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On coupling acceleration and shock waves in a thermoviscoplastic medium II. One-dimensional waves

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THE PAPER contains an analysis of one-dimensional shock waves in a thermoviscoplastic medium. In the linear theory, nonlinear ordinary differential equations were obtained for the thermal and mechanical amplitude. The governing equations, for plane waves, in the form of quasi-linear hyperbolic partial differential equations were solved by using Riemann's method.

Praca poświęcona jest jednowymiarowym falom silnych nieciągłości w ośrodku termolepkoplastycznym. Dla liniowych związków konstytutywnych otrzymano nieliniowe równania różniczkowe zwyczajne opisujące zmianę w czasie amplitudy termicznej i mechanicznej. Dla płaskich fal rozwiązano hiperboliczny układ równań różniczkowych cząstkowych za pomocą metody Riemanna.

Работа посвящена одномерным волнам сильного разрыва в термовязкопластической среде. Для линейных определяющих соотношений получены нелинейные обыкновенные дифференциальные уравнения, описывающие изменение во времени термической и механической амплитуды. Для плоских волн решена гиперболическая система дифференциальных уравнений в частных производных с помощью метода Римана.

1. Introduction

IN PART I of the paper [8] the constitutive equations of thermoviscoplasticity with the Maxwell–Cattaneo relation for the heat flux are formulated. The objective of this paper is to discuss the propagation of one-dimensional waves as well as the acceleration and the shock waves. In the problem of the plane shock waves propagating in a thin rod described by the system of linear equations, the closed form of the solution is obtained.

2. Acceleration waves

The one-dimensional motion is described by a scalar function $\chi(X, t) = x$, which determines the location x at the time t of the material point X, the displacement $u(X, t) = \chi(X, t) - X$, the strain $E(X, t) = \partial_X u(X, t)$ and the temperature gradient $g(X, t) = \partial_X \vartheta(X, t)$. The thermomechanical state G (cf. Part I) contains the functions $\{E, \vartheta, \partial_X \vartheta, \mathbf{a}\}$ where there are four internal parameters $\mathbf{a} = \{\alpha, \beta, \gamma, \varkappa\}$ introduced in the Part I of this paper. Parameter α is the inelastic deformation, β — the thermal parameter, $\varkappa$ — the strain hardening parameter, γ — the viscosity parameter. The material response is described by a set of the four functions: T, ψ, η, q, where T is the stress, ψ — the free energy, η — the entropy and q — the heat flux.

In that case the basic system of equations for the thermoviscoplastic body (cf. Eqs. (3.8) in Part I) reduces to seven equations with respect to $v, E, \vartheta, \alpha, \beta, \gamma, \varkappa$, v is the velocity:

$$\dot{v}-\frac{1}{\varrho_0}\partial_E\mathcal{T}\partial_X E-\frac{1}{\varrho_0}\partial_\vartheta\mathcal{T}\partial_X\vartheta-\frac{1}{\varrho_0}\partial_\alpha\mathcal{T}\partial_X\alpha-\frac{1}{\varrho_0}\partial_\beta\mathcal{T}\partial_X\beta$$
$$-\frac{1}{\varrho_0}\partial_\gamma\mathcal{T}\partial_X\gamma-\frac{1}{\varrho_0}\partial_\varkappa\mathcal{T}\partial_X\varkappa=0,$$
$$\dot{E}-\partial_X v=0,$$
$$\dot{\vartheta}+G_1\partial_X v+G_2\partial_X E+G_3\partial_X\vartheta+G_4\partial_X\alpha+G_5\partial_X\beta+G_6\partial_X\gamma+G_7\partial_X\varkappa+G_8=0,$$
$$\dot{\alpha}-A=0,$$
$$\dot{\beta}-B_1\partial_X\vartheta-B_2=0, \tag{2.1}$$
$$\dot{\gamma}-\Gamma A=0,$$
$$\dot{\varkappa}-\mathcal{K}A=0,$$

where G_i, $i=1,\ldots,8$ are the following scalar functions of the arguments $E, \vartheta, \alpha, \beta, \varkappa, \gamma$ (cf. the denotation (3.9) in Part I):

$$G_1=-c^{-1}\vartheta P,\quad G_2=c^{-1}R,\quad G_3=c^{-1}(W+I),\quad G_4=c^{-1}M,$$
$$G_5=c^{-1}N,\quad G_6=c^{-1}L,\quad G_7=c^{-1}Ł,\quad G_8=c^{-1}H, \tag{2.2}$$

c is the specific heat of the material and $P, R, W, I, M, N, L, Ł, H$ are material functions defined by

$$\varrho_0\vartheta\partial_\vartheta\mathcal{N}=c,\quad \partial_\vartheta\mathcal{T}=P,\quad \varrho_0B_1(\vartheta\partial_\beta\mathcal{N}+\partial_\beta\Psi)=W,\quad \partial_E Q=R,$$
$$\partial_\vartheta Q=I,\quad \partial_\alpha Q=M,\quad \partial_\beta Q=N,\quad \partial_\gamma Q=L,\quad \partial_\varkappa Q=Ł,$$
$$H=\varrho_0[(\vartheta\partial_\beta\mathcal{N}+\partial_\beta\Psi)B_2+(\vartheta\partial_\alpha\mathcal{N}+\partial_\alpha\Psi)A+(\vartheta\partial_\gamma\mathcal{N}+\partial_\gamma\Psi)\Gamma A+(\vartheta\partial_\varkappa\mathcal{N}+\partial_\varkappa\Psi)\mathcal{K}A,$$

$\mathcal{T}, \Psi, \mathcal{N}, Q$ are constitutive functions for stress, free energy, entropy and heat flux.

From now on we assume that the constitutive function for the free energy ψ (cf. Eq. $(2.6)_1$ in Part I) has the simpler form

$$\Psi=\Psi_1(E,\vartheta,\alpha)+\Psi_2(\beta)+\Psi_3(\alpha,\gamma,\varkappa). \tag{2.3}$$

Additionally, the domain of the function B_1 is restricted to

$$B_1=B_1(\dot{\alpha},\beta,\gamma,\varkappa). \tag{2.4}$$

From Eqs. (2.3) and (2.4) it follows that

$$\partial_\beta\mathcal{T}=0,\quad \partial_\gamma\mathcal{T}=0,\quad \partial_\varkappa\mathcal{T}=0,\quad G_2=0,\quad G_3=0 \tag{2.5}$$

and (cf. Eq. $(2.6)_4$ in Part I)

$$q=-\varrho_0\vartheta B_1\partial_\beta\Psi. \tag{2.6}$$

The characteristic Eq. (3.12) given in Part I for the direction $\mathbf{n}=(1,0,0)$ has seven roots λ:

$$\lambda^3\left\{\lambda^4+\lambda^2\left(B_2G_5-\frac{1}{\varrho_0}\partial_E\mathcal{T}+\frac{1}{\varrho_0}G_1\partial_\vartheta\mathcal{T}\right)-\frac{1}{\varrho_0}B_1G_5\partial_E\mathcal{T}\right\}=0. \tag{2.7}$$

The roots are real provided that

$$(2.8)\qquad \begin{gathered} \left(B_1 G_5 - \frac{1}{\varrho_0}\partial_E\mathscr{T} + \frac{1}{\varrho_0}G_1\partial_\vartheta\mathscr{T}\right)^2 + 4\frac{1}{\varrho_0}B_1 G_5 \partial_E\mathscr{T} > 0, \\ -\left(B_1 G_5 - \frac{1}{\varrho_0}\partial_E\mathscr{T} + \frac{1}{\varrho_0}G_1\partial_\vartheta\mathscr{T}\right) > 0, \\ -\frac{1}{\varrho_0}B_1 G_5 \partial_E\mathscr{T} > 0. \end{gathered}$$

These inequalities are satisfied under the following assumption:

$$(2.9)\qquad \partial_E\mathscr{T} > 0, \quad \partial_\vartheta\mathscr{N} > 0, \quad \partial_\vartheta\mathscr{T} < 0, \quad -B_1 G_5 > 0.$$

Additionally, the inequality (2.9) implies

$$(2.10)\qquad \partial_E\mathscr{N} > 0, \quad G_1 > 0.$$

Let us consider two simplifications of Eq. (2.7).

