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On a similarity solution of the Boussinesq problem for elastic dielectrics(*)

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SIMILARITY transformations are constructed and used to obtain an exact solution of the axisymmetric Boussinesq's boundary value problem for an elastic dielectric half space. A concentrated force is applied normal to the plane boundary of the semi-space. Closed-form expressions for components of the displacement and polarization vectors, the potential field, the stress and electrical tensors are obtained. The expressions which are derived for the displacement and stresses, in the absence of polarization and electrical effects, and for the case of a concentrated normal foice applied to the surface of an isotropic elastic semi-space, are found to agree with known results.

Zbudowano transformacje podobieństwa i zastosowano je do otrzymania ścisłego rozwiązania osiowo-symetrycznego zagadnienia brzegowego Boussinesqa dla sprężystej półprzestrzeni dielektrycznej. Do płaskiej powierzchni półprzestrzeni przyłożono siłę skupioną. Otrzymano w postaci zamkniętej wyrażenia dla składowych wektorów przemieszczenia i polaryzacji dla potencjału oraz tensora elektrycznego i tensora naprężenia. Stwierdzono, że wartości przemieszczenia i naprężenia przy zaniedbaniu efektów elektrycznych i polaryzacji, a także w przypadku obciążenia powierzchni izotropowej półprzestrzeni sprężystej normalną siłą skupioną, są zgodne ze znanymi wynikami.

Построены преобразования подобия и они применены для получения точного решения осесимметричной краевой задачи Буссинеска для упругого диэлектрического полупространства. К плоской поверхности полупространства приложена сосредоточенная сила. Получены, в замкнутом виде, выражения для составляющих векторов перемещения и поляризации для потенциала, а также электрического тензора и тензора напряжений. Констатировано, что значения перемещений и напряжений, полученные при пренебрежении электрическими эффектами и эффектами поляризации, а также в случае нагружения поверхности изотропного упругого полупространства нормальной сосредоточенной силой, совпадают с известными результатами.

1. Introduction

RECENT years have witnessed the development of analytical methods for the integration of field equations of various linear and nonlinear theories of continuum mechanics. Some problems can be tackled by similarity methods which are based on nondimensionalization and invariance under Lie's continuous group of transformations of the governing system of differential equations and the associated boundary conditions. Methods of transformation groups have been employed by BIRKHOFF [1] and AMES [2] to linear and nonlinear ordinary and partial differential equations arising from physical phenomena, e.g. transport processes, fluid mechanics, diffusion, heat transfer and related areas. Solutions which are invariant under an appropriate group of transformations are obtained by means of sim-

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ilarity transformations, and the number of dependent and independent variables is reduced.

NA and HANSEN [3] have developed systematic methods for the transformation of the differential equations and boundary conditions into forms more suitable for analysis and solutions, using infinitesimal contact transformations. Similarity methods and their applications to boundary value problems in engineering and other physical fields are further enhanced by the works of AMES [4], BLUMAN and COLE [5], OVSJANNIKOV [6] and HANSEN [7].

In this paper, similarity analysis is applied to the axisymmetric boundary value problem of an elastic dielectric half space subjected to a concentrated force applied normal to its plane surface. Similarity transformations are constructed by nondimensionalization of the variables and by applying invariance to the basic equations and physical and boundary conditions. The transformations are used to reduce the basic partial differential equations to five coupled ordinary differential equations with variable coefficients in the components of the displacement vector, the polarization vector, and the potential field, to which closed form solutions are obtained.

For the particular case, when the polarization and the electrical effects are absent, the problem reduces to an isotropic elastic semi-space subjected to a normal force on its plane boundary. The expressions for the displacement and stresses are derived from the general solutions and are found to agree with known results [8].

2. Basic equations

For a homogeneous isotropic elastic dielectric, with axial symmery and referred to a cylindrical polar coordinate system (r, θ, z), the displacement vector $\bar{u}$, the polarization vector $\bar{P}$ and the potential field ϕ assume the form $(u_r, 0, u_z)$, $(P_r, 0, P_z)$ and $\phi(r, z)$, respectively.

The equations of equilibrium are given by [9]:

$$(2.1)\quad c\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right]+c_{44}\frac{\partial^2 u_r}{\partial z^2}+(c-c_{44})\frac{\partial^2 u_z}{\partial r\,\partial z}$$
$$+d\left[\frac{\partial^2 P_r}{\partial r^2}+\frac{1}{r}\frac{\partial P_r}{\partial r}-\frac{P_r}{r^2}\right]+d_{44}\frac{\partial^2 P_r}{\partial z^2}+(d-d_{44})\frac{\partial^2 P_z}{\partial r\,\partial z}=0,$$

