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# Entirely Lagrangian nonlinear theory of thin shells

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Equations of equilibrium and appropriate four geometric and static boundary conditions are derived for the general nonlinear theory of thin shells. All shell relations are referred to the undeformed shell middle surface. A modified tensor of change of curvature is used, which is a third-degree polynomial with respect to displacements. A new independent parameter describing the finite rotation of the shell boundary element is introduced, upon which works the boundary couple. All shell equations are consistently simplified in the case of elastic shells undergoing small strains but finite rotations and the Hu-Washizu variational principle is constructed.

Wyprowadzono równania równowagi oraz odpowiednie cztery geometryczne i statyczne warunki brzegowe ogólnej nieliniowej teorii powłok cienkich. Wszystkie uzyskane zależności odniesione zostały do nieodkształconej powierzchni środkowej powłoki. W pracy wykorzystano zmodyfikowany tensor zmiany krzywizny, który jest wielomianem trzeciego stopnia względem przemieszczeń. Wprowadzono nowy, niezależny parametr określający obrót skończony elementu brzegowego powłoki. Wyprowadzone zależności konsekwentnie uproszczono dla przypadku małych sprężystych odkształceń, lecz skończonych obrotów oraz zbudowano zasadę wariacyjną Hu-Washizu.

Выведены уравнения равновесия и соответственные четыре геометрические и статические граничные условия общей нелинейной теории тонких оболочек. Все полученные зависимости отнесены к недеформированной срединной поверхности оболочки. В работе использован модифицированный тензор изменения кривизны, который является полиномом третьей степени относительно перемещений. Введен также новый независимый параметр, описывающий конечный поворот граничного элемента оболочки, на котором работает заданный граничный момент. Полученные зависимости последовательно упрощены для случая малых упругих деформаций, но конечных поворотов, а также построен вариационный принцип Ху-Вашицу.

## 1. Introduction

In the nonlinear static problems of thin elastic shells it is often desirable to have them formulated entirely in terms of quantities and equations defined in and referred to the underformed shell middle surface, the geometry of which is known. In the general theory of thin shells such equilibrium equations and appropriate geometric and static boundary conditions, called the Lagrangian shell equations, are usually derived in two steps. First the corresponding simple relations of the Eulerian shell theory are derived. These relations are referred to the unknown deformed shell middle surface. Then appropriate transformation rules are applied to express the Eulerian quantities in terms of corresponding Lagrangian quantities and shell deformation. The vector relations obtained in this way may then be decomposed with reference to the deformed [1-13] or undeformed base vectors of the shell middle surface. Only the latter approach leads to the relations of the Lagrangian nonlinear theory of shells [11-20].

The above geometric interpretation of derivation of the Lagrangian shell equations reveals one weak point concerned with the proper formulation of the fourth boundary condition for the resultant boundary couple. The latter performs work on an appropriate parameter describing the finite rotation of the shell boundary element. Note that in the Eulerian theory this condition requires the assumption, at the deformed boundary, of a couple whose axial vector is tangent to and measured per unit length of the deformed boundary contour. An appropriate transformation rule may modify the couple in such a manner that it can be measured per unit length of the undeformed boundary, but the axial vector of the transformed couple still remains tangent to the unknown deformed boundary. This is why the parameter $\beta_\nu$ used in [12, 13, 21], which together with three displacement components entirely describes an arbitrary deformation of the shell boundary element, was defined with respect to the deformed boundary. However, such form of the boundary condition as not fully Lagrangian is incompatible with other fully Lagrangian shell relations. It makes the proper formulation of the nonlinear shell problems more complicated when a conservative load is applied to the shell lateral boundary surface. In particular, it is difficult to construct appropriate variational principles of the general Lagrangian nonlinear shell theory.

The fully compatible Lagrangian thin shell equations may also be derived directly from the Lagrangian form of the two-dimensional principle of virtual displacements [14]. The direct derivation was actually used in classical works by MARGUERRE [23] and VLASOV [24], who discussed the nonlinear theory of shallow shells, and in the works by MUSHTARI and GALIMOW [8], SANDERS [17], KOITER [4] and PIETRASZKIEWICZ [11–13, 16] concerned with various variants of the nonlinear theory of shells undergoing moderate rotations. The results obtained there allowed to construct various variational principles [25, 26]. Let us note, however, that in kinematic relations of the simple variants of the theory of shells the nonlinear (quadratic) terms appear only in the definition of the surface strain tensor, while the tensor of change of curvature is still a linear function of displacements and their derivatives. As a result, the boundary condition for the couple takes in these cases a form identical with that of the linear shell theory and becomes indistinguishable within the Eulerian or Lagrangian theory. To the best of our knowledge, nobody has succeeded as yet in deriving the fully Lagrangian set of shell equations, together with a proper fourth boundary condition for the boundary couple, within the variants of the nonlinear shell theory more general than that with moderate rotations.

Within the shell theory undergoing small strains but finite (unrestricted) rotations the tensor of change of curvature $\varkappa_{\alpha\beta}$, defined as a difference between the curvature tensors of the deformed and undeformed shell middle surfaces, is a polynomial of the fifth degree with respect to displacements and their surface derivatives [12, 13]. When we introduced it directly into the Lagrangian principle of virtual displacements [14] and applied Stokes' theorem, we obtained in the resulting boundary line integral not only four expected terms with variations of three displacements and of the parameter $\beta_\nu$ — there also appeared some additional terms with variations of derivatives of displacements in the outward normal direction. These additional terms could not have been eliminated through integration by parts along the undeformed boundary. Therefore, it becomes necessary to introduce modified definitions for the tensor of change of curvature and for the fourth parameter describing deformation of the shell boundary element.

In this work a completely consistent set of Lagrangian shell equations is derived for the nonlinear theory of thin shells under the Kirchhoff–Love constraints. Starting from the Lagrangian form of the two-dimensional principle of virtual displacements, the internally compatible equilibrium equations and all four static boundary conditions are given. In comparison with other related works [12–14, 18] we present here three modifications:

1. In place of the parameter $\beta_\nu$ the new parameter $n_\nu$ is introduced. This parameter is related to the undeformed shell boundary and, together with three displacements, describes entirely an arbitrary deformation of the shell boundary element.

2. The tensor of change of curvature $\varkappa_{\alpha\beta}$ is modified by means of some geometric identities so as to obtain in the resulting line integral at the boundary only terms with variations of three displacements and of the parameter $n_\nu$. The variation of $n_\nu$ includes all terms with derivatives of displacements in the outward normal direction.

3. In place of the modified $\varkappa_{\alpha\beta}$ a new modified measure of change of curvature $\chi_{\alpha\beta}$ is introduced which, by definition, is a third-degree polynomial in displacements and their surface derivatives.

In the shell relations an external surface load, applied to the shell middle surface, and the boundary load, applied to the lateral shell boundary surface, are taken into account. In accordance with the Lagrangian theory directions of these loads are assumed to remain constant during the shell deformation. It is shown that even for such a simple boundary load the resultant boundary couple depends in a definite manner on the total finite rotation of the shell boundary element. The work of the variable couple is replaced here by a much simpler formula expressed by means of the constant resultant static moment of the boundary load. This enables us to construct a completely compatible Hu-Washizu type variational principle for the Lagrangian nonlinear theory of thin elastic shells undergoing small strains but finite rotations.

