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Optimal shape design of loaded boundaries

K. DEMS (ŁÓDŹ)

THE PROBLEM of optimal shape design of an elastic structure with unspecified loaded boundary is discussed for the case of mean compliance constraint. The virtual displacement and stress principles for simultaneous variation of the boundary are derived. Next, the optimality conditions are generated for the case of conservative and nonconservative load systems. The optimization of a circular disk with a circular hole is considered in order to illustrate these conditions.

W pracy rozpatrzono problem optymalnego projektowania kształtu brzegów obciążonych konstrukcji sprężystych z punktu widzenia minimalizacji podatności konstrukcji. Wyprowadzono zasadę prac przygotowanych i zasadę uzupełniających prac przygotowanych w przypadku, gdy kształt brzegu ograniczającego ciało może podlegać zmianom. Następnie rozpatrzono warunki optymalności kształtu brzegu obciążonego zachowawczymi i niezachowawczymi układami sił. Jako ilustrację wykorzystania otrzymanych warunków rozpatrzono optymalizację kołowej tarczy z otworem obciążonej stałym ciśnieniem wewnętrznym i zewnętrznym.

В работе рассмотрена проблема оптимального проектирования формы границ нагруженных упругих конструкций с точки зрения минимизации податливости конструкции. Выведен принцип виртуальных работ и принцип дополняющих виртуальных работ в случае, когда форма границы, ограничивающей тело, может подлежать изменениям. Затем рассмотрены условия оптимальности формы границы, нагруженной консервативными и неконсервативными системами сил. Как иллюстрация использования полученных условий рассмотрена оптимизация кругового диска с отверстием, нагруженного постоянными внутренним и внещним давлениями.

1. Introduction

THE PRESENT paper supplements the previous works [1, 2, 3] on optimal shape optimization of structures with unspecified *a priori* external free boundary or the interfaces between particular materials entering into the structure. Whereas in [1] the general optimality conditions were derived for the case of mean compliance design of a nonlinear elastic structure and some numerical examples of disk design were presented, in [2] the optimization of the shape of the interface between different materials entering into the structure was considered. The optimization of cross-sectional shape of prismatic bars under torsion was discussed in [3].

The present work provides first the virtual displacement and stress principles in the case when the displacement or stress variation is accompanied by the variation of the loaded boundary. Next, these principles are applied in generating optimality conditions in the case of mean compliance design. Both the conservative and nonconservative load systems are considered. The optimal design of radii of an elastic circular disk with a circular hole is discussed in order to illustrate the applicability of the optimality conditions. Our analysis will apply to nonlinear elastic materials with stress and strain potentials $W(\sigma_{ij})$ and $U(\varepsilon_{ij})$, so that

$$(1.1)\qquad \varepsilon_{ij} = \frac{\partial W}{\partial \sigma_{ij}}, \quad \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}.$$

Further, it is assumed that $W(\sigma_{ij})$ and $U(\varepsilon_{ij})$ are homogeneous functions of orders $n+1$ and $k+1$, so that

$$(1.2)\qquad \sigma_{ij}\varepsilon_{ij} = \sigma_{ij}\frac{\partial W}{\partial \sigma_{ij}} = (n+1)W(\sigma_{ij}) = \varepsilon_{ij}\frac{\partial U}{\partial \varepsilon_{ij}} = (k+1)U(\varepsilon_{ij}),$$

where $k \cdot n = 1$. For the uniaxial stress state, the stress-strain curve is then described by

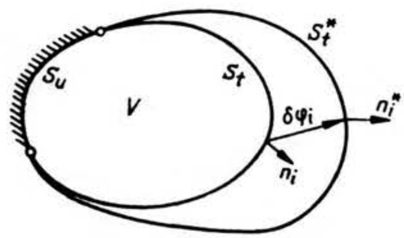

FIG. 1. Body B supported on S_u and loaded on boundary S_t subject to variation.

a power law $\varepsilon = c\sigma^n$ where c and n are material parameters. For $n = 1$, the relations (1.1) correspond to a linear elastic material whereas for $n = \infty$ the perfectly soft behaviour is obtained which is analogous to perfectly plastic behaviour.

2. Principle of virtual displacements for simultaneous variation of a loaded boundary

Consider an elastic body B contained in a domain V and bounded by the boundary $S = S_t \cup S_u$, Fig. 1. On the portion S_t the surface tractions $T_i^0 = \sigma_{ij}n_j$ are prescribed whereas on the portion S_u the displacements $u_i = u_i^0$ are specified.

Consider an infinitesimal variation of configuration by prescribing a continuous and differentiable vector field $\delta\varphi_i = \delta\varphi_i(x)$, so that

$$(2.1)\qquad P \to P^*: x_i^* = x_i + \delta\varphi_i.$$

Thus the domain V is transformed into the domain V^* with the boundary S_t transformed into S_t^*. The function $\delta\varphi_i(x)$ vanishes on S_u so that the shape of the supported boundary is not changed. Let the stresses, strains and displacements of the body B before variation be σ_{ij}, ε_{ij} and u_i. These fields satisfy equilibrium, compatibility and boundary conditions on S_t and S_u. Consider now the variations of the static and kinematic fields. For the displacement field we can write, cf. Fig. 2a [4],

$$(2.2)\qquad u_i^*(x^*) = u_i(x) + \delta u_i(x),$$

where the variation δu_i is defined as follows:

$$(2.3)\qquad \delta u_i = u_i^*(x) - u_i(x) + u_{i,k}(x)\,\delta\varphi_k = \delta\bar{u}_i + u_{i,k}\delta\varphi_k$$

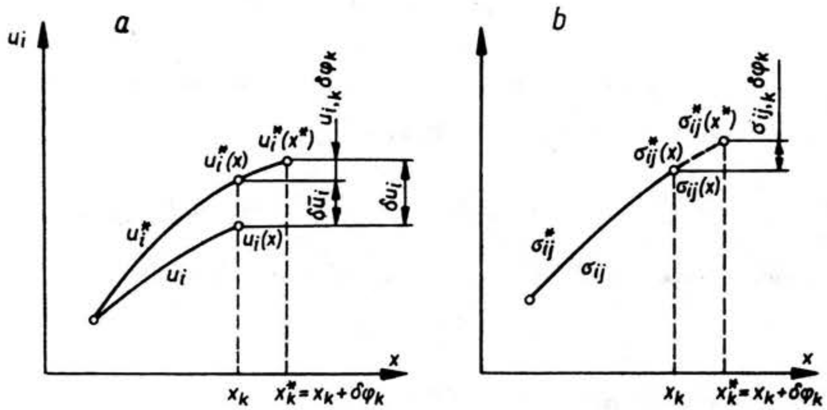

FIG. 2. Variation and continuation of the static and kinematic fields; a) Variation and continuation of the displacement field; b) Continuation of the stress field beyond S_t.