$$(2.2)\quad (c-c_{44})\left[\frac{\partial^2 u_r}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right]+c_{44}\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right]+c\frac{\partial^2 u_z}{\partial z^2}$$
$$+(d-d_{44})\left[\frac{\partial^2 P_r}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial P_r}{\partial z}\right]+d_{44}\left[\frac{\partial^2 P_z}{\partial r^2}+\frac{1}{r}\frac{\partial P_z}{\partial r}\right]+d\frac{\partial^2 P_z}{\partial z^2}=0,$$

$$(2.3)\quad d\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right]+d_{44}\frac{\partial^2 u_r}{\partial z^2}+(d-d_{44})\frac{\partial^2 u_z}{\partial r\partial z}$$
$$+b\left[\frac{\partial^2 P_r}{\partial r^2}+\frac{1}{r}\frac{\partial P_r}{\partial r}-\frac{P_r}{r^2}\right]+b^*\frac{\partial^2 P_r}{\partial z^2}+(b-b^*)\frac{\partial^2 P_z}{\partial r\partial z}-\frac{\partial \psi}{\partial r}=0,$$

$$(2.4)\quad (d-d_{44})\left[\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right]+d_{44}\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right]+d\frac{\partial^2 u_z}{\partial z^2}$$
$$+(b-b^*)\left[\frac{\partial^2 P_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial P_r}{\partial z}\right]+b^*\left[\frac{\partial^2 P_z}{\partial r^2}+\frac{1}{r}\frac{\partial P_z}{\partial r}\right]+b\frac{\partial^2 P_z}{\partial z^2}-\frac{\partial\psi}{\partial z}=0,$$

$$(2.5)\quad \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}-\varepsilon_0^{-1}\left[\frac{\partial P_r}{\partial r}+\frac{1}{r}P_r+\frac{\partial P_z}{\partial z}\right]=0,$$

where

$$(2.6)\quad \operatorname{grad}\psi = a\overline{P}+\operatorname{grad}\phi$$

while ψ is an arbitrary function, the quantities $(a, \varepsilon_0^{-1}) = (a^*, \varepsilon_0^{*-1})(r^2+z^2)^{-1}$, $b_{12}, b_{44}, c_{12}, c_{44}, d_{12}, d_{44}, a^*, \varepsilon^*_0$ are dielectric constants with

$$(2.7)\quad x = x_{12}+2x_{44}\quad (x = b, c, d),\quad b^* = b_{44}+b_{77}.$$

Equation (2.6) leads to the consistency relation

$$(2.8)\quad \frac{\partial}{\partial z}(aP_r) = \frac{\partial}{\partial r}(aP_z).$$

Components of the stress tensor and the electric tensor are given by

$$(2.9)\quad \tau_{rr} = d_{12}\operatorname{div}\overline{P}+2d_{44}\frac{\partial P_r}{\partial r}+c_{12}\operatorname{div}\overline{u}+2c_{44}\frac{\partial u_r}{\partial r},$$

$$(2.10)\quad \tau_{\theta\theta} = d_{12}\operatorname{div}\overline{P}+2d_{44}\frac{P_r}{r}+c_{12}\operatorname{div}\overline{u}+2c_{44}\frac{u_r}{r},$$

$$(2.11)\quad \tau_{zz} = d_{12}\operatorname{div}\overline{P}+2d_{44}\frac{\partial P_z}{\partial z}+c_{12}\operatorname{div}\overline{u}+2c_{44}\frac{\partial u_z}{\partial z},$$

$$(2.12)\quad \tau_{zr} = \tau_{rz} = d_{44}\left(\frac{\partial P_r}{\partial z}+\frac{\partial P_z}{\partial r}\right)+c_{44}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right),$$

$$(2.13)\quad E_{rr} = b_{12}\operatorname{div}\overline{P}+2b_{44}\frac{\partial P_r}{\partial r}+d_{12}\operatorname{div}\overline{u}+2d_{44}\frac{\partial u_r}{\partial r}+b_0,$$

$$(2.14)\quad E_{\theta\theta} = b_{12}\operatorname{div}\overline{P}+2b_{44}\frac{P_r}{r}+d_{12}\operatorname{div}\overline{u}+2d_{44}\frac{u_r}{r}+b_0,$$

$$(2.15)\quad E_{zz} = b_{12}\operatorname{div}\overline{P}+2b_{44}\frac{\partial P_z}{\partial z}+d_{12}\operatorname{div}\overline{u}+2d_{44}\frac{\partial u_z}{\partial z}+b_0,$$

$$(2.16)\quad E_{zr} = b_{44}\left(\frac{\partial P_r}{\partial z}+\frac{\partial P_z}{\partial r}\right)+b_{77}\left(\frac{\partial P_r}{\partial z}-\frac{\partial P_z}{\partial r}\right)+d_{44}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right),$$

$$(2.17)\quad E_{rz} = b_{44}\left(\frac{\partial P_r}{\partial z}+\frac{\partial P_z}{\partial r}\right)+b_{77}\left(\frac{\partial P_z}{\partial r}-\frac{\partial P_r}{\partial z}\right)+d_{44}\left(\frac{\partial u_r}{\partial z}+\frac{du_z}{\partial r}\right),$$

where

$$(2.18)\quad \operatorname{div}\overline{P} = \frac{\partial P_r}{\partial r}+\frac{1}{r}P_r+\frac{\partial P_z}{\partial z},\quad \operatorname{div}\overline{u} = \frac{\partial u_r}{\partial r}+\frac{1}{r}u_r+\frac{\partial u_z}{\partial z}$$

and where we assumed that $b_0 = b_0^*(r^2+z^2)^{-1}$, with b_0^* a constant.