1. There is no thermomechanical coupling, $\partial_\vartheta\mathscr{T} = 0$ $(G_1 = 0)$. There are two kinds of symmetric waves, the thermal one and the mechanical one with the speed $\pm\lambda_T$ and $\pm\lambda_m$:

$$(2.11)\qquad \begin{aligned} \lambda_T^2 &= -B_1 G_5, \\ \lambda_m^2 &= \frac{1}{\varrho_0}\partial_E\mathscr{T}. \end{aligned}$$

The first derivative of temperature is discontinuous while $\dot{v}$, $\partial_X E$ are continuous $\left([\![\dot{\vartheta}]\!] \neq 0, [\![\dot{v}]\!] = 0\right)$ at the thermal wave. On the contrary, $[\![\dot{\vartheta}]\!] = 0$ and $[\![\dot{v}]\!] \neq 0$ at the mechanical wave.

2. In the case of a nonconductor $(Q(\vartheta, \alpha, \beta, \gamma, \varkappa) = 0, G_5 = 0)$ two symmetric waves propagate into the material with the adiabatic speed

$$(2.12)\qquad \lambda_a^2 = \frac{1}{\varrho_0}\partial_E\mathscr{T} - \frac{1}{\varrho_0}G_1\partial_\vartheta\mathscr{T}.$$

Here $[\![\dot{v}]\!] \neq 0$ and $[\![\dot{\vartheta}]\!] \neq 0$.

We can see that the speeds λ_1^2, λ_2^2 which are roots of Eq. (2.7) together with λ_T^2, λ_m^2 and λ_a^2 satisfy the inequalities

$$(2.13)\qquad \begin{aligned} &\lambda_1^2 > \lambda_T^2 > \lambda_2^2, \\ &\lambda_1^2 > \lambda_a^2 > \lambda_m^2 > \lambda_2^2. \end{aligned}$$

3. Propagation of shock waves, linear case

Let us assume that the strain gradient E and temperature ϑ are discontinuous across Σ where

$$(3.1)\qquad \Sigma = \{(X, t): X = Y(t), \quad t \in [0, \infty)\}$$

This means that

$$
(3.2)\qquad
\begin{aligned}
&[\![E]\!](t) = E^-(t) - E^+(t) \neq 0, &\quad& [\![\vartheta]\!](t) = \vartheta^-(t) - \vartheta^+(t) \neq 0,\\
&E^-(t) = \lim_{X\to Y(t)^-} E(X,t), && \vartheta^-(t) = \lim_{X\to Y(t)^-} \vartheta(X,t),\\
&E^+(t) = \lim_{X\to Y(t)^+} E(X,t), && \vartheta^+(t) = \lim_{X\to Y(t)^+} \vartheta(X,t).
\end{aligned}
$$

The derivative $\frac{d}{dt}Y(t) = V(t)$ is called the intrinsic velocity of the wave. Furthermore, let us assume that the free energy Ψ has the biquadratic form

$$
(3.3)\qquad \Psi(E,\vartheta,\alpha,\beta,\gamma,\varkappa) = \frac{1}{\varrho_0}\left[\frac{a_1}{2}(E-\alpha)^2 + a_2(E-\alpha)(\vartheta-\vartheta^{\#})\right.
$$
$$
\left. + \frac{a_3}{2}(\vartheta-\vartheta^{\#})^2 + \frac{a_4}{2}\beta^2 + \frac{a_5}{2}(\gamma-\gamma^{\#})^2 + \frac{a_6}{2}(\varkappa-\varkappa^{\#})^2\right]
$$

where $a_i = \text{const}$, $i = 1, \ldots, 6$, $(a_4 = a\tau)$.

For β we postulate (cf. Eqs. (2.9) and (2.18) in Part I)

$$
(3.4)\qquad \dot{\beta} = \frac{b}{\tau}\partial_X\vartheta - \frac{1}{\tau}\beta.
$$

Under the constitutive equation (3.3) for ψ we obtain the constitutive equations for T, η and q (cf. Eqs. (2.6) in Part I)

$$
(3.5)\qquad
\begin{aligned}
T &= a_1(E-\alpha) + a_2(\vartheta-\vartheta^{\#}),\\
\eta &= -\frac{a_2}{\varrho_0}(E-\alpha) - \frac{a_3}{\varrho_0}(\vartheta-\vartheta^{\#}),\\
q &= -ba\vartheta^{\#}\beta.
\end{aligned}
$$

Assuming that the coefficients of the system (2.1) $G_i = i = 1, \ldots, 7$ are constant and equal to their values at the equilibrium state, the system reduces to

$$
(3.6)\qquad
\begin{aligned}
\dot{v} - \frac{1}{\varrho_0}a_1\partial_X E - \frac{1}{\varrho_0}a_2\partial_X\vartheta + \frac{1}{\varrho_0}a_1\partial_X\alpha &= 0,\\
\dot{E} - \partial_X v &= 0,\\
\dot{\vartheta} + g_1\partial_X v + g_5\partial_X\beta + G_8 &= 0,\\
\dot{\alpha} - A &= 0,\\
\dot{\beta} - \frac{b}{\tau}\partial_X\vartheta + \frac{1}{\tau}\beta &= 0,\\
\dot{\gamma} - \Gamma A &= 0,\\
\dot{\varkappa} - \mathscr{K}A &= 0,
\end{aligned}
$$

where

$$
(3.7)\qquad G_1^{\#} \equiv g_1 = \frac{a_2}{a_3} \qquad G_5^{\#} \equiv g_5 = \frac{ba}{a_3} \qquad G_2^{\#} = G_3^{\#} = G_4^{\#} = G_6^{\#} = G_7^{\#} = 0
$$

and

$$G_8(E, \vartheta, \alpha, \gamma, \varkappa) = -\frac{a_2}{a_3} A(E, \vartheta, \alpha, \gamma, \varkappa) = -\frac{a_2}{a_3}\dot{\alpha}. \tag{3.8}$$

In that case λ as the solutions of Eq. (2.7) are constant. The system (3.9) can be rewritten in the form

$$\partial_t \mathbf{u} + \partial_X \mathscr{F}(\mathbf{u}) + \mathbf{B}(\mathbf{u}) = 0,$$

$$\mathbf{u} = (v, E, \vartheta, \alpha, \beta, \gamma, \varkappa), \quad \partial_X \mathscr{F}(\mathbf{u}) = \mathbf{\Omega}\partial_X \mathbf{u},$$

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\frac{1}{\varrho_0}a_1 & -\frac{1}{\varrho_0}a_2 & \frac{1}{\varrho_0}a_1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_1 & 0 & 0 & 0 & g_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sigma}{\tau} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ G_8 \\ -A \\ \frac{1}{\tau}\beta \\ -\Gamma A \\ -\mathscr{K}A \end{bmatrix}. \tag{3.9}$$

From the continuity of B with respect to $\mathbf{u}$ we have for Eq. (3.9) the Rankine-Hugoniot condition

$$\lambda[\![\mathbf{u}]\!] = [\![\mathscr{F}]\!], \tag{3.10}$$

which for the system (2.1) takes the form

$$\begin{aligned} \varrho_0 \lambda[\![v]\!] &= -a_1[\![E]\!] - a_2[\![\vartheta]\!], \\ \lambda[\![E]\!] &= -[\![v]\!], \\ \lambda[\![\vartheta]\!] &= g_1[\![v]\!] + g_5[\![\beta]\!], \\ \lambda[\![\alpha]\!] &= 0, \\ \lambda[\![\beta]\!] &= -b_1[\![\vartheta]\!], \\ \lambda[\![\gamma]\!] &= \lambda[\![\varkappa]\!] = 0, \quad b_1 \equiv \frac{b}{\tau}. \end{aligned} \tag{3.11}$$