and it satisfies the condition below:

(2.4) $$\delta u_i = 0 \quad \text{on} \quad S_u.$$

Here $\delta\bar{u}_i$ denotes the variation of u_i at the initial positions of material elements and δu_i is the total variation of u_i. The variation of strain is expressed analogously to Eq. (2.3), thus,

(2.5) $$\delta\varepsilon_{ij} = \delta\bar{\varepsilon}_{ij} + \varepsilon_{ij,k}\delta\varphi_k$$

and

(2.6) $$\varepsilon_{ij}^*(x^*) = \varepsilon_{ij}(x) + \delta\varepsilon_{ij}(x).$$

Consider now a static continuation of the stress field defined by the relation, cf. Fig. 2b,

(2.7) $$\sigma_{ij}^*(x^*) = \sigma_{ij}(x) + \sigma_{ij,k}(x)\,\delta\varphi_k.$$

Therefore this stress field is also defined beyond S_t and satisfies the equilibrium equations since [4]

(2.8) $$\sigma_{ij,j}^*(x^*) = \sigma_{ij,j}(x) + \sigma_{ij,kj}(x)\,\delta\varphi_k = 0.$$

The surface tractions on S_t^* are

(2.9) $$T_i^*(x^*) = \sigma_{ij}^*(x^*)n_j^*,$$

where n_j^* denotes the external unit normal vector on S_t^*.

For the configuration V^* we can write

(2.10) $$\int_{V^*} \sigma_{ij}^* \varepsilon_{ij}^* dV^* = \int_{S_u} t_i^* u_i^0 dS_u + \int_{S_t^*} T_i^* u_i^* dS_t^*.$$

Now let us transform the integrals over the domains V^* and S_t^* to the integrals over the initial domains V and S_t. Neglecting higher order terms of $\delta\varphi_k$ in the Jacobian of the transformation (2.1), we find (cf. [4]).

(2.11) $$dV^* = (1 + \delta\varphi_{k,k})dV$$

and the surface element $n_i^* dS_j^*$ is transformed as follows (cf. [5]):

$$n_j^* dS_t^* = (n_j + n_j \delta\varphi_{k,k} - n_k \delta\varphi_{k,j}) dS_t, \tag{2.12}$$

where n_j denotes the external unit normal vector on the initial boundary S_t.

Using Eqs. (2.2) ÷ (2.9), (2.11) and (2.12), Eq. (2.10) can thus be written in the form

$$\int_V (\sigma_{ij} + \sigma_{ij,k}\delta\varphi_k)(\varepsilon_{ij} + \delta\bar{\varepsilon}_{ij} + \varepsilon_{ij,k}\delta\varphi_k)(1 + \delta\varphi_{k,k}) dV \tag{2.13}$$

$$= \int_{S_u} t_i u_i^0 dS_u + \int_{S_t} (\sigma_{ij} + \sigma_{ij,k}\delta\varphi_k)(u_i + \delta\bar{u}_i + u_{i,k}\delta\varphi_k)(n_j + n_j\delta\varphi_{k,k} - n_k\delta\varphi_{k,j}) dS_t.$$

Neglecting higher order terms of $\delta\varphi_k$, $\delta\bar{\varepsilon}_{ij}$ and $\delta\bar{u}_i$ and using the equality

$$\int_V \sigma_{ij}\varepsilon_{ij} dV = \int_{S_u} t_i u_i^0 dS_u + \int_{S_t} T_i^0 u_i dS_t. \tag{2.14}$$

Equation (2.13) can be presented in the form

$$\int_V \sigma_{ij}\delta\bar{\varepsilon}_{ij} dV = \int_{S_t} T_i^0 \delta\bar{u}_i dS_t + \int_{S_t} [(\sigma_{ik}\delta\varphi_j - \sigma_{ij}\delta\varphi_k)u_i]_{,j} n_k dS_t. \tag{2.15}$$

Equation (2.15) represents the required virtual displacement principle. Applying now the Stokes theorem to the last term of Eq. (2.15), we can retransform it to a line integral along the curve Γ bounding the surface S_t, thus,

$$\int_V \sigma_{ij}\delta\bar{\varepsilon}_{ij} dV = \int_{S_t} T_i^0 \delta\bar{u}_i dS_t - \oint_\Gamma e_{jkl}\sigma_{ij}u_i t_k^\Gamma \delta\varphi_l^\Gamma d\Gamma, \tag{2.16}$$

where t_k^Γ denotes the unit vector tangential to the curve Γ, $\delta\varphi_l^\Gamma$ is the variation of S_t on Γ and e_{jkl} denotes the permutation symbol. When the variation $\delta\varphi_l^\Gamma = 0$ on Γ, then the last term of Eq. (2.16) vanishes and the principle of virtual work takes now the form

$$\int_V \sigma_{ij}\delta\bar{\varepsilon}_{ij} dV = \int_{S_t} T_i^0 \delta\bar{u}_i dS_t. \tag{2.17}$$

3. Principle of virtual stress with simultaneous variation of the loaded boundary S_t

Using the notation included in the preceding Section, let us assume that the transformation $V \to V^*$ is accompanied by the stress variation and the stress field σ_{ij}^* is statically admissible and satisfies the boundary conditions. We thus have

$$\sigma_{ij}^*(x^*) = \sigma_{ij}(x) + \delta\sigma_{ij}(x) = \sigma_{ij} + \delta\bar{\sigma}_{ij} + \sigma_{ij,k}\delta\varphi_k, \tag{3.1}$$

where

$$\delta\bar{\sigma}_{ij} = \sigma_{ij}^*(x) - \sigma_{ij}(x) \tag{3.2}$$

and static admissibility requires that

$$\sigma_{ij,j}^{*\,*} = \sigma_{ij,j} + \delta\bar{\sigma}_{ij,j} + \sigma_{ij,kj}\delta\varphi_k = 0. \tag{3.3}$$