The Boussinesq problem consists in solving of the field system of partial differential equations (2.1)–(2.5) for the elastic dielectric half space occupying the region $r \geqslant 0$, $z \geqslant \geqslant 0$ of a cylindrical polar coordinate system (with the positive direction of axis of symmetry $r = 0$ pointing towards the interior) and the plane surface $z = 0$ being subjected to the physical and boundary conditions.

$$(2.19) \quad \text{(i)} \quad \lim_{z\to\infty} [u(\varrho_0, z), w(\varrho_0, z), P_r(\varrho_0, z), P_z(\varrho_0, z), \phi(\varrho_0, z)] = (0, 0, 0, 0, 0)$$

$$(2.20) \quad \text{(ii)} \quad \tau_{zz}(r, 0) = -F\frac{\delta(r)}{r} \quad \text{or} \quad \lim_{z\to 0}\int_0^\infty \int_0^{2\pi} r\tau_{zz}(r, z)drd\theta = -2\pi F,$$

$$(2.21) \quad \text{(iii)} \quad \tau_{zr}(r, 0) = 0,$$

$$(2.22) \quad \text{(iv)} \quad E_{zr}(r, 0) = 0,$$

$$(2.23) \quad \text{(v)} \quad E_{zz}(r, 0) = 0,$$

$$(2.24) \quad \text{(vi)} \quad \left[\varepsilon_0^{-1}P_z - \frac{\partial\phi}{\partial z}\right]_{z=0} = 0,$$

where $\delta(r)$ is the Dirac-delta function.

3. The similarity transformations

Let the nondimensional variables for the problem stated in Sect. 2 be defined by

$$(3.1) \qquad u_r^* = \frac{u_r - u_{r0}}{u_{r1}}, \quad u_z^* = \frac{u_z - u_{z0}}{u_{z1}},$$

$$(3.2) \qquad P_r^* = \frac{P_r - P_{r0}}{P_{r1}}, \quad P_z^* = \frac{P_z - P_{z0}}{P_{z1}},$$

$$(3.3) \qquad \phi^* = \frac{\phi - \phi_0}{\phi_1}, \quad \psi^* = \frac{\psi - \psi_0}{\psi_1},$$

$$(3.4) \qquad r^* = \frac{r}{r_0}, \quad z^* = \frac{z}{z_0},$$

where r_0 and z_0 are arbitrary reference variables and $u_{ri}, u_{zi}, P_{ri}, P_{zi}$ and $\phi_i(r, z)$, $\psi_i (i = 0, 1)$ depend on the nature of the auxiliary conditions.

Transforming Eqs. (2.1)–(2.5) and Eqs. (2.20)–(2.24) by means of Eqs. (3.1)–(3.4), we obtain the equations of equlibrium, the physical and boundary conditions in the form shown in Appendix A.

Absolute invariance demands that each of the factors enclosed in the rectangular enclosures appearing in Eqs. (A.1)–(A.3) should be equal to unity, which leads to

$$(3.5) \qquad \frac{r_0}{z_0} = 1, \quad \frac{u_{z1}}{u_{r1}} = 1, \quad \frac{P_{z1}}{P_{r1}} = 1,$$

$$(3.6) \qquad \frac{P_{r1}}{u_{r1}} = 1, \quad \frac{P_{r1}}{\phi_1 r_0} = 1, \quad \frac{z_0\psi_{z1}}{u_{r1}} = 1, \quad u_{r1}r_0 = 1.$$

Equations (3.5) and (3.6) suggest the form of the similarity transformations as

$$u_r(r,z) = \frac{1}{r}U(\eta), \quad u_z(r,z) = \frac{1}{r}W(\eta), \tag{3.7}$$

$$P_r(r,z) = \frac{1}{r}P(\eta), \quad P_z(r,z) = \frac{1}{r}Q(\eta), \tag{3.8}$$

$$\phi(r,z) = \frac{1}{r^2}\Phi(\eta), \quad \psi(r,z) = \frac{1}{r^2}\Psi(\eta), \tag{3.9}$$