The last equality implies the continuity of α, γ and $\varkappa$ across Σ. For the known magnitude of a jump of one function we can determine another:

$$\begin{aligned} [\![\vartheta]\!] &= -\frac{g_1\lambda^2}{\lambda^2 + b_1 g_5}[\![E]\!], \\ [\![\beta]\!] &= \frac{b_1 g_1 \lambda}{\lambda^2 + b_1 g_5}[\![E]\!], \\ [\![v]\!] &= -\lambda[\![E]\!], \\ [\![\beta]\!] &= -\frac{b_1}{\lambda}[\![\vartheta]\!], \\ [\![T]\!] &= \varrho_0\lambda^2[\![E]\!]. \end{aligned} \tag{3.12}$$

The speeds $\lambda \neq 0$ of shock waves are roots of the polynomial obtained from the system (3.6) provided the condition (3.10) and assumption $[\![\mathbf{u}]\!] \neq 0$ are fulfilled:

$$\lambda^4+\lambda^2\left(b_1 g_5-\frac{1}{\varrho_0}a_1+\frac{1}{\varrho_0}g_1 a_2\right)-\frac{1}{\varrho_0}b_1 g_5 a_4 = 0. \tag{3.13}$$

After ACHENBACH [1] we can write Eq. (3.13) in the dimensionless form:

$$\begin{gathered}\left(\frac{\lambda}{\lambda_m}\right)^4-\left(\frac{\lambda}{\lambda_m}\right)^2(d^2+1+\delta^2)+d^2 = 0,\\ d^2 \equiv \frac{\lambda_T^2}{\lambda_m^2}, \quad \delta = \frac{-\dfrac{1}{\varrho_0}g_1 a_2}{\lambda_m^2}, \quad \lambda_T^2 = -b_1\frac{ab}{a_3}, \quad \lambda_m^2 = \frac{1}{\varrho_0}a_1.\end{gathered} \tag{3.14}$$

The constant δ depends on the coupling of mechanical and thermal effects. For non-conductors we have $d = 0$ ($\lambda_T = 0$), the product $ba = 0$ and then the conductivity coefficient $K = 0$ ($K = b^2a$). In the case of materials for which the Fourier law is valid we have $d \to \infty (\tau = 0)$ (cf. Sect. 2 in Part I).

Now we derive the differential equation for the magnitude of the strain gradient jump using the compatibility conditions (3.11), the Maxwell's theorem of an arbitrary discontinuous function f:

$$\frac{d}{dt}[\![f]\!] = [\![\partial_t f]\!]+\lambda[\![\partial_x f]\!] \tag{3.15}$$

and the jump form of Eqs. (3.6):

$$\begin{aligned}
[\![\dot{v}]\!] &= \frac{1}{\varrho_0}a_1[\![\partial_x E]\!]+\frac{1}{\varrho_0}a_2[\![\partial_x\vartheta]\!]-\frac{1}{\varrho_0}a_1[\![\partial_x\alpha]\!],\\
[\![\dot{E}]\!] &= [\![\partial_x v]\!],\\
[\![\dot{\vartheta}]\!] &= -g_1[\![\partial_x v]\!]-g_5[\![\partial_x\beta]\!]-[\![G_8]\!],\\
[\![\dot{\alpha}]\!] &= [\![A]\!],\\
[\![\dot{\beta}]\!] &= b_1[\![\partial_x\vartheta]\!]+\frac{1}{\tau}[\![\beta]\!],\\
[\![\dot{\gamma}]\!] &= [\![\Gamma A]\!], \quad [\![\dot{\varkappa}]\!] = [\![\mathscr{K}A]\!].
\end{aligned} \tag{3.16}$$

After substituting f for E in Eq. (3.15) and eliminating $\dot{E}$ and $\partial_x E$, we conclude that

$$\frac{d}{dt}[\![E]\!] = -C_1[\![A]\!]-C_2[\![E]\!], \tag{3.17}$$

where C_1 and C_2 are constants:

$$C_1 = \frac{1}{2}\left(\frac{\lambda}{\lambda_m}\right)^2\frac{1-\left(\dfrac{\lambda}{\lambda_T}\right)^2}{1-\dfrac{\lambda^4}{\lambda_T^2\lambda_m^2}} > 0, \quad C_2 = \frac{1}{2\tau}\frac{1-\left(\dfrac{\lambda}{\lambda_m}\right)^2}{1-\dfrac{\lambda^4}{\lambda_T^2\lambda_m^2}} > 0. \tag{3.18}$$

Equation (3.17) describes the change in time of the amplitude $[\![E]\!]$ for two shock waves. Equation (3.17) corresponds to three particular cases.

In the case of an elastic/viscoplastic material we have

$$\frac{d}{dt}[\![E]\!] = -\frac{1}{2}[\![A]\!].$$

Here viscosity leads to the damping of $[\![E]\!]$, what means that $\lim_{t\to\infty}[\![E]\!](t) = 0$.

For a thermoplastic material ($A = 0$), Eq. (3.17) is linear (cf. ACHENBACH [1]) and the amplitude decreases in time:

$$[\![E]\!](t) = le^{-C_2 t}, \quad l = \text{const}. \tag{3.19}$$

We can see that $C_2^1 > C_2^2$(¹) for $d^2 > 1+\delta$ and for $d^2 < 1+\delta$ there is stronger damping at the slow wave. Moreover, for $d^2 < 1 (\lambda_T^2 < \lambda_m^2)$ the slow wave is damped more quickly than the fast one, independently of the quantity δ. However for an elastic material the amplitude is constant along the shock wave, $[\![E]\!](t) = \text{const}$ and is equal to the initial condition for jump $[\![E]\!]\,(t_0)$.

The initial conditions needed for solving Eq. (3.17) can be obtained from the boundary conditions of Eqs. (3.6) in the following way. Suppose that the pressure $p(t)$ and temperature $\theta(t)$ are suddenly applied to the boundary $X = 0$

$$T(0,t) = p(t), \quad \vartheta(0,t) = \theta(t) \quad \text{for} \quad t \geqslant 0 \tag{3.20}$$

with

$$p(0) = p_0 \neq 0, \quad \theta(0) = \theta_0 \neq 0.$$

There are two wave fronts at time $t = 0$, this means that the initial stress and temperature jumps is divided over the two cases and propagate with the speeds λ_1 and λ_2 (cf. ACHENBACH [1]),

$$\begin{aligned} &[\![T]\!](0) = [\![T]\!]_1(0)+[\![T]\!]_2(0) \equiv T^P, \quad T^P = p_0 - T^{\#},\\ &[\![\vartheta]\!](0) = [\![\vartheta]\!]_2(0)+[\![\vartheta]\!]_2(0) \equiv \vartheta^P, \quad \vartheta^P = \theta_0 - \vartheta^{\#}. \end{aligned} \tag{3.21}$$

The indices "1" and "2" in Eq. (3.21) correspond to the first and the second wave, respectively. The jumps $[\![\vartheta]\!]$ and $[\![T]\!]$ are related as follows:

$$[\![\vartheta]\!] = \frac{g_1}{\varrho_0(\lambda^2+b_1 g_5)}[\![T]\!], \quad \lambda = \lambda_1 \quad \text{or} \quad \lambda = \lambda_2. \tag{3.22}$$

This relation leads to

$$\begin{aligned} &[\![T]\!]_1(0) = \frac{\lambda_1^2+b_1 g_5}{\lambda_1^2-\lambda_2^2}\left[\varrho_0\frac{\lambda_2^2+b_1 g_5}{g_1}\vartheta^p + T^p\right],\\ &[\![T]\!]_2(0) = -\frac{\lambda_2^2+b_1 g_5}{\lambda_1^2-\lambda_2^2}\left[\varrho_0\frac{\lambda_1^2+b_1 g_5}{g_1}\vartheta^p + T^p\right],\\ &[\![\vartheta]\!]_1(0) = -\frac{g_1}{\varrho_0(\lambda_1^2-\lambda_2^2)}\left[\varrho_0\frac{\lambda_2^2+b_2 g_5}{g_1}\vartheta^p + T^p\right],\\ &[\![\vartheta]\!]_2(0) = \frac{g_1}{\varrho_0(\lambda_1^2-\lambda_2^2)}\left[\varrho_0\frac{\lambda_1^2+b_1 g_5}{g_1}\vartheta^p + T^p\right]. \end{aligned} \tag{3.23}$$

(¹) C_2^1 means the value of C_2 at the first wave and C_2^2 at the second one.