Hence

$$\delta\bar{\sigma}_{ij,j} = 0 \quad \text{in} \quad V, \tag{3.4}$$

and the surface tractions on S_t^* are

$$T_i^*(x^*) = \sigma_{ij}^*(x^*) n_j^*. \tag{3.5}$$

Denote now the total variation of the surface tractions by

(3.6) $$\delta T_i^0 = T_i^*(x^*) - T_i^0(x) = \delta\sigma_{ij} n_j + \sigma_{ij}\delta n_j.$$

Using Eq. (3.1) and the equality [5]

(3.7) $$\delta n_j = n_j^* - n_j = n_j n_k n_l \delta\varphi_{k,l} - n_k \delta\varphi_{k,j},$$

we obtain from Eq. (3.6)

(3.8) $$\delta\bar{\sigma}_{ij} n_j = \delta T_i^0 - T_i^0 n_k n_l \delta\varphi_{k,l} - \sigma_{ij,k} n_j \delta\varphi_k + \sigma_{ij} n_k \delta\varphi_{k,j} \quad \text{on} \quad S_t.$$

Continuing analytically the displacement and strain fields from V into V^*, we can write

(3.9) $$u_i^*(x^*) = u_i(x) + u_{i,k}(x)\delta\varphi_k,$$
$$\varepsilon_{ij}^*(x^*) = \varepsilon_{ij}(x) + \varepsilon_{ij,k}(x)\delta\varphi_k.$$

Thus, for the configuration V^* we can write

(3.10) $$\int_V \sigma_{ij}^* \varepsilon_{ij}^* dV^* = \int_{S_u} t_i^* u_i^0 dS_u + \int_{S_t^*} T_i^* u_i^* dS_t^*.$$

Following in a similar way as in the previous section, we can transform integration within the domains V^* and S_t^* to the domains V and S_t. Using Eqs. (2.11), (2.12) and (3.1 ÷ 3.9), we can obtain after deleting higher order terms with respect to $\delta\varphi_k$ and $\delta\bar{\sigma}_{ij}$

(3.11) $$\int_V (\sigma_{ij} + \delta\bar{\sigma}_{ij})\varepsilon_{ij} dV = \int_{S_u} t_i^* u_i^0 dS_u + \int_S [T_i^0 u_i + \delta T_i^0 u_i + T_i^0 u_i(\delta\varphi_{k,k} - n_k n_l \delta\varphi_{k,l}) + (\sigma_{ik}\delta\varphi_j - \sigma_{ij}\delta\varphi_k) u_{i,j} n_k] dS_t.$$

Substracting Eq. (2.14) from Eq. (3.11), we obtain the required principle of virtual stress

(3.12) $$\int_V \delta\bar{\sigma}_{ij}\varepsilon_{ij} dV = \int_{S_u} \delta t_i u_i^0 dS + \int_{S_t} [\delta T_i^0 u_i + T_i^0 u_i(\delta\varphi_{k,k} - n_k n_l \delta\varphi_{k,l}) + (\sigma_{ik}\delta\varphi_j - \sigma_{ij}\delta\varphi_k) u_{i,j} n_k] dS_t.$$

Let the surface S_t be parametrized by an orthogonal, curvilinear coordinate system α, β, Fig. 3, coinciding with the lines of principal curvatures of S_t and let a_k, b_k denote the unit vectors tangent to the α-and β-lines, whereas $\delta\varphi_a$, $\delta\varphi_b$ and $\delta\varphi_n$ denote the components of variation of a typical point on S_t in the directions α, β and n. Thus the following equalities hold on S_t:

(3.13) $$\delta\varphi_a = a_k \delta\varphi_k, \qquad \delta\varphi_b = b_k \delta\varphi_k, \qquad \delta\varphi_n = n_k \delta\varphi_k.$$

Furthermore, for any function $f(x)$, continuous and differentiable on S_t, we have

(3.14) $$f_{,k} = \frac{1}{A} f_{,\alpha} a_k + \frac{1}{B} f_{,\beta} b_k + f_{,n} n_k,$$

where A^2 and B^2 are the coefficients of the first quadratic form of the surface S_t. Using Eq. (3.14) we can present Eq. (3.11) in the form

(3.15) $$\int_V \delta\bar{\sigma}_{ij}\varepsilon_{ij} dV = \int_{S_u} \delta t_i u_i^0 dS_u + \int_{S_t} \delta T_i^0 u_i dS_t + \int_{S_t} \{[(T_i^0 u_i)_{,n} - 2T_i^0 u_i H - \sigma_{ij}\varepsilon_{ij}] n_k - T_{i,k}^0 u_i\} \delta\varphi_k dS_t + \int_{S_t} [(T_i^0 u_i n_k \delta\varphi_k)_{,\alpha} + (T_i^0 u_i n_k \delta\varphi_k)_{,\beta}] \frac{1}{AB} dS_t,$$

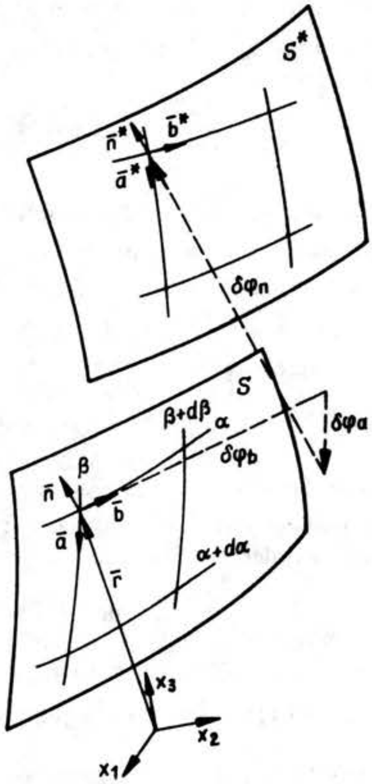

FIG. 3. Parametrization of surface S with the curvilinear coordinate system.

where H denotes the mean curvature of S_t. In writing Eq. (3.15) the following equality was used:

$$2Hn_k = \frac{1}{AB}[(Ba_k)_{,\alpha} + (Ab_k)_{,\beta}]. \tag{3.16}$$

Since the variation $\delta\varphi_k = 0$ on the curve Γ bounding the surface S_t, then the last term on the right-hand side of Eq. (3.15) vanishes and the principle of virtual stresses takes now the form

$$\int_V \delta\bar{\sigma}_{ij}\varepsilon_{ij}dV = \int_{S_u} \delta t_i u_i^0 dS_u + \int_{S_t} \delta T_i^0 u_i dS_t + \int_{S_t} \{[(T_i^0 u_i)_{,n} - 2T_i^0 u_i H - \sigma_{ij}\varepsilon_{ij}]n_k - T_{i,k}^0 u_i\}\,\delta\varphi_k dS_t. \tag{3.17}$$

Let us note that the principle (3.17) (as well as the prinicple (2.17)) holds both in the case of the conservative load system on S_t and in the case of the nonconservative load system.