In the nonlinear shell theory several quantities and relations are defined or derived in the form of differences of two groups of terms of the same order. When strains are assumed to be small, the principal terms of these groups may cancel with each other and the secondary terms may become of primary importance. In our opinion it is the main source of inconsistencies or even errors which have appeared in several papers where the small strain assumption has been introduced at a too early stage of derivation of shell equations. In order to avoid inconsistencies of this kind, we derive our two-dimensional relations for thin shells as far as possible for unrestricted surface strains and rotations. The small strain assumption is introduced only at the end of derivation to simplify the resulting shell equations.

## 2. Notation and basic relations

The notation used in this work follows that of [4, 11–16].

Let $\mathscr{M}$ be a middle surface of the undeformed shell and let $\mathscr{C}$ be its boundary contour. Any point $M$ of $\mathscr{M}$ is uniquely described by assuming at $\mathscr{M}$ two curvilinear coordinates $\theta^1$, $\theta^2$ or a position vector

(2.1) $$\mathbf{r} = \overrightarrow{OM} = x^k\mathbf{i}_k = f^k(\theta^\alpha)\mathbf{i}_k, \quad k = 1,2,3, \quad \alpha = 1,2,$$

where $x^k$ are rectangular coordinates of $M \in \mathscr{M}$ in a Cartesian frame.

Let $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$ be the natural base vectors at $\mathscr{M}$ of the coordinate system $\theta^\alpha$, $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ the components of the surface metric tensor with the determinant $a = |a_{\alpha\beta}|$, $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ the unit vector normal to $\mathscr{M}$, $\varepsilon^{\alpha\beta}$ the components of the permutation tensor, $b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n}$ the components of the curvature tensor of $\mathscr{M}$, and by a symbol $()_{|\alpha}$ let us denote the covariant surface derivative at $\mathscr{M}$. Other geometric quantities of the surface $\mathscr{M}$ are given in [12, 29].

After deformation of the surface $\mathscr{M}$ to a deformed configuration $\overline{\mathscr{M}}$, described by a displacement vector $\mathbf{u} = u^\alpha\mathbf{a}_\alpha + w\mathbf{n}$, the convected coordinates generate analogous quantities on $\overline{\mathscr{M}}$, which are marked with a bar: $\bar{\mathbf{r}}$, $\bar{a}$, $\bar{\mathbf{n}}$, $\bar{\mathbf{a}}_\alpha$, $\bar{a}_{\alpha\beta}$ $\bar{\varepsilon}^{\alpha\beta}$, $\bar{b}_{\alpha\beta}$ etc. For the base vectors the following relations are satisfied [12–14]:

$$\bar{\mathbf{a}}_\alpha = l^\lambda_{\cdot\alpha}\mathbf{a}_\lambda + \phi_\alpha\mathbf{n}, \qquad \bar{\mathbf{n}} = n^\lambda\mathbf{a}_\lambda + n\mathbf{n},$$

$$l_{\alpha\beta} = a_{\alpha\beta} + \theta_{\alpha\beta} - \omega_{\alpha\beta}, \qquad \phi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda,$$

(2.2) $$\theta_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w, \qquad \omega_{\alpha\beta} = \frac{1}{2}(u_{\beta|\alpha} - u_{\alpha|\beta}) = \varepsilon_{\alpha\beta}\phi,$$

$$\sqrt{\frac{\bar{a}}{a}}\,n_\mu \equiv m_\mu = \varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}\phi_\alpha l^\lambda_{\cdot\beta}, \qquad \sqrt{\frac{\bar{a}}{a}}\,n \equiv m = \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}l^\lambda_{\cdot\alpha}l^\mu_{\cdot\beta}.$$

The surface strain tensor $\gamma_{\alpha\beta}$ and the tensor of change of curvature $\varkappa_{\alpha\beta}$ are conventionally defined by

(2.3)$_1$ $$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) = \frac{1}{2}(l^\lambda_{\cdot\alpha}l_{\lambda\beta} + \phi_\alpha\phi_\beta - a_{\alpha\beta}),$$

(2.3)$_2$ $$\varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = -[n(\phi_{\alpha|\beta} + b_{\lambda\beta}l^\lambda_{\cdot\alpha}) + n_\lambda(l^\lambda_{\cdot\alpha|\beta} - b^\lambda_\beta\phi_\alpha) - b_{\alpha\beta}].$$

In general, $\gamma_{\alpha\beta}$ are quadratic polynomials in $u_\alpha$, $w$ and their derivatives, while $\varkappa_{\alpha\beta}$ are nonrational functions of those variables since they contain an invariant $\sqrt{a/\bar{a}}$, where

(2.4) $$\frac{\bar{a}}{a} = 1 + 2\gamma^\alpha_\alpha + 2(\gamma^\alpha_\alpha\gamma^\beta_\beta - \gamma^\alpha_\beta\gamma^\beta_\alpha).$$

Within the small strain theory in the expression $nb_{\lambda\beta}l^\lambda_{\cdot\alpha} - b_{\alpha\beta}$ of Eq. (2.3)$_2$ we should use an approximation $\sqrt{a/\bar{a}} \approx 1 - \gamma^\alpha_\alpha$ since the principal terms cancel with each other. Then $\varkappa_{\alpha\beta}$ may be reduced to polynomials of the fifth degree in $u_\alpha$, $w$ and their derivatives [12, 13].

The boundary contour $\mathscr{C}$ of $\mathscr{M}$ is defined by the equations $\theta^\alpha = \theta^\alpha(s)$ or $\mathbf{r} = \mathbf{r}[\theta^\alpha(s)] = \mathbf{r}(s)$ where $s$ is a length parameter along $\mathscr{C}$. At each $M \in \mathscr{C}$ we define the vectors: $\mathbf{t} = d\mathbf{r}/ds$, a unit tangent to $\mathscr{C}$, and $\boldsymbol{\nu} = \mathbf{t} \times \mathbf{n}$, an outward unit normal to $\mathscr{C}$. For the orthogonal triad $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ we have

(2.5) $$\frac{d\boldsymbol{\nu}}{ds} = \varkappa_t\mathbf{t} - \tau_t\mathbf{n}, \qquad \frac{d\mathbf{t}}{ds} = \sigma_t\mathbf{n} - \varkappa_t\boldsymbol{\nu}, \qquad \frac{d\mathbf{n}}{ds} = \tau_t\boldsymbol{\nu} - \sigma_t\mathbf{t},$$

where $\sigma_t$ is a normal curvature, $\tau_t$ a geodesic torsion and $\varkappa_t$ a geodesic curvature of the surface boundary contour $\mathscr{C}$.

After the shell deforms in a way compatible with the Kirchhoff-Love constraints, the curve $\mathscr{C}$ deforms into $\overline{\mathscr{C}}$ and an undeformed rectilinear boundary surface $\partial\mathscr{P}$, orthogonal to $\mathscr{M}$ along $\mathscr{C}$, deforms into a surface $\partial\overline{\mathscr{P}}$, which is also rectilinear and orthogonal to $\overline{\mathscr{M}}$ along $\overline{\mathscr{C}}$. Both surfaces are described by

$$\mathbf{p} = \mathbf{r}(s)+\zeta\mathbf{n}(s), \quad \overline{\mathbf{p}} = \overline{\mathbf{r}}(s)+\zeta\overline{\mathbf{n}}(s). \tag{2.6}$$

Here $\zeta$ is a distance from $\mathscr{M}$ and $-h/2 \leqslant \zeta \leqslant h/2$, where $h$ is a small shell thickness.