The initial condition for $[\![E]\!]$ (t) is described by Eq. (3.12) such that

$$[\![E]\!]_1(0) = \varrho_0 \lambda_1^2 [\![T]\!]_1(0)$$

and

$$[\![E]\!]_2(0) = \varrho_0 \lambda_2^2 [\![T]\!]_2(0).$$

The initial disturbances can be realized in several ways. One of them is to apply externally discontinuities in the stress only, $[\![\vartheta]\!]$ $(0) = 0$ but $[\![T]\!]$ $(0) = T^p \neq 0$. These disturbances are divided over the two waves for temperature

$$\text{(3.24)} \quad \begin{aligned} [\![\vartheta]\!]_1(0) &= -\frac{g_1}{\varrho_0(\lambda_1^2 - \lambda_2^2)} T^p = \frac{\delta}{a_2\left(\dfrac{\lambda_1^2}{\lambda_m^2} - \dfrac{\lambda_2^2}{\lambda_m^2}\right)} T^p, \\ [\![\vartheta]\!]_2(0) &= \frac{g_1}{\varrho_0(\lambda_1^2 - \lambda_2^2)} T^p = -\frac{\delta}{a_2\left(\dfrac{\lambda_1^2}{\lambda_m^2} - \dfrac{\lambda_2^2}{\lambda_m^2}\right)} T^p \end{aligned}$$

with

$$|[\![\vartheta]\!]_1(0)| = |[\![\vartheta]\!]_2(0)|$$

and for stress

$$\text{(3.25)} \quad \begin{aligned} [\![T]\!]_1(0) &= \frac{\lambda_1^2 + b_1 g_5}{\lambda_1^2 - \lambda_2^2} T^p = \frac{\dfrac{\lambda_1^2}{\lambda_m^2} - d^2}{\dfrac{\lambda_1^2}{\lambda_m^2} - \dfrac{\lambda_2^2}{\lambda_m^2}} T^p, \\ [\![T]\!]_2(0) &= -\frac{\lambda_2^2 + b_1 g_5}{\lambda_1^2 - \lambda_2^2} T^p = \frac{d^2 - \dfrac{\lambda_2^2}{\lambda_m^2}}{\dfrac{\lambda_1^2}{\lambda_m^2} - \dfrac{\lambda_2^2}{\lambda_m^2}} T^p. \end{aligned}$$

From Eq. (2.13) we can conclude that sign $[\![T]\!]_l(0) =$ sign T^p $(l = 1, 2)$, $T^p < 0$ (compression) is compatible with $[\![\vartheta]\!]_1(0) < 0$ and with $[\![\vartheta]\!]_2(0) > 0$, on the other hand, $T^p > 0$ extension is compatible with $[\![\vartheta]\!]_1(0) > 0$ and $[\![\vartheta]\!]_2(0) < 0$.

Also we have the following relation for both waves:

$$\text{(3.26)} \quad \begin{aligned} |[\![T]\!]_1(0)| > |[\![T]\!]_2(0)| &\Leftrightarrow d^2 < 1 + \delta, \\ |[\![T]\!]_1(0)| \leqslant |[\![T]\!]_2(0)| &\Leftrightarrow d^2 \geqslant 1 + \delta. \end{aligned}$$

The second possibility is to apply only heating without compression or extension then $[\![\vartheta]\!](0) = \vartheta^p \neq 0$. Similarly as in the previous case, there are two parts of discontinuities:

$$\text{(3.27)} \quad \begin{aligned} [\![T]\!]_1(0) &= \varrho_0 \frac{(\lambda_1^2 + b_1 g_5)(\lambda_2^2 + b_1 g_5)}{g_1(\lambda_1^2 - \lambda_2^2)} \vartheta^p, \\ [\![T]\!]_2(0) &= -\varrho_0 \frac{(\lambda_1^2 + b_1 g_5)(\lambda_2^2 + b_1 g_5)}{g_1(\lambda_1^2 - \lambda_2^2)} \vartheta^p, \\ |[\![T]\!]_1(0)| &= |[\![T]\!]_2(0)| \end{aligned}$$

and

$$(3.28)\qquad \begin{aligned} [\![\vartheta]\!]_1(0) &= -\frac{\lambda_2^2+b_1g_5}{\lambda_1^2-\lambda_2^2}\vartheta^p, \\ [\![\vartheta]\!]_2(0) &= \frac{\lambda_1^2+b_1g_5}{\lambda_1^2-\lambda_2^2}\vartheta^p. \end{aligned}$$

We can see that: sign $[\![\vartheta]\!]_l(0) = \text{sign}\,\vartheta^p (l = 1, 2)$, sign $[\![T]\!]_1(0) = -\text{sign}\,\vartheta^p$,

$$(3.29)\qquad \begin{aligned} |[\![\vartheta]\!]^1(0)| > |[\![\vartheta]\!]_2(0)| &\Leftrightarrow d^2 > 1+\delta, \\ |[\![\vartheta]\!]_1(0) \leqslant |[\![\vartheta]\!]_2(0)| &\Leftrightarrow d_2 \leqslant 1+\delta \end{aligned}$$

and independently of d^2 and δ

$$(3.30)\qquad |[\![E]\!]_1(0) \leqslant |[\![E]\!]_2(0)|,$$

the equality in Eq. (3.30) appears under the condition that $\lambda_1^2 = \lambda_2^2$.

4. One-dimensional plane waves

One of the possible applications of the theory introduced is one-dimensional state of stress in a semi-infinite ($X \geqslant 0$) rod. Then the material constants in the constitutive equations (3.5) can be expressed as

$$(4.1)\qquad a_1 = J, \quad a_2 = -J\alpha_t, \quad a_3 = -2\frac{1+\nu}{1-2\nu}J\alpha_t^2 - \frac{c}{\vartheta^\#},$$

where J is the Young's modulus, ν — the Poisson's constant, α_t — the coefficient of thermal expansion, c — the specific heat of the material. The coefficients of stress, strain, velocity and inelastic deformation are as follows:

$$T = T_{11} \quad (T_{iK} = 0 \quad \text{for} \quad i \neq 1,\ K \neq 1), \quad E = E_{11},$$

$$E_{22}-\alpha_{22} = E_{33}-\alpha_{33} = -\nu(E_{11}-\alpha_{11})+(1+\nu)\alpha_t(\vartheta-\vartheta^\#), \quad v = v_1, \quad \alpha = \alpha_1.$$

Let the temperature θ_0 and pressure p_0 be suddenly imposed upon the boundary of the rod

$$(4.2)\qquad T(0,t) = p_0 h(t), \quad \vartheta(0,t) = \theta_0 h(t),$$

where $h(t)$ is the Heaviside step function and the material is at the equilibrium state at $t = 0$,

$$(4.3)\qquad \begin{aligned} &T(X,0) = T^\# = \text{const}, \quad v(X,0) = v^\# = \text{const}, \quad E(X,0) = E^\# = \text{const}, \\ &\vartheta(X,0) = \vartheta^\# = \text{const}, \quad \alpha(X,0) = \alpha^\# = 0, \quad \beta(X,0) = \beta^\# = \text{const}. \end{aligned}$$

We have assumed that the material is elastic/viscoplastic without hardening ($\varkappa$ = const) the viscosity parameter γ = const, the function A is linear,

$$(4.4)\qquad A(E,\vartheta,\alpha,\gamma,\varkappa) = \check{A}(T,\gamma,\varkappa) = \begin{cases} \gamma\left(\dfrac{|T|}{\varkappa}-1\right)\dfrac{T}{|T|}, & |T| \geqslant \varkappa, \\ 0, & |T| < \varkappa, \end{cases}$$

and, additionally, the field of the temperature does not depend on the mechanical property, therefore we may neglect the coefficient of $(\alpha_{,t}-v_{,X})$ in Eq. $(3.6)_3$(²). These assumptions involve the system

$$
\begin{aligned}
\dot{v}-\frac{1}{\varrho_0}a_1\partial_X E+\frac{1}{\varrho_0}a_1\partial_X\alpha-\frac{1}{\varrho_0}a_2\partial_X\vartheta &= 0,\\
\dot{E}-\partial_X v &= 0,\\
\dot{\vartheta}+g_5\partial_X\beta &= 0,\\
\dot{\alpha}-A &= 0,\\
\dot{\beta}-b_1\partial_X\vartheta+\frac{1}{\tau}\beta &= 0
\end{aligned}
\tag{4.5}
$$

and Eq. (3.13) simplifies to

$$
\lambda\left(\lambda^2-\frac{1}{\varrho_0}a_1\right)(\lambda^2+b_1g_5)=0. \tag{4.6}
$$

The nonzero roots of Eq. (4.6) correspond to the thermal wave speed $\lambda_1^2=\lambda_T^2=-b_1g_5$ and the mechanical longitudinal wave in the rod $\lambda_2^2=\lambda_m^2=\frac{1}{\varrho_0}a_1$.