where

$$\eta = \frac{z}{r}. \tag{3.10}$$

One can see that

$$\begin{aligned} \frac{\partial u_r}{\partial r} &= -\frac{1}{r^2}\left[U+\eta\frac{dU}{d\eta}\right], \quad \frac{\partial u_r}{\partial z} = \frac{1}{r^2}\frac{dU}{d\eta},\\ \frac{\partial^2 u_r}{\partial r^2} &= \frac{1}{r^3}\left[2U+4\eta\frac{dU}{d\eta}+\eta^2\frac{d^2U}{d\eta^2}\right],\\ \frac{\partial^2 u_r}{\partial r\,\partial z} &= -\frac{1}{r^3}\left[2\frac{dU}{d\eta}+\eta\frac{d^2U}{d\eta^2}\right],\\ \frac{\partial^2 u_r}{\partial z^2} &= \frac{1}{r^3}\frac{d^2U}{d\eta^2}; \end{aligned} \tag{3.11}$$

$$\begin{gathered} \frac{\partial\phi}{\partial r} = -\frac{1}{r^3}\left[2\Phi+\eta\frac{d\Phi}{d\eta}\right], \quad \frac{\partial\phi}{\partial z} = \frac{1}{r^3}\frac{d\Phi}{d\eta},\\ \frac{\partial^2\phi}{\partial z^2} = \frac{1}{r^4}\frac{d^2\Phi}{d\eta^2}, \quad \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{1}{r^4}\left[\eta^2\frac{d^2\Phi}{d\eta^2}+5\eta\frac{d\Phi}{d\eta}+4\Phi\right]. \end{gathered} \tag{3.12}$$

4. Method of solution

Substituting Eqs. (3.7)–(3.10) into Eqs. (2.1–(2.5), one obtains a set of coupled ordinary differential equations with variable coefficients, in U, W, P, Q, Φ, Ψ as

$$\begin{aligned} &c[3\eta U'+\eta^2U'']+c_{44}U''-(c-c_{44})(2W'+\eta W'')\\ &\qquad +d[3\eta P'+\eta^2P'']+d_{44}P''-(d-d_{44})(2Q'+\eta Q'') = 0, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &-(c-c_{44})(U'+\eta U'')+c_{44}(\eta^2W''+3\eta W'+W)+cW''\\ &\qquad -(d-d_{44})(P'+\eta P'')+d_{44}(\eta^2Q''+3\eta Q'+Q)+dQ'' = 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &d(3\eta U'+\eta^2U'')+d_{44}U''-(d-d_{44})(2W'+\eta W'')\\ &\qquad +b(3\eta P'+\eta^2P'')+b^*P''-(b-b^*)(2Q'+\eta Q'')+(2\Psi+\eta\Psi') = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &-(d-d_{44})(U'+\eta U'')+d_{44}(W+3\eta W'+\eta^2W'')+dW''\\ &\qquad -(b-b^*)(P'+\eta P'')+b^*(Q+3\eta Q'+\eta^2Q'')+bQ''-\Psi' = 0, \end{aligned} \tag{4.4}$$

$$(1+\eta^2)\Phi''+5\eta\Phi'+4\Phi = \frac{1}{\varepsilon_0^*(1+\eta^2)}[Q'-\eta P'] \tag{4.5}$$

and the consistency relation (2.8) becomes

$$P'+\eta Q' = \frac{2\eta}{1+\eta^2}P-\frac{3+\eta^2}{1+\eta^2}Q, \tag{4.6}$$

where the prime above the function denotes its derivative with respect to η.

Eqs. (4.2) and (4.4) can be integrated directly and Eqs. (4.1) and (4.3) become integrable when each is multiplied by η. In this manner, the first integration of Eqs. (4.1)–(4.4) leads to

$$(c\eta^3+c_{44}\eta)U'-c_{44}U-(c-c_{44})\eta^2W'+(d\eta^3+d_{44}\eta)P'-d_{44}P -(d-d_{44})\eta^2Q' = A_1, \tag{4.7}$$

$$-(c-c_{44})\eta U'+c_{44}(\eta^2W'+\eta W)+cW'-(d-d_{44})\eta P' +d_{44}(\eta^2Q'+\eta Q)+dQ' = A_2, \tag{4.8}$$

$$d\eta^3U'+d_{44}(\eta U'-U)-(d-d_{44})\eta^2W'+b\eta^3P'+b^*(\eta P'-P) -(b-b^*)\eta^2Q'+\eta^2\Psi' = A_3, \tag{4.9}$$

$$-(d-d_{44})\eta U'+d_{44}(\eta^2W'+\eta W)+dW'-(b-b^*)\eta P'+b^*(\eta^2Q'+\eta Q) +bQ'-\Psi' = A_4, \tag{4.10}$$

where $A_i(i = 1-4)$ are arbitrary constants.