The surface $\partial\overline{\mathscr{P}}$ may be uniquely described by assuming at $\mathscr{C}$ the values of $\mathbf{u}$ and of a parameter $\beta_\nu$, [12, 13, 21]:

$$\begin{gathered}\mathbf{u} = u_\nu\boldsymbol{\nu}+u_t\mathbf{t}+w\mathbf{n},\\ \beta_\nu = \frac{(\overline{\mathbf{n}}-\mathbf{n})\cdot\overline{\mathbf{a}}_\nu}{1+2\gamma_{tt}} = -\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\overline{a}}{a}}\,\nu_\alpha\overline{a}^{\alpha\beta}\mathbf{u}_{,\beta}\cdot\mathbf{n},\\ \overline{\mathbf{a}}_t = \frac{d\overline{\mathbf{r}}}{ds}, \quad \overline{\mathbf{a}}_\nu = \overline{\mathbf{a}}_t\times\overline{\mathbf{n}}, \quad |\overline{\mathbf{a}}_t| = |\overline{\mathbf{a}}_\nu| = \sqrt{1+2\gamma_{tt}}.\end{gathered} \tag{2.7}$$

Note that the components of $\mathbf{u}$ in Eq. $(2.7)_1$ are given with reference to the basis of the undeformed boundary, but the parameter $\beta_\nu$ is the component of $\overline{\mathbf{n}}-\mathbf{n}$ with reference to the vector $\overline{\mathbf{a}}_\nu$ of the deformed boundary. Within the small strain theory $\beta_\nu \approx -\phi_\nu = -\phi_\alpha\nu^\alpha$, and the parameter describes then the linear rotation of the shell boundary element [13]. This is why $\beta_\nu$ may be used with success in the linear theory, in the Eulerian nonlinear theory and in the Lagrangian nonlinear theory of shells undergoing moderate rotations. This is so since in the last case as well it is enough to take into consideration only the linear part of the tensor of change of curvature [12, 13]. However, in the Lagrangian theory of shells undergoing finite (or even only large) rotations the nonlinear terms describing the deformation of the shell boundary element cannot be expressed in terms of $\mathbf{u}$ and $\beta_\nu$ only (see Eqs. (7.3) and (7.4)).

### 3. Modified relations

Let us differentiate covariantly an identity $\overline{\mathbf{a}}_\alpha\cdot\overline{\mathbf{n}} = n_\lambda l^\lambda_{\cdot\alpha}+n\phi_\alpha \equiv 0$. Introducing the result into Eq. $(2.3)_2$ we obtain an equivalent form for the tensor of change of curvature:

$$\varkappa_{\alpha\beta} = (n_{\lambda|\beta}-b_{\lambda\beta}n)l^\lambda_{\cdot\alpha}+(b^\lambda_\beta n_\lambda+n_{|\beta})\phi_\alpha+b_{\alpha\beta}. \tag{3.1}$$

Note that this form of $\varkappa_{\alpha\beta}$ contains covariant derivatives of $n_\lambda$ and $n$, while the form $(2.3)_2$ has contained covariant derivatives of $l^\lambda_{\cdot\alpha}$ and $\phi_\alpha$. This change of $\varkappa_{\alpha\beta}$ affects considerably later transformations of the boundary integral in the Lagrangian virtual work (see p. 4).

Let us introduce a new tensor of change of curvature, defining it by

$$\chi_{\alpha\beta} = -\left(\sqrt{\frac{\overline{a}}{a}}\,\overline{b}_{\alpha\beta}-b_{\alpha\beta}\right)+b^\varkappa_{\alpha\beta}\gamma^\varkappa_\varkappa. \tag{3.2}$$

Taking into account the identity

$$(3.3)\qquad \sqrt{\frac{\bar{a}}{a}}\,(n^{\lambda}|_{\beta}l_{\lambda\alpha}+n_{|\beta}\phi_{\alpha}) = m^{\lambda}|_{\beta}l_{\lambda\alpha}+m_{|\beta}\phi_{\alpha}$$

and introducing it together with Eq. (3.1) into Eq. (3.2), we obtain

$$(3.4)\qquad \chi_{\alpha\beta} = (m^{\lambda}|_{\beta}-b^{\lambda}_{\beta}m)l_{\lambda\alpha}+(b_{\lambda\beta}m^{\lambda}+m_{|\beta})\phi_{\alpha}+b_{\alpha\beta}(1+\gamma^{\varkappa}_{\varkappa}).$$

The advantage of Eq. (3.4) is that even for unrestricted strains $\chi_{\alpha\beta}$ is by definition a third-degree polynomial in $u_{\alpha}$, $w$ and their derivatives. Note also that $\varkappa_{\alpha\beta}$ and $\chi_{\alpha\beta}$ have identical linear parts. The definition similar to Eq. (3.2) but with additional terms $1/2(b^{\lambda}_{\alpha}\gamma_{\lambda\beta}+b^{\lambda}_{\beta}\gamma_{\lambda\alpha})$ in the right side of Eq. (3.2) was used by BUDIANSKY [18].

The deformed shell boundary surface $\partial\bar{\mathscr{P}}$, defined by Eq. $(2.6)_2$, can also be described uniquely by assuming at $\mathscr{C}$ three displacements $u_{\nu}$, $u_t$, $w$ and three components of the vector $\bar{\mathbf{n}} = n_{\nu}\boldsymbol{\nu}+n_t\mathbf{t}+n\mathbf{n}$, where $n_{\nu} = n^{\alpha}\nu_{\alpha}$ and $n_t = n^{\alpha}t_{\alpha}$. In contrast to the parameter $\beta_{\nu}$ defined in Eq. $(2.7)_2$, the components $n_{\nu}$, $n_t$ and $n$ are taken with reference to base vectors of the undeformed shell boundary.

The vector $\bar{\mathbf{a}}_t$, tangent to $\bar{\mathscr{C}}$, can be expressed in terms of $\mathbf{u}$ by the relations

$$(3.5)\qquad \begin{aligned} \bar{\mathbf{a}}_t &= \mathbf{t}+\frac{d\mathbf{u}}{ds} = c_{\nu}\boldsymbol{\nu}+c_t\mathbf{t}+c\mathbf{n}, & c_{\nu} &= \frac{du_{\nu}}{ds}+\tau_t w-\varkappa_t u_t,\\ c_t &= 1+\frac{du_t}{ds}+\varkappa_t u_{\nu}-\sigma_t w, & c &= \frac{dw}{ds}+\sigma_t u_t-\tau_t u_{\nu}. \end{aligned}$$

Using the set of identities

$$(3.6)\qquad \bar{\mathbf{n}}\cdot\bar{\mathbf{n}} = n_{\nu}^2+n_t^2+n^2 = 1, \quad \bar{\mathbf{a}}_t\cdot\bar{\mathbf{n}} = c_{\nu}n_{\nu}+c_t n_t+cn = 0$$

and Eq. $(2.7)_2$ the components $n_t$ and $n$ may be expressed in terms of $\mathbf{u}$ and $n_{\nu}$ according to

$$(3.7)\qquad \begin{aligned} n_t &= -\frac{1}{1+2\gamma_{tt}-c_{\nu}^2}\left[c_{\nu}c_t n_{\nu}+c\sqrt{(1+2\gamma_{tt})(1-n_{\nu}^2-c_{\nu}^2}\right],\\ n &= -\frac{1}{1+2\gamma_{tt}-c_{\nu}^2}\left[c_{\nu}cn_{\nu}-c_t\sqrt{(1+2\gamma_{tt})(1-n_{\nu}^2-c_{\nu}^2}\right]. \end{aligned}$$

Thus only four independent parameters $u_{\nu}$, $u_t$, $w$ and $n_{\nu}$ describe completely an arbitrary deformation of the shell boundary element.