4. Optimality conditions for the surface S_t

Consider now the problem of optimal design for an elastic body with an unspecified, in advance, loaded boundary S_t. Our discussion will be limited to mean compliance (maximum stiffness) design with a prescribed upper bound on the total material cost of the structure.

This type of design was proposed first by WASIUTYŃSKI [8] and discussed in general terms by MRÓZ [6, 7], who derived global optimality conditions for the shape of the free boundary. Multiparameter formulation of optimal shape problems for external free boundary and for internal surface between particular materials entering into the body was presented by DEMS and MRÓZ [1, 2]. Here we derive the optimality conditions for the external boundary loaded by conservative and nonconservative surface tractions. Both the stress and displace...ent approaches will be used. Let us consider a loaded boundary S_t shown in Fig. 1 and derive the optimality conditions for minimum compliance design. The total cost of the structure is

$$C = cV, \tag{4.1}$$

where c is the specific cost of the material and V denotes the volume of the structure.

Assume the complementary energy as a measure of mean compliance

$$\Pi_\sigma = \int_V W(\sigma_{ij})dV - \int_{S_u} t_i u_i^0 dS_u. \tag{4.2}$$

Let us note that for a homogenous stress energy function, of order $n+1$, in view of Eq. (1.2) there is

$$\Pi_\sigma = \frac{1}{n+1}\int_{S_t} T_i^0 u_i dS_t \quad \text{for} \quad S_u = 0, \tag{4.3}$$

$$\Pi_\sigma = -\frac{1}{n+1}\int_{S_u} t_i u_i^0 dS_u \quad \text{for} \quad S_t = 0 \tag{4.4}$$

and the complementary energy is proportional to the work of surface tractions on S_t or S_u. The optimization problem

$$\text{minimize } \Pi_\sigma, \text{ subject to } C \leqslant C_0, \tag{4.5}$$

where C_0 is the upper bound on the material cost, is now reduced to investigating the conditions for stationarity of the Lagrange functional

$$\Pi'_\sigma(\sigma_{ij}, \varphi_k, \lambda) = \Pi_\sigma + \lambda(C - C_0), \tag{4.6}$$

where λ is a positive Lagrange multiplier. The first variation of Eq. (4.6) with respect to σ_{ij}, φ_k and λ now equals [4]

$$\delta\Pi'_\sigma = \int_V \delta\bar{\sigma}_{ij}\frac{\partial W}{\partial\sigma_{ij}}dV + \int_{S_t} Wn_k\delta\varphi_k dS_t - \int_{S_u}\delta t_i u_i^0 dS_u + \lambda c\int_{S_t} n_k\delta\varphi_k dS_t + \delta\lambda(C - C_0). \tag{4.7}$$

Using the virtual stress equation (3.17), we have the stationarity condition

$$\delta\Pi'_\sigma = \int_{S_t}\{[W + (T_i^0 u_i)_{,n} - 2T_i^0 u_i H - \sigma_{ij}\varepsilon_{ij}]n_k - T^0_{i,k}u_i\}\delta\varphi_k dS_t + \int_{S_t}\delta T_i^0 u_i dS_t + \lambda c\int_{S_t} n_k\delta\varphi_k dS_t + \delta\lambda(C - C_0) = 0. \tag{4.8}$$

Consider now the variation of the surface tractions δT_i^0. For the conservative load system we can write

$$T_i^0 = \partial \Pi_T[u_i(x)]/\partial u_i, \tag{4.9}$$

where Π_T denotes the potential of external forces. Thus the variation of surface tractions, due to variation of boundary configuration, takes the form

$$T_i^0 = T_{i,k}^0 \delta\varphi_k. \tag{4.10}$$

Using now Eq. (4.10) in the stationarity condition (4.8) and taking into account Eq. (3.13), we obtain

$$\delta \Pi_\sigma' = \int_{S_t} [W + (T_i^0 u_i)_{,n} - 2T_i^0 u_i H - \sigma_{ij}\varepsilon_{ij} + \lambda c]\,\delta\varphi_n dS_t + \delta\lambda(C - C_0) = 0. \tag{4.11}$$

Since $\delta\varphi_n$ and $\delta\lambda$ are arbitrary variations, Eq. (4.11) yields the local conditions

$$\begin{gathered} \sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n} = \lambda c \quad \text{on} \quad S_t, \\ C = C_0. \end{gathered} \tag{4.12}$$

Consider now the parameter constrained variation of S_t [1]. Let the boundary modfication function $\varphi_k(x)$ be specified to within a set of L parameters a_l,

$$\varphi_k = \varphi_k(x, a_l), \quad \delta\varphi_k = \frac{\partial\varphi_k}{\partial a_l}\delta a_l, \quad k = 1, 2, 3, \quad l = 1, 2, \ldots, L. \tag{4.13}$$

The stationarity conditions of Π_σ' now take the form

$$\begin{gathered} \int_{S_t} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}] n_k \varphi_{k,a_l} dS_t = \lambda c \int_{S_t} n_k \varphi_{k,a_l} dS_t, \\ C = C_0 \end{gathered} \tag{4.14}$$

and constitute a set of algebraic equations from which the parameters a_l can be determined. The most typical cases of bundary variations will be discussed in Sect. 5.