Multiplying Eqs. (4.8) and (4.10) each by η^2 and adding the results to Eqs. (4.7) and (4.9) respectively, one finds that

$$\eta(\eta^2+1)U'-U+\eta^2(\eta^2+1)W'+\eta^3W = \alpha[d_{44}(A_4\eta^2+A_3)-b^*(A_2\eta^2+A_1)], \tag{4.11}$$

$$\eta(\eta^2+1)P'-P+\eta^2(\eta^2+1)Q'+\eta^3Q = -\alpha[c_{44}(A_4\eta^2+A_3)-d_{44}(A_2\eta^2+A_1)], \tag{4.12}$$

where

$$\alpha^{-1} = d_{44}^2-c_{44}b^*. \tag{4.13}$$

Dividing Eqs. (4.11) and (4.12) each by $\eta^2\sqrt{1+\eta^2}$ makes the resulting equations integrable and one can show that

$$U+\eta W = \alpha\left[(d_{44}A_4-b^*A_2)\frac{\eta}{\sqrt{1+\eta^2}}\sinh^{-1}\eta-(d_{44}A_3-b^*A_1)\right]+\frac{\eta}{\sqrt{\eta^2+1}}A_5, \tag{4.14}$$

$$P+\eta Q = -\alpha\left[(c_{44}A_4-d_{44}A_2)\frac{\eta}{\sqrt{1+\eta^2}}\sinh^{-1}\eta-(c_{44}A_3-d_{44}A_1)\right] +\frac{\eta}{\sqrt{\eta^2+1}}A_6. \tag{4.15}$$

Substituting the expression for U, P from Eqs. (4.14) and (4.15) into Eq. (4.7), after lengthy algebra and integration, one finds

$$cW+dQ = A_2\frac{\sinh^{-1}\eta}{\sqrt{\eta^2+1}}+\frac{\alpha}{2}[k_1A_2+(c_{12}d_{44}-d_{12}c_{44})A_4] \times\left[\frac{1+2\eta^2}{1+\eta^2}\frac{\sinh^{-1}\eta}{\sqrt{\eta^2+1}}-\frac{\eta}{\eta^2+1}\right]-\frac{1}{2}[(c-c_{44})A_5+(d-d_{44})A_6]\frac{1}{(1+\eta^2)^{3/2}} +\frac{A_7}{\sqrt{1+\eta^2}}, \tag{4.16}$$

where

$$k_1 = -\alpha^{-1}+dd_{44}-b^*c.$$

The physical condition on $[cW+dQ] = r[cu_z+dP_z]$ for finite ϱ_0 and $z \to \infty$ leads to the conditions

$$A_2 = A_4 = 0, \tag{4.17}$$

From Eqs. (4.15) and the consistency relation (4.6), we determine P, Q. The resulting expression for Q when used in Eq. (4.16) enables one to determine W. The expression for U is given by Eq. (4.14). The detailed evaluations for Φ are outlined in Appendix B. The expressions for U, W, P, Q and Φ are thus found and are given by

$$U = \left[A_5+\frac{dA_6-A_7}{c}\right]\frac{\eta}{\sqrt{\eta^2+1}}+\frac{(c-c_{44})A_5-(2d+d_{44})A_6}{zc}\frac{\eta}{(\eta^2+1)^{3/2}} - \frac{d}{c}\frac{A^*}{\eta^2+1}+K, \tag{4.18}$$

$$W = -\frac{dA_6-A_7}{c}\frac{1}{\sqrt{\eta^2+1}}-\frac{(c-c_{44})A_5-(2d+d_{44})A_6}{2c}\frac{1}{(\eta^2+1)^{3/2}} - \frac{d}{c}A^*\frac{\eta}{\eta^2+1}, \tag{4.19}$$

$$P = \frac{A^*}{\eta^2+1}+\frac{3}{2}\frac{\eta}{(\eta^2+1)^{3/2}}A_6, \tag{4.20}$$

$$Q = \frac{\eta}{\eta^2+1}A^*+\left[\frac{1}{\sqrt{\eta^2+1}}-\frac{3}{2}\frac{1}{(\eta^2+1)^{3/2}}\right]A_6, \tag{4.21}$$

$$\Phi = \frac{\eta}{(\eta^2+1)^{3/2}}B_1+\varepsilon_0^{*-1}\left[\frac{A^*}{2}\frac{1}{\eta^2+1}+\frac{A_6}{3}\frac{\eta}{(\eta^2+1)^{3/2}}\log(\eta^2+1)\right], \tag{4.22}$$

where

$$A^* = \alpha(c_{44}A_3-d_{44}A_1), \tag{4.23}$$

$$K = \frac{d}{c}A^*-\alpha(d_{44}A_3-b^*A_1). \tag{4.24}$$

The expressions for the essential stresses and electrical stresses when Eqs. (4.18)–(4.22) are substituted into Eqs. (2.9)–(2.17) are given by