Using the definition $\bar{\mathbf{n}} = 1/2\,\bar{\varepsilon}^{\alpha\beta}\bar{\mathbf{a}}_{\alpha}\times\bar{\mathbf{a}}_{\beta}$ and keeping in mind that

$$(3.8)\qquad \bar{\mathbf{a}}_{\alpha} = \nu_{\alpha}\left(\boldsymbol{\nu}+\frac{d\mathbf{u}}{ds_{\nu}}\right)+t_{\alpha}\left(\mathbf{t}+\frac{d\mathbf{u}}{ds}\right),$$

we obtain for the parameter $n_{\nu}$ the following formula:

$$(3.9)\qquad n_{\nu} = \sqrt{\frac{a}{\bar{a}}}\left(\frac{d\mathbf{u}}{ds}\times\boldsymbol{\nu}-\mathbf{n}\right)\cdot\frac{d\mathbf{u}}{ds_{\nu}}.$$

It is evident that the definition of $n_{\nu}$ includes derivatives of $\mathbf{u}$ along $\mathscr{C}$ and in the outward normal direction. The latter derivative is the one that causes major difficulties in transforming the line integral of the virtual work at the shell boundary. Therefore, the inclusion of $d\mathbf{u}/ds_\nu$ into the definition of $n_\nu$, treated as the fourth independent parameter of deformation of the shell boundary element, is an important step in the modifications proposed in this work. In particular, it allows to formulate properly all static boundary conditions of the Lagrangian nonlinear theory of shells.

From Eqs. $(2.7)_2$ and (3.5) we obtain the formula

$$c_t n_\nu = (1+2\gamma_{tt})\beta_\nu + c_\nu n_t, \tag{3.10}$$

which relates the Lagrangian parameter $n_\nu$ to the parameter $\beta_\nu$.

## 4. Internal virtual work

Let us discuss a thin shell in equilibrium state. For any additional virtual displacement field $\delta\mathbf{u} = \delta u_\alpha \mathbf{a}^\alpha + \delta w\mathbf{n}$ subject to geometric constraints the internal virtual work IVW, performed by the internal stress and couple resultant tensors on variations of corresponding strain measures, should be equal to the external virtual work EVW, performed on variations of appropriate displacemental variables by the external surface and boundary loads: IVW = EVW.

Under the Kirchhoff–Love constraints the shell deformation is described entirely by deformation of its middle surface. The Lagrangian internal virtual work can be put in the form [12]

$$\mathrm{IVW} = \iint_{\mathcal{M}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\chi_{\alpha\beta})dA = \iint_{\mathcal{M}} \delta W_i dA, \tag{4.1}$$

where $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the symmetric (2nd Piola–Kirchhoff type) internal stress and couple resultant tensors.

From Eqs. $(2.3)_1$ and (3.4) we obtain variations of the shell strain measures $\delta\gamma_{\alpha\beta}$ and $\delta\chi_{\alpha\beta}$ expressed in terms of variations of displacements and their derivatives:

$$\begin{aligned} \delta\gamma_{\alpha\beta} &= \frac{1}{2}(l^\lambda_{\cdot\alpha}\delta l_{\lambda\beta} + l^\lambda_{\cdot\beta}\delta l_{\lambda\alpha} + \phi_\alpha\delta\phi_\beta + \phi_\beta\delta\phi_\alpha), \\ \delta\chi_{\alpha\beta} &= l_{\lambda\alpha}\delta m^\lambda|_\beta + \phi_\alpha\delta m|_\beta + b_{\alpha\beta}a^{\varkappa\varrho}(l^\lambda_{\cdot\varkappa}\delta l_{\lambda\varrho} + \phi_\varkappa\delta\phi_\varrho) \\ &\quad + (m^\lambda|_\beta - b^\lambda_\beta m)\delta l_{\lambda\alpha} + (m_{|\beta} + b_{\lambda\beta}m^\lambda)\delta\phi_\alpha \\ &\quad + \varepsilon^{\varkappa\varrho}\varepsilon^{\lambda\mu}[(b_{\mu\beta}\phi_\alpha\phi_\varkappa + b^\gamma_\beta l_{\gamma\alpha}l_{\mu\varkappa})\delta l_{\lambda\varrho} + b_{\lambda\beta}\phi_\alpha l_{\mu\varkappa}\delta\phi_\varrho]. \end{aligned} \tag{4.2}$$

When Eqs. (4.2) are introduced into Eq. (4.1), we obtain after some transformations

$$\delta W_i = [M^{\alpha\beta}(l_{\lambda\alpha}\delta m^\lambda + \phi_\alpha\delta m)]_{|\beta} + T^{\lambda\beta}\delta l_{\lambda\beta} + T^\beta\delta\phi_\beta, \tag{4.3}$$

where

$$\begin{aligned} T^{\lambda\beta} &= l^\lambda_{\cdot\alpha}(N^{\alpha\beta} + a^{\alpha\beta}b_{\varkappa\varrho}M^{\varkappa\varrho}) + (m^\lambda|_\alpha - b^\lambda_\alpha m)M^{\alpha\beta} \\ &\quad + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\{l_{\mu\alpha}[(\phi_\varkappa M^{\varkappa\varrho})|_\varrho + l_{\gamma\varkappa}b^\gamma_\varrho M^{\varkappa\varrho}] - \phi_\alpha[(l_{\mu\varkappa}M^{\varkappa\varrho})|_\varrho - \phi_\varkappa b_{\mu\varrho}M^{\varkappa\varrho}]\}, \\ T^\beta &= \phi_\alpha(N^{\alpha\beta} + a^{\alpha\beta}b_{\varkappa\varrho}M^{\varkappa\varrho}) + (m|_\alpha + b^\lambda_\alpha m_\lambda)M^{\alpha\beta} + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}l_{\lambda\alpha}[(l_{\mu\varkappa}M^{\varkappa\varrho})|_\varrho - \phi_\varkappa b_{\mu\varrho}M^{\varkappa\varrho}]. \end{aligned} \tag{4.4}$$

Using again the identity $l_{\lambda\alpha}n^{\lambda}+\phi_{\alpha}n = 0$ and Eq. (2.2)$_4$ we can show that

$$M^{\alpha\beta}(l_{\lambda\alpha}\,\delta m^{\lambda}+\phi_{\alpha}\,\delta m) = \sqrt{\frac{\bar{a}}{a}}\,M^{\alpha\beta}(l_{\lambda\alpha}\,\delta n^{\lambda}+\phi_{\alpha}\,\delta n) \tag{4.5}$$

and so Eq. (4.3) can be expressed in terms of variations of $\mathbf{u}$ and $\bar{\mathbf{n}}$

$$\delta W_i = \left[\sqrt{\frac{\bar{a}}{a}}\,M^{\alpha\beta}(l_{\lambda\alpha}\,\delta n^{\lambda}+\phi_{\alpha}\,\delta n)+T^{\alpha\beta}\delta u_{\alpha}+T^{\beta}\delta w\right]_{|\beta} -(T^{\alpha\beta}|_{\beta}-b^{\alpha}_{\beta}\,T^{\beta})\,\delta u_{\alpha}-(T^{\beta}|_{\beta}+b_{\alpha\beta}\,T^{\alpha\beta})\,\delta w. \tag{4.6}$$