As an example of a nonconservative load system consider now the surface tractions given in the form

$$T_i^0 = p(x_k) n_i, \tag{4.15}$$

where n_i denotes the external unit normal vector on S_t and $p(x_k)$ is a given function of position. Therefore Eq. (4.15) represents, for example, loading by a pressurized fluid. By using Eq. (3.7), the variation of Eq. (4.15) due to the variation of the boundary configuration can be presented as follows:

$$\delta T_i^0 = \delta p(x_k) n_i + p(x_k)\delta n_i = p_{,k} n_i \delta\varphi_k + p(n_i n_k \delta\varphi_{k,n} - n_k \delta\varphi_{k,i}), \tag{4.16}$$

whereas the work of the force variations on the diplacements u_i can be expressed in the form

$$\int_{S_t} \delta T_i^0 u_i dS_t = \int_{S_t} \{[2pn_i u_i H - (pn_i u_i)_{,n} + (pu_i)_{,i}] n_k + (pn_i)_{,k} u_i\}\,\delta\varphi_k dS_t. \tag{4.17}$$

Using now Eqs. (4.15) and (4.17) in Eq. (4.8) and taking into account Eq. (3.13), we obtain the stationarity condition in the form

$$\delta \Pi_\sigma' = \int_{S_t} [W - \sigma_{ij}\varepsilon_{ij} + (pu_i)_{,i} + \lambda c]\,\delta\varphi_n dS_t + \delta\lambda(C - C_0) = 0. \tag{4.18}$$

The local necessary optimality conditions follow directly from Eq. (4.18):

$$\begin{aligned} \sigma_{ij}\varepsilon_{ij} - W - (pu_i)_{,i} &= \lambda c \quad \text{on} \quad S_t, \\ C &= C_0. \end{aligned} \tag{4.19}$$

In the case of the parameter-constrained variation of S_t (4.13), the global stationarity conditions are similar to Eq. (4.14):

$$\begin{gathered} \int_{S_t} [\sigma_{ij}\varepsilon_{ij} - W - (pu_i)_{,i}] n_k \varphi_{k,a_l} dS_t = \lambda c \int_{S_t} n_k \varphi_{k,a_l} dS_t, \\ C = C_0. \end{gathered} \tag{4.20}$$

The derivation of optimality conditions using the potential energy follows similar steps. Assume the potential energy

$$\Pi_u = \int_V U(\varepsilon_{ij}) dV - \int_{S_t} T_i^0 u_i dS_t \tag{4.21}$$

as a measure of structure stiffness. The optimization problem is now formulated as follows:

$$\text{maximize } \Pi_u, \text{ subject to } C \leqslant C_0. \tag{4.22}$$

The stationarity conditions are derived by considering the functional

$$\Pi_u'(u_i, T_i^0, \varphi_k, \lambda) = \Pi_u - \lambda(C - C_0) \tag{4.23}$$

whose first variation equals [4]

$$\begin{aligned} \delta\Pi_u' = \int_V \frac{\partial U}{\partial \varepsilon_{ij}} \delta\varepsilon_{ij} dV + \int_{S_t} U n_k \delta\varphi_k dS_t - \delta \int_{S_t} T_i^0 u_i dS_t \\ - \lambda c \int_{S_t} n_k \delta\varphi_k dS_t - \delta\lambda(C - C_0) = 0. \end{aligned} \tag{4.24}$$

The variation of the work of surface tractions can be expressed as follows:

$$\delta \int_{S_t} T_i^0 u_i dS_t = \int_{S_t} \delta T_i^0 u_i dS_t + T_i^0 \delta u_i dS_t + T_i^0 u_i \delta(dS_t). \tag{4.25}$$

By using Eq. (2.3) and the equality [5]

$$\delta(dS_t) = (\delta\varphi_{k,k} - n_k \delta\varphi_{k,n}) dS_t, \tag{4.26}$$

Eq. (4.25) can be transformed to the form

$$\begin{aligned} \delta \int_{S_t} T_i^0 u_i dS_t = \int_{S_t} \delta T_i^0 u_i dS_t + T_i^0 \delta\bar{u}_i dS_t + [(T_i^0 u_i)_{,n} - 2T_i^0 u_i H] n_k \delta\varphi_k dS_t \\ - T_{i,k}^0 u_i \delta\varphi_k dS_t + \frac{1}{AB} [(T_i^0 u_i B a_k \delta\varphi_k)_{,\alpha} + (T_i^0 u_i A b_k \delta\varphi_k)_{,\beta}] dS_t, \end{aligned} \tag{4.27}$$

where the last term on the right-hand side equals zero when the variation $\delta\varphi_k = 0$ on the curve Γ bounding the surface S_t. Using the virtual work principle (2.17) and Eq. (4.27) in Eq. (4.24), the stationarity condition of Π_u' can be presented as follows:

$$(4.28)\qquad \delta\Pi'_u = \int_{S_t} \{[U-(T_i^0 u_i)_{,n}+2T_i^0 u_i H]n_k + T^0_{i,k}u_i\}\,\delta\varphi_k dS_t - \int_{S_t} \delta T_i^0 u_i dS_t - \lambda c \int_{S_t} n_k\delta\varphi_k dS_t - \delta\lambda(C-C_0) = 0.$$

When the surface S_t is loaded by the conservative load system (4.9), we obtian from Eq. (4.28) the local optimality conditions

$$(4.29)\qquad \begin{gathered} U-(T_i^0 u_i)_{,n}+2T_i^0 u_i H = \lambda c \quad \text{on } S_t, \\ C = C_0 \end{gathered}$$

or for the parameter-constrained variation of S_t (4.13) the global conditions

$$(4.30)\qquad \begin{gathered} \int_{S_t} [U-(T_i^0 u_i)_{,n}+2T_i^0 u_i H]n_k\varphi_{k,a_l} dS_t = \lambda c \int_{S_t} n_k \varphi_{k,a_l} dS_t, \\ C = C_0. \end{gathered}$$

In the case of the nonconservative load system (4.15), that variation being described by Eqs. (4.16) and (4.17), the local optimality conditions that follow from Eq. (4.28) take the form

$$(4.31)\qquad \begin{gathered} U-(pu_i)_{,i} = \lambda c \quad \text{on } S_t, \\ C = C_0. \end{gathered}$$

The global conditions for the parameter-dependent variation of S_t will be presented as follows:

$$(4.32)\qquad \begin{gathered} \int_{S_t} [U-(pu_i)_{,i}]n_k\varphi_{k,a_l} dS_t = \lambda c \int_{S_t} n_k \varphi_{k,a_l} dS_t, \\ C = C_0. \end{gathered}$$

Let us note that the equivalence of the optimality conditions derived by means of the stress energy and the potential energy functions follows directly from the equality

$$(4.33)\qquad U(\varepsilon_{ij}) + W(\sigma_{ij}) = \sigma_{ij}\varepsilon_{ij}.$$

5. Parameter-constrained simple boundary variations

The derived optimality conditions provide equations for the function $\varphi_i(x)$ defining the loaded boundary S_t for any three-dimensional structure. In this Section we restrict our discussion to a plane case when the stress state in the direction x_3 normal to the plane $x_1 x_2$ is uniform or vanishes and the structure shape in the $x_1 x_2$-plane is to be determined. We consider several simpler cases depending on a set of shape parameters a_l.