$$\begin{aligned} r^2\tau_{zz} &= -d_{12}\eta P'+dQ'-c_{12}\eta U'+cW' \\ &= \frac{2(cd_{12}-dc_{12})}{c}A^*\frac{\eta^2}{(\eta^2+1)^2}+\frac{c_{44}}{c}\left[-c_{12}\frac{\eta}{(\eta^2+1)^{3/2}}+3(c-c_{44})\frac{\eta}{(\eta^2+1)^{5/2}}\right]A_5 \\ &+\frac{3}{2}\left\{\left[(-2d_{12}+d_{44})+\frac{c_{12}}{c}(2d+d_{44})\right]\frac{\eta}{(\eta^2+1)^{5/2}}+\left[3d_{12}-\frac{c_{12}}{c}(3d_{12}+7d_{44})\right.\right. \\ &\left.\left.\times\frac{\eta}{(\eta^2+1)^{3/2}}\right\}A_6+\left(\frac{c_{12}}{c}-1\right)\frac{\eta}{(\eta^2+1)^{3/2}}A_7, \end{aligned} \tag{4.25}$$

$$(4.26)\quad r^2\tau_{zr} = d_{44}[P'-Q-\eta Q']+c_{44}[U'-W-\eta W'] = \left(-d_{44}+\frac{d}{c}c_{44}\right)A^*\frac{4\eta}{(\eta^2+1)^2}$$
$$+c_{44}\left[\left(-1+\frac{2c_{44}}{c}\right)\frac{1}{(\eta^2+1)^{3/2}}+\frac{3}{c}(c-c_{44})\frac{1}{(\eta^2+1)^{5/2}}\right]A_5$$
$$+\left\{\left[\left(-7d_{44}+\frac{c_{44}}{c}(6d+2d_{44}\right)\right]\frac{1}{(\eta^2+1)^{3/2}}+\left[9d_{44}-\frac{3c_{44}}{c}(2d+d_{44})\right]$$
$$\times\frac{1}{(1+\eta^2)^{5/2}}\right\}A_6-\frac{2c_{44}}{c}A_7\frac{1}{(\eta^2+1)^{3/2}},$$

$$(4.27)\quad r^2E_{zr} = (b_{44}+b_{77})P'-(b_{44}-b_{77})(Q+\eta Q')+d_{44}(U'-W-\eta W')$$
$$=\left(-4b_{44}+\frac{2d}{c}\right)A^*\frac{\eta}{(\eta^2+1)^2}+d_{44}\left[\left(-1+\frac{2c_{44}}{c}\right)\frac{1}{(\eta^2+1)^{3/2}}+3\left(1-\frac{c_{44}}{c}\right)\right.$$
$$\left.\times\frac{1}{(\eta^2+1)^{5/2}}\right]A_5+\left[\left(-7b_{44}+b_{77}+6d_{44}\frac{d}{c}+\frac{2d_{44}^2}{c}\right)\frac{1}{(\eta^2+1)^{3/2}}\right.$$
$$\left.+\left(9b_{44}-\frac{3d_{44}}{c}(2d+d_{44})\right)\frac{1}{(\eta^2+1)^{5/2}}\right]A_6-\frac{2}{c}d_{44}A_7\frac{1}{(\eta^2+1)^{3/2}},$$

$$(4.28)\quad r^2E_{zz} = -b_{12}\eta P'+bQ'-d_{12}\eta U'+dW'+\frac{b_0^*}{\eta^2+1}$$
$$=\left[\frac{b_{12}-2b_{44}-\frac{d}{c}(d_{12}-2d_{44})}{\eta^2+1}+4\frac{b_{44}-d_{44}\frac{d}{c}}{(\eta^2+1)^2}\right]A^*$$
$$-\left[\frac{d_{12}c_{44}}{c}\frac{\eta}{(\eta^2+1)^{3/2}}+\frac{3(c-c_{44})d_{44}}{c}\frac{\eta}{(\eta^2+1)^{5/2}}\right]A_5$$
$$+\left[\left[2(b_{12}+b_{44})-\frac{3dd_{12}+d_{12}d_{44}-d^2}{c}\right]\frac{\eta}{(\eta^2+1)^{3/2}}\right.$$
$$\left.+\left[9b_{44}-\frac{3d_{44}}{c}(2d+d_{44})\right]\frac{\eta}{(\eta^2+1)^{5/2}}\right]A_6-\frac{2d_{44}}{c}A_7\frac{\eta}{(\eta^2+1)^{3/2}}+\frac{b_0^*}{\eta^2+1},$$

$$(4.29)\quad r^3\left[\varepsilon_0^{-1}P_z-\frac{\partial\phi}{\partial z}\right]=\left[\frac{\eta}{\eta^2+1}-\frac{\eta}{(\eta^2+1)^2}\right]\frac{A^*}{\varepsilon_0^*}+\left[\frac{2}{(\eta^2+1)^{3/2}}-\frac{3}{(\eta^2+1)^{5/2}}\right]B_1$$
$$+\varepsilon_0^{*-1}\left[\frac{1}{(\eta^2+1)^{1/2}}-\frac{13}{6}\frac{1}{(\eta^2+1)^{3/2}}+\frac{2}{3}\frac{1}{(\eta^2+1)^{5/2}}\right.$$
$$\left.+\left(\frac{2/3}{(\eta^2+1)^{3/2}}-\frac{1}{(\eta^2+1)^{5/2}}\right)\log(1+\eta^2)\right].$$