When Eq. (4.6) is introduced into IVW and the Stokes' theorem is applied to the differentiated terms in Eq. (4.6), we obtain

$$\mathrm{IVM} = -\iint_{\mathcal{M}} \mathbf{T}^{\beta}|_{\beta}\cdot\delta\mathbf{u}\,dA+\int_{\mathcal{C}}[(\mathbf{T}^{\beta}\nu_{\beta})\cdot\delta\mathbf{u}+R_{\nu\nu}\,\delta n_{\nu}+R_{t\nu}\,\delta n_t+R_{\nu}\,\delta n]ds, \tag{4.7}$$

where

$$\begin{gathered}\mathbf{T}^{\beta} = T^{\alpha\beta}\mathbf{a}_{\alpha}+T^{\beta}\mathbf{n},\\ R_{\nu\nu} = \sqrt{\frac{\bar{a}}{a}}\,\nu^{\lambda}l_{\lambda\alpha}M^{\alpha\beta}\nu_{\beta},\quad R_{t\nu} = \sqrt{\frac{\bar{a}}{a}}\,t^{\lambda}l_{\lambda\alpha}M^{\alpha\beta}\nu_{\beta},\quad R_{\nu} = \sqrt{\frac{\bar{a}}{a}}\,\phi_{\alpha}M^{\alpha\beta}\nu_{\beta}.\end{gathered} \tag{4.8}$$

The line boundary integral in Eq. (4.7) has been expressed in terms of six variations: three components of $\delta\mathbf{u}$ and three components of $\delta\bar{\mathbf{n}}$ with reference to the undeformed base. It follows from Eq. (3.7) that $\delta n_t$ and $\delta n$ are dependent variables, which may be determined from the set of equations

$$\begin{aligned} n_t\,\delta n_t+n\delta n &= -n_{\nu}\,\delta n_{\nu},\\ c_t\,\delta n_t+c\delta n &= -c_{\nu}\,\delta n_{\nu}-n_{\nu}\,\delta c_{\nu}-n_t\,\delta c_t-n\delta c\end{aligned} \tag{4.9}$$

following from the set (3.6). Solving Eqs. (4.9) we obtain

$$\begin{aligned}\delta n_t &= d_{\nu}\,\delta u_{\nu}+d_t\,\delta u_t+d\delta w+f\delta n_{\nu}+g_{\nu}\frac{d}{ds}\,\delta u_{\nu}+g_t\frac{d}{ds}\,\delta u_t+g\frac{d}{ds}\,\delta w,\\ \delta n &= h_{\nu}\,\delta u_{\nu}+h_t\,\delta u_t+h\delta w+k\delta n_{\nu}+r_{\nu}\frac{d}{ds}\,\delta u_{\nu}+r_t\frac{d}{ds}\,\delta u_t+r\frac{d}{ds}\,\delta w,\end{aligned} \tag{4.10}$$

where

$$\begin{gathered} d_{\nu} = \frac{n}{D}(\varkappa_t n_t-\tau_t n),\qquad d_t = \frac{n}{D}(\sigma_t n-\varkappa_t n_{\nu}),\qquad d = \frac{n}{D}(\tau_t n_{\nu}-\sigma_t n_t),\\ g_{\nu} = \frac{n_{\nu}n}{D},\qquad g_t = \frac{n_t n}{D},\qquad g = \frac{n^2}{D},\\ h_{\nu} = \frac{n_t}{D}(\tau_t n-\varkappa_t n_t),\qquad h_t = \frac{n_t}{D}(\varkappa_t n_{\nu}-\sigma_t n),\qquad h = \frac{n_t}{D}(\sigma_t n_t-\tau_t n_{\nu}),\\ r_{\nu} = -\frac{n_{\nu}n_t}{D},\qquad r_t = -\frac{n_t^2}{D},\qquad r = -\frac{n_t n}{D},\end{gathered} \tag{4.11}$$

(4.11) [cont.]
$$f = \frac{1}{D}(c_\nu n - c n_\nu), \qquad k = \frac{1}{D}(c_t n_\nu - c_\nu n_t),$$
$$D = c n_t - c_t n = -\sqrt{(1+2\gamma_{tt})(1-n_\nu^2) - c_\nu^2}.$$

If we introduce Eqs. (4.10) into Eq. (4.7), we obtain the final form for the Lagrangian internal virtual work of the shell:

$$\text{IVW} = -\iint\limits_{\mathscr{M}} \mathbf{T}^\beta|_\beta \cdot \delta\mathbf{u} dA + \int\limits_{\mathscr{C}} (\mathbf{P}\cdot\delta\mathbf{u} + M\delta n_\nu) ds + \sum_k \mathbf{F}_k \cdot \delta\mathbf{u}_k, \tag{4.12}$$

where

$$\mathbf{P} = \mathbf{T}^\beta \nu_\beta + \mathbf{Q}, \qquad M = R_{\nu\nu} + fR_{t\nu} + kR_\nu,$$
$$\mathbf{Q} = \left[d_\nu R_{t\nu} + h_\nu R_\nu - \frac{d}{ds}(g_\nu R_{t\nu} + r_\nu R_\nu)\right]\boldsymbol{\nu} + \left[d_t R_{t\nu} + h_t R_\nu - \frac{d}{ds}(g_t R_{t\nu} + r_t R_\nu)\right]\mathbf{t}$$
$$+ \left[dR_{t\nu} + hR_\nu - \frac{d}{ds}(gR_{t\nu} + rR_\nu)\right]\mathbf{n}, \tag{4.13}$$
$$\mathbf{F} = (g_\nu R_{t\nu} + r_\nu R_\nu)\boldsymbol{\nu} + (g_t R_{t\nu} + r_t R_\nu)\mathbf{t} + (gR_{t\nu} + rR_\nu)\mathbf{n},$$
$$\mathbf{F}_k = \mathbf{F}(s_k+0) - \mathbf{F}(s_k-0)$$

and $M_k$, $k = 1, 2, 3, \ldots, K$, are corner points of the boundary contour $\mathscr{C}$.

Thus the internal virtual work (4.1) of the shell has been expressed entirely in terms of Lagrangian quantities. The introduction of $n_\nu$ as a fourth idependent parameter of deformation of the shell boundary element has allowed to express the boundary line integral as a sum of works of the generalized boundary forces $\mathbf{P}$ and $M$ performed on appropriate variations of $\mathbf{u}$ and $n_\nu$. Additionally, some expected work terms have appeared in the corner points of $\mathscr{C}$ as a result of reducing the number of independent parameters at the boundary from six to four.

## 5. External virtual work

Let a shell with a simply connected middle surface be in equilibrium under the middle surface load $\mathbf{p} = p^\alpha \mathbf{a}_\alpha + p\mathbf{n}$, per unit area of the undeformed middle surface $\mathscr{M}$, and under the boundary surface load $\mathbf{f}$, per unit area of the undeformed lateral boundary surface $\partial\mathscr{P}$. In accordance with the Lagrangian theory, we assume here that the directions of the loads $\mathbf{p}$ and $\mathbf{f}$ remain constant during the shell deformation.

Within the Kirchhoff–Love shell theory the boundary load $\mathbf{f}$ is usually assumed through a resultant boundary force $\mathbf{N} = N_\nu \boldsymbol{\nu} + N_t \mathbf{t} + N\mathbf{n}$ and a resultant boundary couple $\mathbf{k} = k_\nu \bar{\mathbf{a}}_\nu + k_t \bar{\mathbf{a}}_t$, both per unit length of $\mathscr{C}$, defined by

$$\int\limits_{\mathscr{C}} \mathbf{N} ds = \iint\limits_{\partial\mathscr{P}} \mathbf{f} dA, \qquad \int\limits_{\mathscr{C}} \mathbf{k} ds = \iint\limits_{\partial\mathscr{P}} \bar{\mathbf{n}} \times \mathbf{f}\zeta dA. \tag{5.1}$$

Note that the so defined $\mathbf{k}$ depends upon the shell deformation through the vector $\bar{\mathbf{n}}$. In the Lagrangian theory it is convenient to introduce a static moment $\mathbf{H} = H_\nu \boldsymbol{\nu} + H_t \mathbf{t} + H\mathbf{n}$, per unit length of $\mathscr{C}$, by the relation

$$\int_{\mathscr{C}} \mathbf{H} ds = \iint_{\partial\mathscr{P}} \mathbf{f}\zeta dA, \quad \mathbf{k} = \overset{\bar{h}}{\bar{\mathbf{n}}} \times \mathbf{H}. \tag{5.2}$$

For the constant $\mathbf{f}$ the vector $\mathbf{H}$ does not depend upon the shell deformation.