The characteristic lines for the system (4.5) are

$$
\begin{aligned}
X &= \pm\lambda_m t+\text{const},\\
X &= \pm\lambda_T t+\text{const},\\
X &= 0.
\end{aligned}
\tag{4.7}
$$

In our case the characteristic lines $X=\lambda_T t$ and $X=\lambda_m t$ are shock waves and the jump conditions can be written as

$$
\begin{aligned}
\frac{a_2}{\varrho_0}[\![\vartheta]\!] &= (\lambda^2-\lambda_m^2)[\![E]\!],\\
[\![\beta]\!] &= \frac{\lambda}{g_5}[\![\vartheta]\!],\\
[\![v]\!] &= -\lambda[\![E]\!],\\
[\![T]\!] &= \varrho_0\lambda^2[\![E]\!],
\end{aligned}
\tag{4.8}
$$

where λ should be replaced by λ_T or λ_m. It is essential that for $\lambda^2=\lambda_m^2$ the jump $[\![\vartheta]\!]=0$ and $[\![\beta]\!]=0$, but the functions E, v, T are discontinuous, while the functions E, v, T, ϑ and β are discontinuous at the thermal wave, $\lambda^2=\lambda_T^2$. For both waves $[\![\alpha]\!]_{1,2}=0$.

The boundary conditions (4.2) determine the initial stress and temperature jumps (cf. Eqs. (3.21), (3.23)) assigned here by the indices "1" and "2",

$$
\begin{aligned}
&[\![\vartheta]\!]_1(0)=\vartheta^p, \qquad [\![\vartheta]\!]_2(0)=0,\\
&[\![T]\!]_1(0)=\frac{\lambda_T^2a_2}{\lambda_T^2-\lambda_m^2}\vartheta^p, \qquad [\![T]\!]_2(0)=T^p-\frac{\lambda_T^2a_2}{\lambda_T^2-\lambda_m^2}\vartheta^p.
\end{aligned}
\tag{4.9}
$$

(²) Several thermomechanical problems with temperature governed by the Fourier law have been solved in Nowacki's book [6].

The whole initial temperature propagates with the thermal speed λ_T but the stress jump appears as a result of ϑ^p, independently of pressure $p_0(T^p)$. The pressure p_0 influences the stress jump only at the wave $X = \lambda_m t$. This is the reason why these waves are called thermal and mechanical, respectively.

For the magnitude of the propagating temperature and stress we have the following ordinary differential equations:

$$\frac{d}{dt}[\![\vartheta]\!]_1 = -\frac{1}{2\tau}[\![\vartheta]\!]_1 \tag{4.10}$$

with the initial condition $(4.9)_1$ and

$$\frac{d}{dt}[\![T]\!]_2 = -\frac{1}{2}\frac{\gamma}{\varkappa}a_1[\![T]\!]_2 \tag{4.11}$$

with the condition $(4.9)_4$.

The straight lines $X = \lambda_T t$ and $X = \lambda_m t$ divide the phase plane $(X > 0, t > 0)$ into three subregions $\mathscr{D}_0$, $\mathscr{D}_1$ and $\mathscr{D}_2$ (Fig. 1). Region $\mathscr{D}_0$ is the undisturbed region where

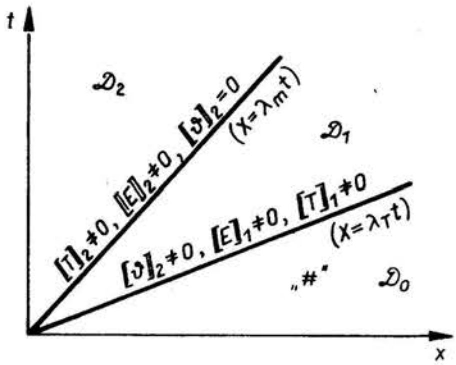

FIG. 1.

$(T, v, E, \vartheta, \alpha, \beta) = (T^\#, v^\#, E^\#, \vartheta^\#, 0, 0)$. Here we have assumed that the thermal wave was faster than the mechanical one, $\lambda_T^2 > \lambda_m^2$, which is reasonable only at the beginning of the wave propagation processes(³). In fact, the thermal disturbances propagate with varying speed and decrease in time such that after some time the mechanical signal is faster and propagates into the undisturbed matrial. This effect cannot be described in the linear theory where λ_T and λ_m are constants. For different material constants LORD and SHULMAN [4], KUKUDŽANOV [2] have assumed that $\lambda_T^2 > \lambda_m^2$ but LORD, LOPEZ [3], NORWOOD, WARREN [5], that $\lambda_m^2 > \lambda_T^2$. In this paper the mechanical wave is slower than the thermal one and propagates into the disturbed region $\mathscr{D}_1$. The solution of Eq. (4.10) together with Eqs. (4.8) provides all unknown solutions for $X = \lambda_T t$,

(³) This consequence in the case of a conductor satisfying Fourier's law was obtained by RANIECKI in [7]

$$\vartheta_1^-(t) = \vartheta^p e^{-\frac{1}{2\tau}t} + \vartheta^\#, \quad T_1^-(t) = \frac{\lambda_T^2 a_2}{\lambda_T^2 - \lambda_m^2}\vartheta^p e^{-\frac{1}{2\tau}t} + T^\#,$$

$$E_1^-(t) = \frac{\frac{1}{\varrho_0}a_1}{\lambda_T^2 - \lambda_m^2}\vartheta^p e^{-\frac{1}{2\tau}t} + E^\#, \quad v_1^-(t) = -\frac{\lambda_T^2 \frac{1}{\varrho_0} a_2}{\lambda_T^2 - \lambda_m^2}\vartheta^p e^{-\frac{1}{2\tau}t} + v^\#, \tag{4.12}$$

$$\beta_1^-(t) = \frac{\lambda_T}{g_5}\vartheta^p e^{-\frac{1}{2\tau}t}, \quad \alpha_1^-(t) = 0.$$

These functions decrease in time and for $t \to \infty$ reach the equilibrium state, for example $\lim\limits_{t\to\infty} \vartheta_1^-(t) = \vartheta^\#$. To determine $T_2^-(t)$ at $X = \lambda_m t$ from Eq. (4.11) the stress $T(X, t)$ in the region $\mathscr{D}_1$ is needed:

$$T_2^-(t) = [\![T]\!]_2(0)e^{-\frac{1}{2}\frac{\gamma}{\varkappa}a_1 t} + T_2^+(t), \tag{4.13}$$

where

$$T_2^+(t) = \lim_{X\to\lambda_m t^+} T(X, t).$$

Similarly, to obtain $v_2^-(t)$ and $E_2^-(t)$ from Eq. (4.2), the velocity and strain in region $\mathscr{D}_1$ are required

$$v_2^-(t) = -\frac{1}{\varrho_0\lambda_m}[\![T]\!]_2(0)e^{-\frac{1}{2}\frac{\gamma}{\varkappa}a_1 t} + v_2^+(t),$$

$$E_2^-(t) = \frac{1}{a_1}[\![T]\!]_2(0)e^{-\frac{1}{2}\frac{\gamma}{\varkappa}a_1 t} + E_2^+(t). \tag{4.14}$$

For the remainder of this section we note that the temperature ϑ and the parameters α and β are continuous at $X = \lambda_m t$, $\vartheta_2^-(t) = \vartheta_2^+(t), \alpha_2^-(t) = \alpha_2^+(t)$, $\beta_2^-(t) = \beta_2^+(t)$.