In the following we shall assume that the optimization problem is formulated by using the stress energy function and the structure is loaded by conservative surface tractions on S_t. Thus the optimality conditions (4.14) will be used for determining the shape parameters. Other cases of the optimization problem can be considered in a similar manner.

5.1. Piecewise linear boundary

Consider a boundary composed of a finite number of linear segments, Fig. 4, forming a polygon of r sides. Let boundary modification be performed by describing a displacement vector $\varphi_i^{(j)}$ to each polygon vertex. Since after modification each boundary segment should

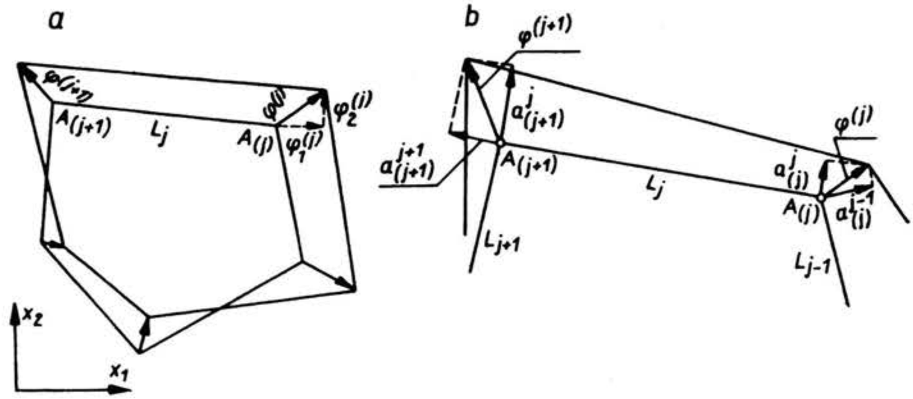

FIG. 4. Piecewise linear boundary; a) Variation of boundary; b) Decomposition of vertex displacements and shape parameters of the boundary.

remain linear, the boundary displacement function for the j-th segment φ_i^j takes the form

$$(5.1)\quad \varphi_i^j(s) = \frac{1}{L_j}\left[(L_j - s)\varphi_i^{(j)} + s\varphi_i^{(j+1)}\right], \quad 0 \leqslant s \leqslant L_j, \quad i = 1, 2, \quad j = 1, 2, \ldots, r,$$

where L_j denotes the length of the side j.

Assume now that the displacement components of the vertices $A_{(j)}$, normal to the sides $j-1$ and j and denoted by $a_{(j)}^{j-1}$, $a_{(j)}^j$, are the shape parameters and should be determined from the optimality conditions. Thus the boundary modification function for the j-th segment can be expressed as follows:

$$(5.2)\quad \begin{aligned} \varphi_1^j &= \frac{1}{L_j}\left[(L_j - s)\frac{a_{(j)}^{j-1}n_2^j - a_{(j)}^j n_2^{j-1}}{n_1^{j-1}n_2^j - n_2^{j-1}n_1^j} + s\,\frac{a_{(j+1)}^j n_2^{j+1} - a_{(j+1)}^{j+1} n_2^j}{n_1^j n_2^{j+1} - n_2^j n_1^{j+1}}\right], \\ \varphi_2^j &= \frac{1}{L_j}\left[-(L_j - s)\frac{a_{(j)}^{j-1}n_1^j - a_{(j)}^j n_1^{j-1}}{n_1^{j-1}n_2^j - n_2^{j-1}n_1^j} - s\,\frac{a_{(j+1)}^j n_1^{j+1} - a_{(j+1)}^{j+1} n_1^j}{n_1^j n_2^{j+1} - n_2^j n_1^{j+1}}\right], \\ &0 \leqslant s \leqslant L_j, \quad j = 1, 2, \ldots, r, \end{aligned}$$

where $a_{(j)}^{j-1}$ and $a_{(j)}^j$ form a set of $2r$ shape parameters.

Using now Eq. (5.2) in the optimality conditions (4.14), we obtain a set of $2r$ equations:

$$(5.3)\quad \begin{aligned} \frac{1}{L_j}\int_0^{L_j} \left[\sigma_{ij}\varepsilon_{ij} - W - (T_i^0 u_i)_{,n}\right](L_j - s)\,ds &= \frac{1}{2}\lambda c L_j, \\ \frac{1}{L_j}\int_0^{L_j} \left[\sigma_{ij}\varepsilon_{ij} - W - (T_i^0 u_i)_{,n}\right] s\,ds &= \frac{1}{2}\lambda c L_j \end{aligned}$$

from which $2r$ parameters $a^k_{(j)}$ defining the shape of the optimal boundary can be determined. The Lagrange multiplier λ is found from the condition of the constant material cost of the polygon.

5.2. Rigid-body translation of a closed contour

Consider now a translation of a closed boundary where each point undergoes the same displacement, Fig. 5. Assuming that the two independent parameters a_1, a_2 define the

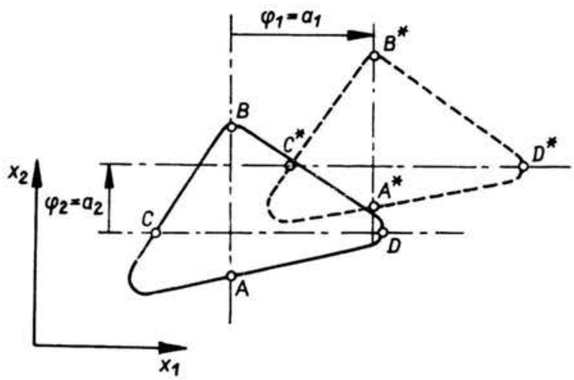

FIG. 5. Translation of a closed contour.

position of the domain enclosed by the surface S_t, the boundary modification function can be presented in the form

$$\varphi_i = a_i = \text{const}, \quad i = 1, 2 \tag{5.4}$$

and from Eq. (4.14) we obtain two stationarity conditions:

$$\begin{gathered}\int\limits_{\widehat{ACB}} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}]dx_2 = \int\limits_{\widehat{ADB}} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}]dx_2,\\ \int\limits_{\widehat{CAD}} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}]dx_1 = \int\limits_{\widehat{CDB}} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}]dx_1,\end{gathered} \tag{5.5}$$

where integration is performed on portions ACB, BDA, CAD and DBC, respectively.