The total finite rotation of the boundary element is given by a proper orthogonal tensor [31]

$$\mathbf{R}_t = \frac{1}{\bar{a}_t} (\bar{\mathbf{a}}_\nu \otimes \boldsymbol{\nu} + \bar{\mathbf{a}}_t \otimes \mathbf{t}) + \bar{\mathbf{n}} \otimes \mathbf{n}. \tag{5.3}$$

If the process of shell deformation is described by a parameter $\tau$, the angular velocity vector $\boldsymbol{\omega}$ of the boundary element is an axial vector of the skew-symmetric tensor $(d\mathbf{R}_t/d\tau)\mathbf{R}_t^T$, [32], that is $(d\mathbf{R}_t/d\tau)\mathbf{R}_t^T = \boldsymbol{\omega} \times \mathbf{1}$, where $\mathbf{1}$ is a metric tensor of the three-dimensional Euclidean space. Let us introduce a skew-symmetric tensor of the boundary couple $\mathbf{K}$ which, according to Eq. $(5.2)_2$, can be given by

$$\mathbf{K} = \mathbf{k} \times \mathbf{1} = \mathbf{H} \otimes \mathbf{R}_t \mathbf{n} - \mathbf{R}_t \mathbf{n} \otimes \mathbf{H}. \tag{5.4}$$

Then the elementary work performed by $\mathbf{k}$ on an infinitesimal rotation angle $d\boldsymbol{\omega} = \boldsymbol{\omega} d\tau$ is calculated according to

$$dW_r = \mathbf{k} \cdot d\boldsymbol{\omega} = \frac{1}{2} \mathbf{K} \cdot (d\mathbf{R}_t \mathbf{R}_t^T) = (\mathbf{H} \otimes \mathbf{n} \mathbf{R}_t^T) \cdot (d\mathbf{R}_t \mathbf{R}_t^T) = (\mathbf{H} \otimes \mathbf{n}) \cdot d\mathbf{R}_t, \tag{5.5}$$

where the scalar product in the tensor space is performed according to the rule $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr}(\mathbf{A}^T\mathbf{B}) = A^{ij}B_{ij}$.

Since the tensor $\mathbf{H} \otimes \mathbf{n}$ is constant during the shell deformation, we can integrate Eq. (5.5) and calculate the total work performed on the rotational part of the boundary deformation by the surface boundary load:

$$W_r = (\mathbf{H} \otimes \mathbf{n}) \cdot \int_{\mathbf{1}}^{\mathbf{R}_t} d\mathbf{R}_t = \mathbf{H} \cdot (\bar{\mathbf{n}} - \mathbf{n}). \tag{5.6}$$

Note the simplicity of this formula expressed in terms of the constant $\mathbf{H}$. The analogous formula expressed in terms of the variable $\mathbf{k}$ would be more complex.

Now it is easy to see that in terms of $\mathbf{p}$, $\mathbf{N}$ and $\mathbf{H}$ the Lagrangian external virtual work is calculated according to

$$\mathrm{EVW} = \iint_{\mathscr{M}} \mathbf{p} \cdot \delta\mathbf{u} dA + \int_{\mathscr{C}} (\mathbf{N} \cdot \delta\mathbf{u} + H_\nu \delta n_\nu + H_t \delta n_t + H \delta n) ds, \tag{5.7}$$

which is analogous to IVW given in Eq. (4.7). When we introduce Eqs. (4.10) into Eq. (5.7) it takes the final form

$$\mathrm{EVW} = \iint_{\mathscr{M}} \mathbf{p} \cdot \delta\mathbf{u} dA + \int_{\mathscr{C}_f} (\mathbf{P}^* \cdot \delta\mathbf{u} + M^* \delta n_\nu) ds + \sum_j \mathbf{F}_j^* \cdot \delta\mathbf{u}_j, \tag{5.8}$$

where

$$\mathbf{P}^* = \mathbf{N} + \mathbf{Q}^*, \quad M^* = H_\nu + f H_t + k H, \tag{5.9}$$

(5.9) [cont.]
$$\mathbf{Q}^* = \left[d_\nu H_t + h_\nu H - \frac{d}{ds}(g_\nu H_t + r_\nu H)\right]\boldsymbol{\nu} + \left[d_t H_t + h_t H - \frac{d}{ds}(g_t H_t + r_t H)\right]\mathbf{t} + \left[dH_t + hH - \frac{d}{ds}(gH_t + rH)\right]\mathbf{n},$$
$$\mathbf{F}^* = (g_\nu H_t + r_\nu H)\boldsymbol{\nu} + (g_t H_t + r_t H)\mathbf{t} + (gH_t + rH)\mathbf{n},$$
$$\mathbf{F}_j^* = \mathbf{F}^*(s_j+0) - \mathbf{F}^*(s_j-0)$$

and $\mathscr{C}_f$ is the part of $\mathscr{C}$ on which at least one component of $\mathbf{P}^*$ or $M^*$ is prescribed, while $M_j$, $j = 1, 2, \ldots J \leqslant K$, are those corner points of $\mathscr{C}$ where at least one component of $\mathbf{F}_j^*$ is prescribed.

## 6. Lagrangian shell equations

Let us compare Eq. (4.12) and Eq. (5.8) and note the identical structure of both virtual works. Then from IVW = EVW we obtain the following Lagrangian equilibrium equations and appropriate static boundary conditions:

$$\begin{aligned} &\mathbf{T}^\beta|_\beta + \mathbf{p} = \mathbf{0} && \text{in } \mathscr{M}, \\ &\mathbf{P} = \mathbf{P}^* \text{ and } M = M^* && \text{on } \mathscr{C}_f, \\ &\mathbf{F}_j = \mathbf{F}_j^* && \text{at each } M_j \in \mathscr{C}_f. \end{aligned} \tag{6.1}$$

The appropriate geometric boundary conditions take the form

$$\begin{aligned} &\mathbf{u} = \mathbf{u}^* \text{ and } n_\nu = n_\nu^* && \text{on } \mathscr{C}_u, \\ &\mathbf{u}_i = \mathbf{u}_i^* && \text{at each } M_i \in \mathscr{C}_u, \end{aligned} \tag{6.2}$$

where $\mathscr{C}_u$ is the part of $\mathscr{C}$ on which at least one component of $\mathbf{u}^*$ or $n_\nu^*$ is prescribed while $M_i$, $i = 1, 2, \ldots I \leqslant K$, are those corner points of $\mathscr{C}$ where at least one component of $\mathbf{u}_i^*$ is prescribed. In the general case of mixed (mutually complementary) boundary conditions $\mathscr{C}_u$ and $\mathscr{C}_f$ coincide. They may become separated only if all four boundary conditions are geometric or static, respectively. Likewise for the corner points $M_i$ and $M_j$.