Now we change the system of Eqs. (4.5) to two differential equations of the second order. After differentiating Eq. $(4.5)_3$ with respect to t and Eq. $(4.5)_5$ with respect to X and eliminating the parameter β, we obtain a hyperbolic equation for ϑ describing a damped plane wave:

$$\vartheta_{,XX} - \frac{1}{\lambda_T^2}\vartheta_{,tt} - \frac{1}{\tau\lambda_T^2}\vartheta_{,t} = 0. \tag{4.15}$$

But the velocity v fulfills the so-called telegraph equation for a damped and forced plane wave:

$$v_{,XX} - \frac{1}{\lambda_m^2}v_{,tt} - \frac{\gamma}{\varkappa}\varrho_0 v_{,t} + \frac{a_2}{a_1}\vartheta_{,Xt} = 0 \tag{4.16}$$

obtained from Eqs. $(4.5)_{1,2,4}$. Here the following condition was used:

$$\alpha_{,tX} = \frac{\gamma}{\varkappa}\varrho_0 v_{,t}, \tag{4.17}$$

which results from the linear evolution equation (4.4) for α. The nonhomogeneous term in Eq. (4.16), $\frac{a_2}{a_1}\vartheta_{,Xt}$ will be treated as a known function being the solution of Eq. (4.15).

At the beginning it is natural to look for the solution of Eq. (4.15) in the whole region between line $X = 0$ and $X = \lambda_T t$ (Fig. 1).

A. Straight lines $X = \pm\lambda_T t + \text{const}$ are characteristic lines of Eq. (4.15). Due to the fact that the temperature ϑ is a continuous function at $X = \lambda_m t$, this fact transforms our problem of solving Eq. (4.15) to Picard's problem with the initially boundary conditions

$$\vartheta(0, t) = \theta_0,$$

$$\vartheta(X, t)|_{X=\lambda_T t} = \vartheta^p e^{-\frac{1}{2\tau}t} + \vartheta^{\#}. \tag{4.18}$$

Here $\vartheta(X, t)|_{X=\lambda_T t}$ means $\vartheta_1^-(t)$, which is a known function (cf. Eq. $(4.12)_1$). We will look for a closed-form of the solution by using Riemann's method. First we change the variables (X, t) to (z, w) through

$$z = \frac{1}{4\tau\lambda_T}(X - \lambda_T t), \quad w = \frac{1}{4\tau\lambda_T}(X + \lambda_T t) \tag{4.19}$$

and introduce the new unknown functions $\bar{u}(z, w)$,

$$\bar{\vartheta}(z, w) = \bar{u}(z, w)e^{z\ \ w}, \tag{4.20}$$

where $\bar{u}(z, w) = u(z(X, t), w(X, t))$ and $\vartheta(X, t) = u(X, t)e^{-\frac{1}{2\tau}t}$ Then Eq. (4.15) simplifies to

$$\bar{u}_{,zw} + \bar{u} = 0. \tag{4.21}$$

This equation has the characteristic lines $z = \text{const}$ and $w = \text{const}$. Let us notice that $X = 0$ and $X = \lambda_T t$ transform under Eq. (4.19) to the lines $w = -z$ and $z = 0$.

The solution of Eq. (4.21) which we will construct for any point A_1 belongs to the region $\mathscr{D}$ (Fig. 2) where the coefficients are (z, w). Now (z, w) are parameters and (ζ, μ) indicate the coefficients of the coordinates. The points B and D are points of intersection

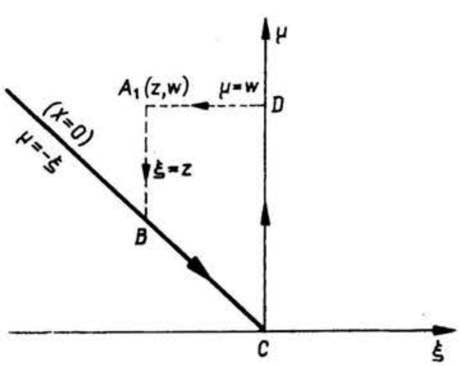

FIG. 2.

of the characteristic line $\mu = -\xi$ with line $w = -\mu$ and $w = \mu$ with $\xi = 0$. The Riemann function v^* for Eq. (4.21) is the solution of the following characteristic problem:

$$
\begin{aligned}
& v^*_{,zw}+v^* = 0,\\
(4.22)\qquad & v^*(\xi,\mu)|_{\xi=z} = 1,\\
& v^*(\xi,\mu)|_{\mu=w} = 1.
\end{aligned}
$$

The solution consists of Bessel's function

$$
(4.23)\qquad v^*(\zeta,\mu;z,w) = J_0\left(2\sqrt{(\xi-z)(\mu-w)}\right).
$$

Then we conclude from Eqs. (4.22) and (4.21) that u and v^* must satisfy

$$
(4.24)\qquad (v^*\bar{u}_{,w})_{,z}-(\bar{u},v^*_{,z})_{,w} = 0.
$$

Finally, by integrating Eq. (4.24) over the region A_1BCD we can see that the known value of $\bar{u}$ at lines BC and DC determine $\bar{u}$ at the arbitrarily choosen point A_1

$$
(4.25)\qquad \bar{u}(A_1) = \frac{1}{2}\bar{u}(B)+\frac{1}{2}\bar{u}(D)+\frac{1}{2}\int\limits_{BC}[\bar{u}(v^*_{,\xi}+v^*_{,\mu})-v^*(\bar{u}_{,\xi}+\bar{u}_{,\mu})]d\xi + \frac{1}{2}\int\limits_{CD}(v^*\bar{u}_{,\mu}-\bar{u}v^*_{,\mu})d\mu.
$$

Returning to the coordinate (X, t) we arrive at line BC,

$$
(4.26)\qquad
\begin{aligned}
\bar{u}_{,\xi} &= 2\tau(u_{,t}+\lambda_T u_{,X}) = 2\tau\frac{d}{dt}u(\lambda_T t, t),\\
\bar{u}_{,\mu} &= 2\tau(\lambda_T u_{,X}-u_{,t})
\end{aligned}
$$

and deduce that

$$
(4.27)\qquad
\begin{aligned}
u(0,t) &= \theta_0 e^{\frac{1}{2\tau}t},\\
u(X,t)|_{X=\lambda_T t} &= \vartheta^p+\vartheta^{\#}e^{\frac{1}{2\tau}t}.
\end{aligned}
$$