The boundary conditions (2.19)–(2.24) lead to the following system of algebraic equations in the constants A_5, A_6, A_7, B_1 and A^*:

$$(4.30)\quad \frac{2c-c_{44}}{c}A_5+\left(2\frac{d_{44}}{c_{44}}-\frac{d_{44}}{c}\right)A_6-\frac{2}{c}A_7=0,$$

(4.31) $$\frac{2c-c_{44}}{c}A_5+\left(\frac{b_{44}+b^*}{d_{44}}-\frac{d_{44}}{c}\right)A_6-\frac{2}{c}A_7=0,$$

(4.32) $$A^*=\frac{cb_0^*}{d^2-bc},$$

(4.33) $$2\varepsilon_0^* B_1+A_6=0,$$

(4.34) $$c_{44}A_5+\frac{2dc_{44}-cd_{44}}{c}A_6-\frac{2c_{44}}{c}A_7+\frac{cd_{12}-dc_{12}}{c}A^*=-F.$$

The detailed evaluation of the integral boundary condition (2.20) which leads to Eq. (4.34) is given in Appendix C. The solution of the system of Eqs. (4.30)–(4.34) leads to

(4.35) $$A_6=0,\quad B_1=0,$$

(4.36) $$A_5=\frac{c}{c_{44}(c-c_{44})}F+\frac{(cd_{12}-dc_{12})cb_0^*}{c_{44}(c-c_{44})(d^2-bc)},$$

(4.37) $$A_7=\frac{2c-c_{44}}{2c_{44}}\left[\frac{c}{(c-c_{44})}F+\frac{(cd_{12}-dc_{12})cb_0^*}{(c-c_{44})(d^2-bc)}\right],$$

(4.38) $$A^*=\frac{cb_0^*}{d^2-bc}.$$

5. The classical solution

When the homogeneous, isotropic elastic semi-space is subjected to a concentrated force normal to its surface and the electric polarization effects are neglected, the problem reduces to the classical Boussinesq problem.

Allowing the dielectric constants d_{12}, d_{44}, d, b^* to vanish, the constants given by Eqs. (4.35)–(4.38) assume the form

(5.1) $$A^*=A_6=B_1=0,$$

(5.2) $$A_5=\frac{c}{c_{44}(c-c_{44})}F,\quad A_7=\frac{c(2c-c_{44})}{2c_{44}(c-c_{44})}.$$

Substituting the expressions (5.1) and (5.2) into Eqs. (4.18)–(4.24) and Eqs. (4.25)–(4.29), we find that the expressions for displacements and stresses are given by

(5.3) $$u(r,z)=\frac{F}{2c_{44}}\left\{\frac{rz}{(r^2+z^2)^{3/2}}-\frac{c_{44}}{c_{12}+c_{44}}\left(\frac{1}{r}-\frac{z}{r(r^2+z^2)^{1/2}}\right)\right\},$$

(5.4) $$w(r,z)=\frac{F}{2c_{44}}\left\{\frac{z^2}{(r^2+z^2)^{3/2}}+\frac{c}{c_{12}+c_{44}}\frac{1}{(r^2+z^2)^{1/2}}\right\},$$

(5.5) $$\tau_{zz}=-F\frac{3z^3}{(r^2+z^2)^{5/2}},$$

(5.6) $$\tau_{zr}=-F\frac{3rz^2}{(r^2+z^2)^{5/2}}$$

which, with minor change in notation, agree with those arrived at by SNEDDON [8] by means of HANKEL transforms.

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Appendix A

The transformed system of the equilibrium equations, the physical and boundary conditions in the nondimensional variables are given by

$$\text{(A.1)}\quad c\left[\frac{\partial^2 u_r^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial u_r^*}{\partial r^*}-\frac{u_r^*}{r^{*2}}\right]+c_{44}\frac{\partial^2 u_r^*}{\partial z^{*2}}\boxed{\frac{r_0^2}{z_0^2}}+(c-c_{44})\frac{\partial^2 u_z^*}{\partial r^*\partial z^*}\boxed{\frac{r_0 u_{z1}}{z_0 u_{r1}}}$$
$$+\left\{d\left[\frac{\partial^2 P_r^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial P_r^*}{\partial r^*}-\frac{P_r^*}{r^{*2}}\right]+d_{44}\frac{\partial^2 P_r^*}{\partial z^{*2}}\boxed{\frac{r_0^2}{z_0^2}}+(d-d_{44})\frac{\partial^2 P_z^*}{\partial r^*\partial z^*}\right.$$
$$\left.\times\boxed{\frac{r_0 P_{z1}}{z_0 P_{r1}}}\right\}\left(\frac{P_{r1}}{u_{r1}}\right)=0,$$