When Eqs. (4.8) and (4.4) are introduced into the conditions (6.1), the equilibrium equations $(6.1)_1$ become linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ and quadratic in $u_\alpha$, $w$. The static boundary conditions $(6.1)_{2,3}$ then become linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but nonrational in $u_\alpha$, $w$ and $n_\nu$ since in Eqs. $(4.8)_2$, (3.7) and (4.11) there are square roots of polynomials in $u_\alpha$, $w$ and $n_\nu$. The structure of our equilibrium equations $(6.1)_1$ agrees with the one of the equations derived by BUDIANSKY [18] with the use of a more complex tensor of change of curvature. However, in [18] only the operator form of equilibrium equations was presented and the problem of appropriate boundary conditions for the bending shell theory was not discussed. In fact, we do not know any paper in which the general form of four fully Lagrangian boundary conditions of the type $(6.1)_{2,3}$ and (6.2) would be given for the $K-L$ type theory of shells.

The shell equations (6.1) and (6.2) and other shell relations obtained here are derived for unrestricted middle surface strains and rotations. They do not depend upon the material properties of the shell. Therefore, all the relations describe the so-called restricted Cosserat surface model of shell behaviour [9]. They should be supplemented by some two-dimensional constitutive equations satisfying appropriate invariance requirements which would describe the behaviour of a material of which the shell is composed. Even for the simple case of an isotropic elastic Cosserat surface these constitutive equations are quite complex polynomials of the surface strain measures whose coefficients should be determined on the basis of experimental tests.

## 7. Small elastic strains

The Lagrangian shell equations (6.1) and (6.2) may also be treated as an approximation to the three-dimensional shell problem. In such a case the results obtained here under the Kirchhoff–Love constraints are meaningless for large strains since the first-order effect of change of the shell thickness has been ignored. Nevertheless, all the relations are still valuable as a basis for the derivation of appropriately simplified equations of the Lagrangian nonlinear theory of thin elastic shells undergoing small strains but finite rotations.

For an elastic shell there exists a strain energy $\Sigma$, per unit area of $\mathcal{M}$, which can be consistently simplified [16, 27, 28] in the case of small strains and isotropic material behaviour. Within the consistent first approximation the strain energy function is given, to within a small relative error, by the quadratic expression

$$
\begin{aligned}
&\Sigma \simeq \frac{h}{2} H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu}+\frac{h^2}{12}\chi_{\alpha\beta}\chi_{\lambda\mu}\right),\\
&H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu}+a^{\alpha\mu}a^{\beta\lambda}+\frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right),
\end{aligned}
\tag{7.1}
$$

where $E$ and $\nu$ are the Young's modulus and the Poissons ratio, respectively.

The formula $(7.1)_1$ includes the main contributions to the elastic strain energy of a shell caused by stretching and bending of the shell middle surface. The effect of change of the shell thickness is also included into Eq. $(7.1)_1$ by using there the modified shell elasticity tensor $H^{\alpha\beta\lambda\mu}$. From Eq. (7.1) we obtain the linear constitutive equations

$$
\begin{aligned}
&N^{\alpha\beta} = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} \simeq \frac{Eh}{1-\nu^2}\left[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma^{\lambda}_{\lambda}\right],\\
&M^{\alpha\beta} = \frac{\partial\Sigma}{\partial\chi_{\alpha\beta}} \simeq \frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\chi^{\alpha\beta}+\nu a^{\alpha\beta}\chi^{\lambda}_{\lambda}\right].
\end{aligned}
\tag{7.2}
$$

Under small strains $\sqrt{\bar{a}/a} \simeq 1+\gamma^{\lambda}_{\lambda} \simeq 1$ and $\chi_{\alpha\beta}$ used here differs from $\varkappa_{\alpha\beta}$ defined by Eq. (3.1) only by small terms of the order of $\varkappa\gamma$ or $\gamma^2/R$, where $\varkappa$ and $\gamma$ are greater eigenvalues of $\varkappa_{\alpha\beta}$ and $\gamma_{\alpha\beta}$, respectively, while $R$ is the smallest radius of curvature of $\mathcal{M}$ at the point under consideration. Differences of this order in the definition of the change of curvature do not influence the accuracy of the elastic strain energy $(7.1)_1$ of a shell within the first-approximation theory [27].

Under small strains the parameters $n_\nu$, $n_t$ and $n$ become quadratic polynomials of displacements:

$$
\begin{aligned}
&n_\nu \simeq m_\nu = -\phi_\nu(1+\theta_{tt})+\phi_t(\theta_{\nu t}+\phi),\\
&n_t \simeq m_t = \phi_\nu(\theta_{\nu t}-\phi)-\phi_t(1+\theta_{\nu\nu}),
\end{aligned}
\tag{7.3}
$$

(7.3) [cont.]
$$n \simeq m = 1+\theta_{\nu\nu}+\theta_{tt}+\phi^2+\theta_{\nu\nu}\theta_{tt}-\theta_{\nu t}^2$$
$$\simeq 1-\frac{1}{2}(\phi_\nu^2+\phi_t^2)+\theta_{\nu\nu}\theta_{tt}-\frac{1}{2}\theta_{\nu\nu}^2-2\theta_{\nu t}^2-\frac{1}{2}\theta_{tt}^2,$$

where [13]

$$\phi_\nu = \frac{dw}{ds_\nu}+\sigma_\nu u_\nu-\tau_t u_t, \quad \phi_t = \frac{dw}{ds}-\tau_t u_\nu+\sigma_t u_t,$$

$$\phi \equiv \omega_{\nu t} = \frac{1}{2}\left(\frac{du_t}{ds_\nu}-\varkappa_\nu u_\nu-\frac{du_\nu}{ds}+\varkappa_t u_t\right) = -\frac{du_\nu}{ds}+\varkappa_t u_t-\tau_t w+\theta_{\nu t}, \tag{7.4}$$

$$\theta_{\nu\nu} = \frac{du_\nu}{ds_\nu}+\varkappa_\nu u_t-\sigma_\nu w, \quad \theta_{tt} = \frac{du_t}{ds}+\varkappa_t u_\nu-\sigma_t w,$$

$$\theta_{\nu t} = \frac{1}{2}\left(\frac{du_t}{ds_\nu}-\varkappa_\nu u_\nu+\frac{du_\nu}{ds}-\varkappa_t u_t\right)+\tau_t w.$$

Here $\sigma_\nu$, $\tau_\nu \equiv \tau_t$ and $\varkappa_\nu$ are the normal curvature, the geodesic torsion and the geodesic curvature, respectively, of the curve $\mathscr{C}_\nu$ orthogonal to $\mathscr{C}$ and $s_\nu$ is a length parameter along $\mathscr{C}_\nu$.

Note that differentiation with respect to $s_\nu$ appears in Eqs. (7.4) not only in $\phi_\nu$ but also in $\phi$, $\theta_{\nu\nu}$ and $\theta_{\nu t}$. If $\beta_\nu \simeq -\phi_\nu$ were taken as the fourth independent parameter of the shell deformation, after the introduction of Eqs. (7.3) into Eqs. (4.7) and (5.7) we would not be able to eliminate terms differentiated with respect to $s_\nu$ from the line integral. This is why the parameter $\beta_\nu$ is not adequate for the Lagrangian formulation of the theory of shells undergoing finite rotations. However, if $n_\nu$ is taken as the fourth independent parameter, then using $1+2\gamma_{tt} \simeq 1$ the parameters $n_t$ and $n$ can be given in alternative approximate forms following from Eqs. (3.7) to be

$$n_t \simeq -\frac{c_\nu c_t n_\nu - cD}{1-c_\nu^2}, \quad n \simeq -\frac{c_\nu c n_\nu + c_t D}{1-c_\nu^2}, \quad D \simeq -\sqrt{1-c_\nu^2-n_\nu^2}. \tag{7.5}$$

However, when calculating the approximations for the expressions like $n-1$, $dn/ds$ or $n/D$ the parameters $n$ and $D$ should be approximated with a higher accuracy directly from $(3.7)_1$ and $(4.11)_6$, see [33].