Now the solution of Eq. (4.15) takes the form

$$
(4.28)\qquad \vartheta(X,t) = \frac{1}{2}\theta_0 e^{-\frac{1}{2\tau}\lambda_T t}+\frac{1}{2}J_0\left(\frac{1}{2\tau\lambda_T}\sqrt{X^2-\lambda_T^2t^2}\right)\theta_0 e^{-\frac{1}{2\tau}t}+\frac{1}{2}\lambda_T e^{-\frac{1}{2\tau}t}\int\limits_{t-\frac{1}{\lambda_T}X}^{0}\left\{\theta_0 e^{\frac{1}{2\tau}}\cdot J_0'\left(\frac{1}{2\tau\lambda_T}\sqrt{X^2-\lambda_T^2(\omega-t)^2}\right)\frac{X}{2\tau\lambda_T\sqrt{X^2-\lambda_T^2(\omega-t)^2}} + J_0\left(\frac{1}{2\tau\lambda_T}\sqrt{X^2-\lambda_T^2(\omega-t)^2}\right)e^{\frac{1}{2\tau}\omega}\frac{\partial\vartheta}{\partial\sigma}(0,\omega)\right\}d\omega + \frac{1}{2\tau}\vartheta^{\#}e^{-\frac{1}{2\tau}t}\int\limits_{0}^{1/2\left(t+\frac{1}{\lambda_T}X\right)} e^{\frac{1}{2\tau}\omega}J_0\left(\frac{1}{2\tau\lambda_T}\sqrt{(\lambda_T\omega-X)^2-\lambda_T^2(\omega-t)^2}\right)d\omega.
$$

where $\frac{\partial \vartheta}{\partial \sigma}(0, \omega)$ is found from Eq. (4.28) as

$$(4.29)\quad \frac{\partial \vartheta}{\partial X}(0, t) = \frac{1}{2\tau}\left\{\frac{1}{\lambda_T}\left(\vartheta^{\#} e^{-\frac{1}{4\tau}t} + \theta_0\right) + e^{-\frac{1}{2\tau}t}\theta_0 \int\limits_0^t e^{\frac{1}{2\tau}\omega} \frac{J_0'\left(\frac{1}{2\tau\lambda_T}\sqrt{-\lambda_T^2(\omega-t)^2}\right)}{\sqrt{-\lambda_T^2(\omega-t)^2}}\, d\omega \right.$$

$$\left. -\frac{1}{\tau} e^{-\frac{1}{2\tau}t}\vartheta^{\#} \int\limits_0^{1/2t} e^{-\frac{1}{2\tau}\omega} \frac{J_0'\left(\frac{1}{2\tau\lambda_T}\sqrt{\lambda_T^2 t(2\omega-t)}\right)}{\sqrt{\lambda_T^2 t(2\omega-t)}}\, \omega d\omega\right\}.$$

B. When we have temperature in the whole region D we can start solving Eq. (4.16) for velocity rewritten in the most useful form:

$$(4.30)\quad v_{,XX} - \frac{1}{\lambda_m^2} v_{,tt} - k v_{,t} + g^*(X, t) = 0$$

Where $k \equiv \frac{\gamma}{\varkappa}\varrho_0 > 0$ and g^* is a known function of (X, t) and $g^*(X, t) \equiv \frac{a_2}{a_1}\vartheta_{,Xt}(X, t)$.

Our procedure is divided into two steps: the first step corresponds to the solving of the Cauchy problem in the region D_1 and the second to the Picard problem in the region D_2 (Fig. 1). The straight lines $X = \pm\lambda_m t + \text{const}$ are characteristics of Eq. (4.30). For the Cauchy problem we know the value of v and its derivative at the line $X = \lambda_T t$, which is not a characteristic of Eq. (4.30):

$$(4.31)\quad \begin{aligned} v(X, t)|_{X=\lambda_T t} &\equiv \varphi_1(t) = -\frac{\lambda_T \frac{1}{\varrho_0} a_2}{\lambda_T^2 - \lambda_m^2}\vartheta^p e^{-\frac{1}{2\tau}t} + v^{\#}, \\ \frac{\partial v}{\partial t}(X, t)|_{X=\lambda_T t} &\equiv \varphi_2(t) = -\frac{\lambda_T \lambda_m^2 a_2}{(\lambda_T^2 - \lambda_m^2)^2}\vartheta^p e^{\frac{1}{2\tau}t}\left(\frac{1}{\tau\varrho_0} + \frac{\gamma}{\varkappa}\lambda_T\right) \\ &+ \frac{\lambda_T^2 a_2 \frac{1}{\varrho_0}}{\lambda_T^2 - \lambda_m^2}\frac{\partial \vartheta}{\partial X}\Big|_{X=\lambda_T t} + \frac{\lambda_m^2}{\lambda_T^2 - \lambda_m^2}\left(\frac{\gamma}{\varkappa}T^{\#} - s\gamma\right), \quad s \equiv \operatorname{sign} T. \end{aligned}$$

The formula for $\varphi_1(t)$ implies from Eq. $(4.12)_4$, for $\varphi_2(t)$ from Eqs. $(4.5)_{1,2,3}$ and from the following equality

$$(4.32)\quad \frac{dv}{dt}(\lambda_T t, t) = \left(\frac{\partial v}{\partial t} + \lambda_T \frac{\partial v}{\partial x}\right)\Big|_{X=\lambda_T t}$$

The derivative of temperature in Eq. $(4.31)_2$ is calculated from Eq. (4.8)

$$(4.33)\quad \frac{\partial \vartheta}{\partial X}(X, t)|_{X=\lambda_T t} = \frac{1}{2\tau\lambda_T}\left[\frac{1}{8\tau}(\vartheta^{\#} - \vartheta^p)t - \vartheta^p\right] e^{-\frac{1}{2\tau}t}.$$

Now the same procedure will be used for solving Eq. (4.30) with the conditions (4.31). Introducing the new variables by

$$(4.34)\quad z = \frac{1}{4}k\lambda_m(X - \lambda_m t), \quad w = \frac{1}{4}k\lambda_m(X + \lambda_m t)$$

and the new unknown function which is given in the form

$$\bar{v}(z, w) = \bar{u}(z, w)e^{z-w} \quad \text{or} \quad v(X, t) = u(X, t)e^{\frac{1}{2}k\lambda_m^2 t},$$

we obtain the nonhomogeneous equation (cf. Eq. (4.21))

$$\bar{u}_{,zw} + \bar{u} + f(z, w) = 0, \tag{4.35}$$

where

$$f(z, w) = \frac{4}{k^2\lambda_m^2}\bar{g}^*(z, w)e^{w-z} = \frac{1}{4}\lambda_m\frac{a_2}{a_1}(\bar{\vartheta}_{,ww} - \bar{\vartheta}_{,zz})e^{w-z}.$$

Riemann's function v^* is the same as that given by Eq. (4.23) and fulfills the similar conditions at the characteristic $\xi = z$ and $\mu = w$ lines A_1B and A_1C, (Fig. 3).

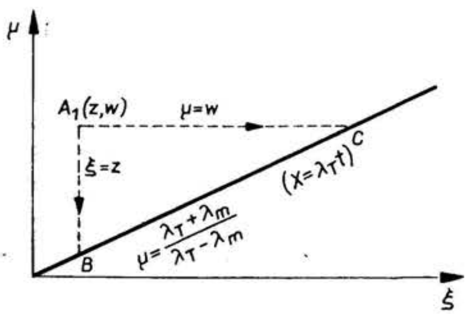

FIG. 3.

Integrating the equation

$$(v^*\bar{u}_{,w})_{,z} - (\bar{u}v_z^*)_{,w} + fv^* = 0 \tag{4.36}$$

over the region $\bar{\Omega}$ of the triangle A_1BC (Fig. 3)(4), we determine the value of the function $\bar{u}$ at the arbitrary point A_1.

$$\bar{u}(A_1) = \frac{1}{2}\bar{u}(B) + \frac{1}{2}\bar{u}(C) + \frac{1}{2}\int\limits_{BC}\left[\bar{u}\left(v^*_{,\xi} - \frac{\lambda_T + \lambda_m}{\lambda_T - \lambda_m}v^*_{,\mu}\right)\right. \\ \left. - v^*\left(\bar{u}_{,\xi} - \frac{\lambda_T + \lambda_m}{\lambda_T - \lambda_m}\bar{u}_{,\mu}\right)\right]d\xi + \iint\limits_{\Omega} fv^* d\xi d\mu. \tag{4.37}$$

Returning to the variable (X, t) and using the relations

$$\begin{aligned} u(X, t)|_{X=\lambda_T t} &= \varphi_1(t)e^{\frac{1}{2}k\lambda_m^2 t}, \\ \frac{\partial u}{\partial t}(X, t)|_{X=\lambda_T t} &= \left(\varphi_2(t) + \frac{1}{2}k\lambda_m^2\varphi_1(t)\right)e^{\frac{1}{2}k\lambda_m^2 t}, \end{aligned} \tag{4.38}$$

