II}}(z_s) = \mathbf{F}\left({}^a\boldsymbol{\varphi}^b\overset{z_s}{\underset{0}{(z'')}}, z_s-z''\right), \tag{A.2}$$

where we have the set $z_s = z_{s+b}-z_b$. On the other hand for specimen I of Fig. 1

$$\boldsymbol{\rho}^{\mathrm{I}}(z_t) = \mathbf{F}\left(\boldsymbol{\varphi}^b\overset{z_t}{\underset{0}{(z')}}, z_t-z'\right). \tag{A.3}$$

Since $\boldsymbol{\varphi}^b$ is identical to ${}^a\boldsymbol{\varphi}^b$, we see that the two responses will be equal if the intrinsic times z_s and z_t are equal in both cases. Formally, an intrinsic time formulation represented by Eq. (8.2) will *exhibit invariable heredity* if

$$z_t = z_s \quad \text{for} \quad t = s. \tag{A.4}$$

Before we proceed further we want to remark that the use of convolution integrals in intrinsic time immediately implies additivity under disjoint support, i.e. Eq. (4.6) holds. The repository for history dependence in the sense of plasticity rests entirely in the violation of Eq. (4.7) affected by the intrinsic time z [see Eq. (A4)].

In [29] the following intrinsic time z is postulated, see p.p. 859 and 860:

$$dz = \varkappa[d\zeta^2+g^2d\tau^2]^{1/2} \tag{A.5}$$

with

(A.6) $$d\zeta = \frac{d\xi}{f(\xi)}$$

and

(A.7) $$d\xi = (\dot{\mathbf{E}}\cdot \mathbf{P}\cdot \dot{\mathbf{E}})^{1/2}d\tau = (\dot{\boldsymbol{\varphi}}\cdot \mathbf{P}\cdot \dot{\boldsymbol{\varphi}})^{1/2}d\tau,$$

where $\dot{\mathbf{E}}$ is the material derivative of the finite strain tensor and $\mathbf{P}$ is a positive definite tensor which may depend on $\mathbf{E}$. Since also as tress-based time scale is proposed in [29], we have introduced our forcing function tensor $\boldsymbol{\varphi}$ to cover both cases. Integration of Eq. (A. 5) together with Eqs. (A. 6) and (A. 7) yields

(A.8) $$z_s = \int_0^s \varkappa\left[\left(\frac{d\xi(\tau')}{d\tau'}\,\frac{1}{f\big(\xi(a)+\xi(\tau')\big)}\right)^2+g^2\right]^{1/2} d\tau'.$$

On the other hand we obtain for z in Eq. (A. 3)

(A.9) $$z_t = \int_0^t \varkappa\left[\left(\frac{d\xi(\tau)}{d\tau}\,\frac{1}{f\big(\xi(\tau)\big)}\right)^2+g^2\right]^{1/2} d\tau.$$

A comparison of Eqs. (A. 8) and (A. 9) shows that $z_s = z_t$ for $t = s$ if

(A.10) $$f\big(\xi(a)+\xi(\tau)\big) = f\big(\xi(\tau)\big), \quad 0 \leqslant \tau \leqslant s.$$

The condition (A. 10) is certainly true if f is a constant.

The same procedure can be repeated for the "old endochronic" time proposed in [22, 23] where

(A.11) $$dz = \frac{1}{f(\zeta)}\,\frac{d\zeta}{d\tau}\,d\tau$$

with

$$d\zeta = [d\boldsymbol{\varphi}\cdot \mathbf{P}\cdot d\boldsymbol{\varphi}+g^2 d\tau^2]^{1/2}$$

to yield

(A.12) $$z_s = \int_0^s \frac{1}{f\big(\zeta(b)+\zeta(\tau')\big)}\,[{}^a\dot{\boldsymbol{\varphi}}^b\cdot \mathbf{P}\cdot {}^a\dot{\boldsymbol{\varphi}}^b+g^2]^{1/2}d\tau'$$

and

(A.13) $$z_t = \int_0^t \frac{1}{f\big(\zeta(\tau)\big)}\,[\dot{\boldsymbol{\varphi}}^b\cdot \mathbf{P}\cdot \dot{\boldsymbol{\varphi}}^b+g^2]^{1/2}d\tau.$$

Comparison of Eqs. (A. 12) and (A. 13) yields, noting that $\boldsymbol{\varphi}^b = {}^a\boldsymbol{\varphi}^b$,

(A.14) $$z_s = z_t \quad \text{for} \quad t = s$$

if

$$f\big(\zeta(\tau)\big) = f\big(\zeta(b)+\zeta(\tau)\big), \quad 0 \leqslant \tau \leqslant s.$$

Again the condition (A. 14) is true if f is introduced as a constant.

We can conclude that the introduction of an intrinsic time (A. 7) alone does not guarantee a multiple convolution integral series in $z-z'$ to represent history dependence in the sense of plasticity. The functions f introduced in Eqs. (A. 6) and (A. 11) play a crucial role. Only if f is not a constant can stimulus-induced variable heredity or history dependence in the sense of plasticity be reproduced. This observation coincides with the findings in [23], p. 537 where it is shown that the choice of $f =$ constant precludes the modelling of cross hardening, a true plasticity effect, and which conforms to the definition of history dependence in the sense of plasticity. The material tensor $\mathbf{P}$ in Eq. (A. 7) or Eq. (A. 11) can be set to unity without affecting the outcome of our results.

Further, note that $1/f(\zeta)$ multiplies the expression in square brackets in Eq. (A. 12). Consider now the rate-independent case in Eq. (A. 12), i.e. $g = 0$. To reproduce plasticity f should not be a constant so that $f = f(\zeta)$ is required. Now consider $\mathbf{P} = \mathbf{0}$ but $g \neq 0$, i.e. rate-dependence without plasticity. Then the expressions (A. 12) and (A. 13) will differ. As a consequence the convolution integral series will represent aging in real time. This result may be unacceptable. The new intrinsic time (A. 5) does not suffer from this difficulty, see the discussion on p. 860 of [29].

It is of interest further that $f = f(\xi)$ in Eq. (A. 8) or $f = f(\zeta)$ in Eq. (A. 12) will make the convolution integral series represent *stimulus-induced variable heredity and not stimulus-induced variable hereditary response* as required by our necessary condition. Plasticity effects are therefore introduced for every loading. Elasticity is then excluded form such representations.

Since the function f is absent in [24], see Eqs. (52)–(54) of [24], the functionals employed there represent invariable hereditary. History dependence in the sense of plasticity cannot be represented by this theory of "Thermo-Plastic Materials with Memory".

The importance of the function $f(\zeta)$ in Eq. (A. 11) was also recognized in [33]. It is stated on p. 169 of [33] that a linear function $G[\zeta]$ will preclude the modelling of "cross-hardening". (The derivative of the function $G(\zeta)$ of [33] is equal to $l/f(\zeta)$ of this paper.)

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