Note that $n_t$ and $n$ depend here upon $\mathbf{u}$, $d\mathbf{u}/ds$ and $n_\nu$ and do not depend explicity upon terms differentiated with respect to $s_\nu$. When introduced into Eqs. (4.7) and (5.7) the parameters $(7.5)_1$ allow to perform the transformations (4.9)–(4.11). Since under small strains $D$ in Eq. (7.5) is still a nonrational square-root function of $\mathbf{u}$, $d\mathbf{u}/ds$ and $n_\nu$, all parameters (7.5) and (4.11) are also nonrational functions of those variables.

The boundary couples $(4.8)_2$ may be approximated by

$$\begin{aligned} R_{\nu\nu} &\simeq (1+\theta_{\nu\nu})M_{\nu\nu}+(\theta_{\nu t}-\phi)M_{\nu t}, \\ R_{t\nu} &\simeq (\theta_{\nu t}+\phi)M_{\nu\nu}+(1+\theta_{tt})M_{\nu t}, \\ R_\nu &\simeq \phi_\nu M_{\nu\nu}+\phi_t M_{\nu t}. \end{aligned} \tag{7.6}$$

When Eqs. (7.5), (4.11) and (7.6) are introduced into Eqs. (4.13) and (5.9), we obtain a simplified form of the static boundary conditions for the Lagrangian nonlinear theory of shells undergoing small strains but finite rotations. The conditions are linear in $N^{\alpha\beta}$ $M^{\alpha\beta}$ but still nonrational in displacemental parameters.

If we introduce Eqs. (7.2), $(2.3)_1$ and (3.4) into the conditions (6.1), we obtain the displacemental form of equilibrium equations which become fifth-degree polynomials with respect to displacements and their surface derivatives.

## 8. Hu-Washizu variational principle

In the paper [26] a general approach to the derivation of variational principles was given for the geometrically nonlinear theory of thin elastic shells undergoing moderate rotations. A set of sixteen basic free functionals was constructed. Here we apply the same approach and derive the most general Hu-Washizu variational functional for the geometrically nonlinear theory of thin elastic shells undergoing finite rotations.

In the case of an elastic shell the virtual work (4.1) can be expressed as a variation of the shell strain energy function: $N^{\alpha\beta}\delta\gamma_{\alpha\beta}+M^{\alpha\beta}\delta\chi_{\alpha\beta} = \delta\Sigma(\gamma_{\alpha\beta},\chi_{\alpha\beta})$. In the case of dead loads assumed here there exist potential functions $\Phi(\mathbf{u}) = -\mathbf{p}\cdot\mathbf{u}$ and $\Psi(\mathbf{u}, n_\nu) = -[\mathbf{N}\cdot\mathbf{u}+ +\mathbf{H}\cdot(\bar{\mathbf{n}}-\mathbf{n})]$ such that their variations constitute the external virtual work (5.7) or (5.8) with approximations implied by Eqs. (7.5). Therefore, in this case the principle of virtual displacements can be transformed into a variational principle $\delta I = 0$ for the functional

$$I = \iint_{\mathscr{M}} [\Sigma(\gamma_{\alpha\beta}, \chi_{\alpha\beta})-\mathbf{p}\cdot\mathbf{u}]dA - \int_{\mathscr{C}_f} [\mathbf{N}\cdot\mathbf{u}+\mathbf{H}\cdot(\bar{\mathbf{n}}-\mathbf{n})]\,ds, \tag{8.1}$$

where strain — displacement relations $(2.3)_1$ and (3.4), geometric boundary and corner conditions (6.2) and geometric relations (7.5) at the boundary have to be imposed as subsidiary conditions. The variational principle $\delta I = 0$ states that among all possible values of $\gamma_{\alpha\beta}$, $\chi_{\alpha\beta}$, $\mathbf{u}$ in $\mathscr{M}$ and $\mathbf{u}$, $n_\nu$, $n_t$, $n$ on $\mathscr{C}$, which are subject to the subsidiary conditions, the actual solution renders the functional $I$ stationary.

Let us introduce the subsidiary conditions $(2.3)_1$, (3.4), (7.5) and (6.2) of $I$ into the functional itself by use of the Lagrange multiplier method. Then we obtain the free functional

$$\begin{aligned} I_1 = \iint_{\mathscr{M}} \{\Sigma(\gamma_{\alpha\beta}, \chi_{\alpha\beta})-\mathbf{p}\cdot\mathbf{u}-N^{\alpha\beta}[\gamma_{\alpha\beta}-\gamma_{\alpha\beta}(\mathbf{u})]-M^{\alpha\beta}[\chi_{\alpha\beta}-\chi_{\alpha\beta}(\mathbf{u})]\}dA \\ - \int_{\mathscr{C}_f} \{\mathbf{N}\cdot\mathbf{u}+\mathbf{H}\cdot(\bar{\mathbf{n}}-\mathbf{n})-\lambda_t[n_t-n_t(\mathbf{u}, n_\nu)]-\lambda[n-n(\mathbf{u}, n_\nu)]\}ds \\ - \int_{\mathscr{C}_u} [\mathbf{P}\cdot(\mathbf{u}-\mathbf{u}^*)+M(n_\nu-n_\nu^*)]ds - \sum_i \mathbf{F}_i\cdot(\mathbf{u}_i-\mathbf{u}_i^*). \end{aligned} \tag{8.2}$$

The functional $I_1$ is defined in terms of the following independent free variables subject to variation: three displacements $\mathbf{u}$ in $\mathscr{M}$, six displacemental parameters $\mathbf{u}, n_\nu$, $n_t$ and $n$ on $\mathscr{C}$, three displacements $\mathbf{u}_k$ at each corner $M_k \in \mathscr{C}$, six strain components $\gamma_{\alpha\beta}$ and $\chi_{\alpha\beta}$ in $\mathscr{M}$, six Lagrange multipliers $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in $\mathscr{M}$, two Lagrange multipliers $\lambda_t$ and $\lambda$ on $\mathscr{C}_f$, four Lagrange multipliers $\mathbf{P}$ and $M$ on $\mathscr{C}_u$ and three Lagrange multipliers $\mathbf{F}_i$ at each corner $M_i \in \mathscr{C}_u$. The associated variational principle $\delta I_1 = 0$ states that among all possible values of the variables, which are not restricted by any subsidiary conditions, the actual solution renders the functional $I_1$ stationary.

Taking the variation of $I_1$, we obtain as its stationarity conditions all basic shell relations: the equilibrium equations $(6.1)_1$, the static boundary and corner conditions $(6.1)_{2,3}$, the strain-displacement relations $(2.3)_1$ and (3.4), the geometric boundary and corner conditions (6.2) together with additional relations which show the Lagrange multipliers to be indeed those already described by their symbols. These additional relations are constitutive equations (6.3), definitions of the effective boundary forces and the moment $(4.13)_1$ on $\mathscr{C}_u$, the definition of the effective corner force $(4.13)_4$ at all $M_i \in \mathscr{C}_u$ and the definitions $\lambda_t = H_t$, $\lambda = H$ for the Lagrange multipliers on $\mathscr{C}_f$.

The variational principle $\delta I_1 = 0$ is the Hu-Washizu principle for the Lagrangian geometrically nonlinear theory of thin elastic shells. Following [26, 34] from $I_1$ a number of other free functionals and associated Lagrangian variational principles may be generated.

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