(4) $\bar{\Omega}$ and Ω are regions related by

$$\bar{\Omega} = \left\{(z, w): z = \frac{1}{4}k\lambda_m(X - \lambda_m t),\ w = \frac{1}{4}k\lambda_m(X + \lambda_m t),\ (X, t)\Omega\right\}.$$

we have

$$(4.39)\quad v(X,t) = \frac{1}{2}\varphi_1\left(\frac{X-\lambda_m t}{\lambda_T-\lambda_m}\right)e^{\frac{1}{2}k\lambda_m^2\frac{X-\lambda_m t}{\lambda_T-\lambda_m}} + \frac{1}{2}\varphi_1\left(\frac{X+\lambda_m t}{\lambda_T+\lambda_m}\right)e^{\frac{1}{2}k\lambda_m^2\frac{X-\lambda_T t}{\lambda_T+\lambda_m}}$$

$$+\frac{1}{4}k\lambda_m e^{-\frac{1}{2}k\lambda_m^2 t}\int_B^C e^{\frac{1}{2}k\lambda_m^2\omega}\Bigg[\lambda_m\varphi_1(\omega)\frac{J_0'\left(\frac{1}{2}k\lambda_m\sqrt{(\lambda_T\omega-X)^2-\lambda_m^2(\omega-t)^2}\right)}{\sqrt{(\lambda_T\omega-X)^2-\lambda_m^2(\omega-t)^2}}(X-\lambda_T t)$$

$$+\frac{2}{k\lambda_T}J_0\left(\frac{1}{2}k\lambda_m\sqrt{(\lambda_T\omega-X)^2-\lambda_m^2(\omega-t)^2}\right)\left(\frac{d\varphi_1}{d\omega}(\omega)+\frac{\lambda_T^2 k}{2}\varphi_1(\omega)+\frac{\lambda_T^2-\lambda_m^2}{\lambda_m^2}\varphi_2(\omega)\right)\Bigg]d\omega$$

$$+\frac{a_2}{2a_1}\lambda_m e^{-\frac{1}{2}k\lambda_m^2 t}\iint_\Omega J_0\left(\frac{1}{2}k\lambda_m\sqrt{(\sigma-X)^2-\lambda_m^2(\omega-t)^2}\right)\vartheta_{,\sigma\omega}e^{\frac{1}{2}k\lambda_m^2\omega}\,d\sigma d\omega.$$

This solution is valid for $(X,t)\in\mathscr{D}_2$ (Fig. 1). For $(X,t)\in\mathscr{D}_2$ we construct the solution of Eq. (4.30) in a similar way as the temperature at point A_1, with the condition at he characteristic $X=\lambda_m t$ and line $X=0$:

$$(4.40)\quad \begin{aligned} u_{,\sigma|\sigma=0} &= (v_{,\sigma}e^{\frac{1}{2}k\lambda_m^2\omega})_{\sigma=0} = e^{\frac{1}{2}k\lambda_m^2\omega}\left(\frac{\gamma}{\varkappa}p_0-s\gamma\right),\\ u_{|\sigma=\lambda_m\omega} &= v_{|\sigma=\lambda_m\omega}e^{\frac{1}{2}k\lambda_m^2\omega} = v_2^-(\omega)e^{\frac{1}{2}k\lambda_m^2\omega} = v_2^p+v_2^+(\omega)e^{\frac{1}{2}k\lambda_m^2\omega}, \end{aligned}$$

where

$$v_2^p = \text{const} = -\frac{1}{\varrho_0\lambda_m}[\![T_2]\!](0).$$

These equalities were obtained from the boundary conditions (4.2), the constitutive function $(3.5)_1$ and Eq. $(4.5)_1$ at point $(0,t)$; let us note that $E_{,t}(0,t)=\gamma/\varkappa p_0-s\gamma = v_{,X}(0,t)$.

Then, generally for any point (X,t) the solution is

$$(4.41)\quad v(X,t)=\frac{1}{2}v\left(0,t-\frac{1}{\lambda_m}X\right)e^{-\frac{1}{2}k\lambda_m X}+e^{-\frac{1}{2}k\lambda_m^2 t}\Bigg\{\frac{1}{2}J_0\left(\frac{1}{2}k\lambda_m\sqrt{X^2-\lambda_m^2 t}\right)$$

$$\times\left(v_2^p+v_2^+(0)\right)+\frac{1}{2}\lambda_m\int_{t-\frac{1}{\lambda_m}X}^{0}e^{\frac{1}{2}k\lambda_m^2\omega}\left(\frac{1}{2}k\lambda_m v(0,\omega)J_0'\left(\frac{1}{2}k\lambda_m\sqrt{X^2-\lambda_m^2(\omega-t)^2}\right)\frac{X}{\sqrt{X^2-\lambda_m^2(\omega-t)^2}}\right.$$

$$\left.+J_0\left(\frac{1}{2}k\lambda_m\sqrt{X^2-\lambda_m^2(\omega-t)^2}\right)\left(\frac{\gamma}{\varkappa}p_0-s\gamma\right)\right)d\omega+\int_0^{\frac{1}{2}\left(t+\frac{1}{\lambda_m}X\right)}\left(\frac{d}{d\omega}v_2^+(\omega)\right.$$

$$\left.+\frac{1}{2}k\lambda_m^2 v_2^+(0)\right)e^{\frac{1}{2}k\lambda_m^2\omega}J_0\left(\frac{1}{2}k\lambda_m\sqrt{(\lambda_m\omega-X)^2-\lambda_m^2(\omega-t)^2}\right)d\omega+\frac{1}{2}\frac{a_2}{a_1}\lambda_m e^{-\frac{1}{2}k\lambda_m^2 t}$$

$$\times\iint_\Omega J_0\left(\frac{1}{2}k\lambda_m\sqrt{(\sigma-X)^2-\lambda_m^2(\omega-t)^2}\right)\vartheta_{,\sigma\omega}e^{\frac{1}{2}k\lambda_m^2\omega}\,d\sigma d\omega\Bigg\}.$$

Putting into Eq. (4.41) $X = 0$, we calculate

$$
\begin{aligned}
(4.42)\quad v(0,t) = e^{-\frac{1}{2}k\lambda_m^2 t}\Bigg\{ & J_0\left(\frac{1}{2}k\lambda_m\sqrt{-\lambda_m^2 t}\right)\Big((v_2^p+v_2^+(0)) + \lambda_m\int_t^0 e^{\frac{1}{2}k\lambda_m^2\omega} \\
& \times J_0\left(\frac{1}{2}k\lambda_m\sqrt{-\lambda_m^2(\omega-t)^2}\right)\left(\frac{\gamma}{\varkappa}p_0 - s\gamma\right)d\omega + 2\int_0^{\frac{1}{2}t}\Big(\frac{dv_2^+}{d\omega}(\omega) \\
& + \frac{1}{2}k\lambda_m^2 v_2^+(\omega)\Big)e^{\frac{1}{2}k\lambda_m^2\omega}J_0\left(\frac{1}{2}k\lambda_m\sqrt{\lambda_m^2 t(2\omega-t)}\right)d\omega \\
& + \frac{a_2}{a_1}\lambda_m\iint_\Omega \vartheta_{,\sigma\omega}J_0\left(\frac{1}{2}k\lambda_m\sqrt{\sigma^2-\lambda_m^2(\omega-t)}\right)e^{\frac{1}{2}k\lambda_m^2\omega}\,d\sigma d\omega\Bigg\}.
\end{aligned}
$$

The solution obtained in this section is valid only if the stress fulfills the inequality, $|T| \geqslant \varkappa$.

This implies some restrictions on the boundary conditions (4.2)

$$
(4.43)\qquad
\begin{aligned}
|T_1^-(0)| &= \left|\frac{\lambda_T^2 a_2}{\lambda_T^2-\lambda_m^2}\vartheta^p + T^\#\right| > \varkappa, \\
|T_2^-(0)| &= |T^p + T^\#| > \varkappa.
\end{aligned}
$$

Generally, it is well known that there are several plastic and elastic zones at the phase space dependent on the boundary and initial conditions. But it is essential that the procedure of solving this problem should be the same thanks to generalization of the classical Fourier heat-conduction equation.

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