5.3. Rotation of a closed contour

Consider now the rotation of a closed boundary around a point 0, Fig. 6. The displacements of the typical boundary point P equal

$$\begin{aligned}\varphi_1 &= -x_1^0(1-\cos\omega) - x_2^0\sin\omega,\\ \varphi_2 &= x_1^0\sin\omega - x_2^0(1-\cos\omega),\end{aligned} \tag{5.6}$$

where x_1^0, x_2^0 are the initial coordinates of the point P and ω denotes the angle of rotation,

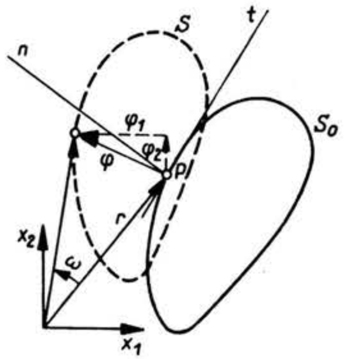

FIG. 6. Rotation of a closed contour.

which plays the role of the shape parameter of boundary modification. By using Eq. (5.6) in Eq. (4.14), the stationarity condition now takes the form

(5.7) $$\int_{S_t} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}](x_1 dx_1 + x_2 dx_2) = 0.$$

5.4. Expansion and contraction of the closed contour

Let us give a family of closed curves described by the equation

(5.8) $$F(x_1, x_2, k) = 0$$

and, furthermore, let F be a homogeneous function of order p of its arguments, such that

(5.9) $$F(tx_1, tx_2, tk) = t^p F(x_1, x_2, k).$$

For $k = k_0$ Eq. (5.8) describes the optimal boundary S_t. Let us assume that the coefficient k can be expressed as follows:

(5.10) $$k = k_0 + a = k_0\left(1 + \frac{a}{k_0}\right),$$

where a is the shape parameter of boundary modification. For $a > 0$ the boundary undergoes an expansion, whereas for $a < 0$ — contraction. By using Eqs. (5.9) and (5.10), the boundary modification function can thus be presented in the form

(5.11) $$\varphi_i = \frac{x_i}{k_0} a_i, \quad i = 1, 2,$$

where x_i are the coordinates of the boundary points. Using now Eq. (5.11), the stationarity condition (5.11) yields

(5.12) $$\int_{S_t} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}](x_1 dx_2 - x_2 dx_1) = 2\lambda c A,$$

where A denotes the area of the surface bounded by the curve $F(x_1, x_2, k_0) = 0$. The equations of the form (5.12) together with the condition of the constant material cost of the structure constitute a set of algebraic equations from which the parameters k_0 for all considered boundaries and the Lagrange multiplier λ can be determined.

Let now Eq. (5.8) represent the family of concentric circles described by the equation

$$x_1^2+x_2^2-k^2 = 0, \tag{5.13}$$

where k denotes a radius of the circle. In such a case the boundary modification function (5.11) represents the translation of the boundary points along radial directions, and the stationarity condition (5.12) takes the form

$$\int_{S_t} \left[\sigma_{ij}\varepsilon_{ij}-W+\frac{1}{k_0}T_i^0u_i-(T_i^0u_i)_{,n}\right]ds = 2\pi\lambda ck_0, \tag{5.14}$$

where k_0 is the required radius of the optimal boundary.

5.5. General modification of a boundary

The discussed boundary modifications contained several simpler transformations of optimized boundaries. Consider now a more general parameter modification. Let us assume

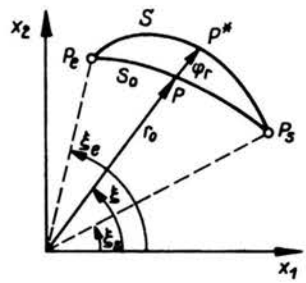

FIG. 7. General modification of a boundary.

to this end that the optimized boundary can be described in a polar coordinate system (r, ξ). Fig. 7, by the equation

$$r = r_0(\xi)+\varphi_r(\xi), \quad \xi_s \leqslant \xi \leqslant \xi_e, \tag{5.15}$$

where $r_0(\xi)$ is a reference shape function. The function $\varphi_r(\xi)$ modifying the boundary along radial direction can be expressed as follows:

$$\varphi_r(\xi) = \sum_{l=1}^{L} a_l f_l(\xi), \tag{5.16}$$

where f_l are smooth functions each satisfying relevant end conditions and a_l denote the L shape parameters.

Transforming φ_r to the Cartesian coordinate system (x_1, x_2), the boundary modification functions take the form

$$\begin{aligned} \varphi_1 &= \sum_{l=1}^{L} a_l f_l(\xi)\cos\xi, \\ \varphi_2 &= \sum_{l=1}^{L} a_l f_l(\xi)\sin\xi \end{aligned} \tag{5.17}$$

and the optimality conditions (4.14) constitute a set of $L+1$ equations:

$$(5.18)\qquad \int_{\xi_s}^{\xi_e} [\sigma_{ij}\varepsilon_{ij} - W + 2T_i^0 u_i H - (T_i^0 u_i)_{,n}]\Big(r_0 + \sum_{l=1}^{L} a_l f_l\Big) f_k d\xi$$

$$= \lambda c \int_{\xi_s}^{\xi_e} \Big(r_0 + \sum_{l=1}^{L} a_l f_l\Big) f_k d\xi, \quad k = 1, 2, \ldots, L,$$

$$C = C_0$$

from which the shape parameters a_l and the Lagrange multiplier λ can be determined.

6. Optimal design of a circular disk with a circular hole

As simple illustration of utilization of the stationarity conditions obtained in Sect. 4, let us consider the optimal design problem of a circular disk with a circular hole that is loaded by uniform internal pressure p_i and external pressure p_e, Fig. 8. The disk with an inner radius r_i and an outer one r_e is made of a linearly elastic material. The optimization

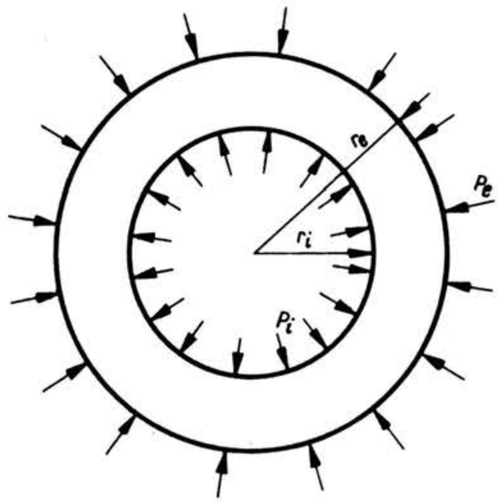

FIG. 8. Circular disk with a hole subject to uniformly distributed pressures.

problem is now reduced to determining these radii under the condition of constant material cost of the disk. Moreover, let us assume that the state of plane stress is considered.

The cost of the disk is assumed to be proportional to

$$(6.1)\qquad C = c\pi(r_e^2 - r_i^2).$$

The complementary energy of the disk equals

$$(6.2)\qquad \Pi_\sigma = \frac{1}{2E}\int_{r_i}^{r_e} (\sigma_r^2 - 2\nu\sigma_r\sigma_t + \sigma_t^2) r\,dr,$$

where σ_r and σ_t are the radial and circumferential stress components, while E and ν denote elastic constants. The equilibrium equation

$$\frac{d}{dr}(r\sigma_r)-\sigma_t = 0 \tag{6.3}$$

should be accompanied by boundary conditions:

$$\begin{aligned} T_r^0 &= -\sigma_r = p_i, \quad T_t^0 = 0 \quad \text{for} \quad r = r_i, \\ T_r^0 &= \sigma_r = -p_e, \quad T_t^0 = 0 \quad \text{for} \quad r = r_e \end{aligned} \tag{6.4}$$

and stationarity conditions (4.19) on the surfaces $r = r_i$ and $r = r_e$, which, expressed in terms of stress components, could be written in the form

$$\begin{aligned} (\sigma_t+p_i)^2-2(1-\nu)p_i^2 &= 2\lambda cE \quad \text{for} \quad r = r_i, \\ (\sigma_t+p_e)^2-2(1-\nu)p_e^2 &= 2\lambda cE \quad \text{for} \quad r = r_e. \end{aligned} \tag{6.5}$$

Equation (6.3) is satisfied for the stress field

$$\sigma_r = \frac{A}{r^2}+B, \qquad \sigma_t = -\frac{A}{r^2}+B \tag{6.6}$$

and the boundary conditions (6.4) are satisfied when

$$A = \frac{r_i^2 r_e^2}{r_e^2-r_i^2}(p_e-p_i), \qquad B = \frac{p_i r_i^2-p_e r_e^2}{r_e^2-r_i^2}. \tag{6.7}$$

The optimality conditions (6.5), in view of Eqs. (6.6) and (6.7), take the form

$$\begin{aligned} \frac{2r_e^4}{(r_e^2-r_i^2)^2}(p_i-p_e)^2-(1-\nu)p_i^2 &= \lambda cE, \\ \frac{2r_i^4}{(r_e^2-r_i^2)^2}(p_i-p_e)^2-(1-\nu)p_e^2 &= \lambda cE. \end{aligned} \tag{6.8}$$

The constraint on the cost of the disk in view of Eq. (6.1) can be expressed as follows:

$$r_e^2-r_i^2 = q, \tag{6.9}$$

where $q > 0$ is the prescribed relative cost of the design.

Equations (6.8) and (6.9) constitute a set of equations with 3 unknowns r_i, r_e and λ. The optimal radii determined from this set are then

$$r_i = \frac{1}{2}\sqrt{q\frac{(3-\nu)p_e-(1+\nu)p_i}{p_i-p_e}}, \qquad r_e = \frac{1}{2}\sqrt{q\frac{(3-\nu)p_i-(1+\nu)p_e}{p_i-p_e}} \tag{6.10}$$

under the condition

$$1 < \frac{p_i}{p_e} < \frac{3-\nu}{1+\nu}. \tag{6.11}$$

If the condition (6.11) is not valid, then there is no real solution of the optimality conditions (6.8) and the complementary energy of the disk does not attain the optimal value in the sense of the formulation above. In that case, however, the mean compliance of the disk of constant material cost decreases together with the value of the inner radius r_i tending to zero.

The mean compliance of the disk (6.2) in view of Eqs. (6.6), (6.7) and (6.9) can now be presented in the form

$$\Pi_\sigma = \frac{1}{2qE}\left[(1+\nu)r_i^2 r_e^2(p_e-p_i)^2+(1-\nu)(p_i r_i^2-p_e r_e^2)^2\right]. \tag{6.12}$$

A similar consideration can be made for the case of the state of plane strain. The solution of the optimization problem is once again described by Eqs. (6.10)÷(6.12), in which the Young's modulus E should be replaced by $E/(1-\nu^2)$ and the Poisson's ratio ν by $\nu/(1-\nu)$.

Figure 9 shows the variation of the disk compliance (6.12) as a function of the radius r_i for a given value of cost of the design q. Both the states of plane stress and plane strain

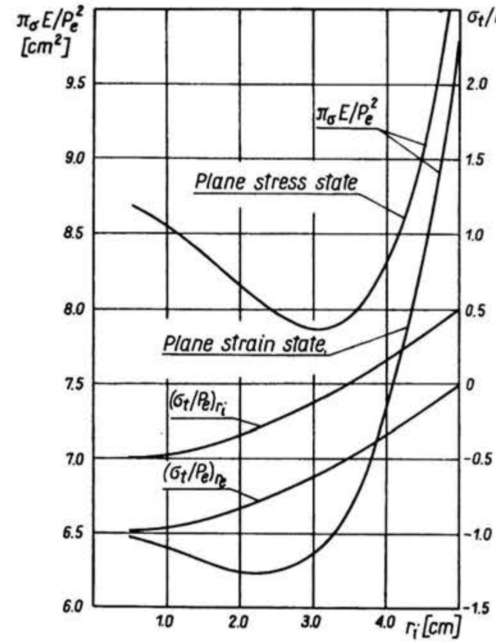

FIG. 9. Mean disk compliance and circumferential stresses versus inner radius ($\nu = 0.3$, $q = 25.0$ cm², $p_i/p_e = 1.5$).

are considered. Moreover, the change of circumferential stress σ_t on the inner and outer edge of the disk is shown. It is easy to see that the values of r_i and r_e satisfying Eqs. (6.70) correspond to a global minimum of the mean disk compliance.

7. Conclusions

The derived optimality conditions generate the nonlinear set of equations which determine the shape parameters of the loaded boundaries. The solution of this set is possible, in general, through the iterative procedure analysis-synthesis similar to that already discussed in [1, 3], where the finite element formulation of optimal shape design was presented. So far, our analysis has been confined to mean compliance design, but other behavioural constraints can easily be incorporated.

The derived virtual displacement and stress principles can thus constitute a foundation for a more general class of problems of optimal structural synthesis which will be discussed in consecutive papers.

The optimality conditions for a free boundary are automatically generated by assuming the surface tractions on a part of S_t as vanishing.

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