$$\text{(A.2)}\quad (c-c_{44})\left[\frac{\partial^2 u_z^*}{\partial r^*\partial z^*}+\frac{1}{r^*}\frac{\partial u_r^*}{\partial z^*}\right]+c_{44}\left[\frac{\partial^2 u_z^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial u_z^*}{\partial r^*}\right]\boxed{\frac{u_{z1}}{u_{r1}}\frac{z_0}{r_0}}$$
$$+c\frac{\partial^2 u_z^*}{\partial z^{*2}}\boxed{\frac{u_{z1} r_0}{u_{r1} z_0}}+\left\{(d-d_{44})\left[\frac{\partial^2 P_r^*}{\partial r^*\partial z^*}+\frac{1}{r^*}\frac{\partial P_r^*}{\partial z^*}\right]+d_{44}\left[\frac{\partial^2 P_z^*}{\partial r^{*2}}\right.\right.$$
$$\left.\left.+\frac{1}{r^*}\frac{\partial P_z^*}{\partial r^*}\right]\boxed{\frac{P_{z1}}{P_{r1}}\frac{z_0}{r_0}}+d\frac{\partial^2 P_z^*}{\partial z^{*2}}\boxed{\frac{P_{z1} r_0}{u_{r1} z_0}}\right\}\left(\frac{P_{r1}}{u_{r1}}\right)=0,$$

$$\text{(A.3)}\quad d\left[\frac{\partial^2 u_r^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial u_r^*}{\partial r^*}-\frac{u_r^*}{r^{*2}}\right]+d_{44}\frac{\partial^2 u_z^*}{\partial z^{*2}}\boxed{\frac{r_0^2}{z_0^2}}+(d-d_{44})\frac{\partial^2 u_z^*}{\partial r^*\partial z^*}\boxed{\frac{r_0 z_{z1}}{z_0 u_{r1}}}$$
$$+\left\{b\left[\frac{\partial^2 P_r^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial P_r^*}{\partial r^*}-\frac{P_r^*}{r^{*2}}\right]+b^*\frac{\partial^2 P_r^*}{\partial z^{*2}}\boxed{\frac{r_0^2}{z_0^2}}+(b-b^*)\frac{\partial^2 P_z^*}{\partial r^*\partial z^*}\right.$$
$$\left.\times\boxed{\frac{r_0 P_{z1}}{z_0 P_{r1}}}\right\}\left(\frac{P_{r1}}{u_{r1}}\right)-\boxed{\frac{r_0\psi_1}{u_{r1}}}\left(\frac{\partial\psi^*}{\partial r^*}\right)=0,$$

$$\text{(A.4)}\quad (d-d_{44})\left[\frac{\partial^2 u_r^*}{\partial r^*\partial z^*}+\frac{1}{r^*}\frac{\partial u_r^*}{\partial z^*}\right]+d_{44}\left[\frac{\partial^2 u_z^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial u_z^*}{\partial r^*}\right]\boxed{\frac{u_{z1}}{u_{r1}}\frac{z_0}{r_0}}$$
$$+d\frac{\partial^2 u_z^*}{\partial z^{*2}}\boxed{\frac{u_{z1}}{u_{r1}}\frac{r_0}{z_0}}+\left\{(b-b^*)\left[\frac{\partial^2 P_r^*}{\partial r^*\partial z^*}+\frac{1}{r^*}\frac{\partial P_r^*}{\partial z^*}\right]+b^*\left[\frac{\partial^2 P_z^*}{\partial r^{*2}}\right.\right.$$
$$\left.\left.+\frac{1}{r^*}\frac{\partial P_z^*}{\partial r^*}\right]\boxed{\frac{P_{z1}}{P_{r1}}\frac{z_0}{r_0}}+b\frac{\partial^2 P_z^*}{\partial z^{*2}}\boxed{\frac{P_{z1}}{P_{r1}}\frac{r_0}{z_0}}\right\}\left(\frac{P_{r1}}{u_{r1}}\right)-\boxed{\frac{z_0\psi_1}{u_{z1}}}\left(\frac{\partial\psi_1^*}{\partial^* z}\right)=0,$$

$$(A.5)\qquad \frac{\partial\phi^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial\phi^*}{\partial r^*}+\boxed{\frac{r_0^2}{z_0^2}}\frac{\partial\phi^{*2}}{\partial z^{*2}}-\varepsilon_0^{*-2}\left(1+\frac{z^{*2}}{r^{*2}}\boxed{\frac{z_0^2}{r_0^2}}\right)^{-1}\left(\frac{\partial P_r^*}{\partial r^*}+\frac{1}{r^*}P_r^*\right.$$

$$\left.+\boxed{\frac{P_{z1}}{P_{r1}}\frac{r_0}{z_0}}\frac{\partial P_z^*}{\partial z^*}\right)\boxed{\frac{P_{r1}}{r_0\phi_1}}=0,$$

$$(i)\qquad (u_r^*,u_z^*,P_r^*,P_z^*,\phi^*,\psi^*)|_\infty=\left(\frac{u_r|_\infty-u_{r0}}{u_{r1}},\quad\frac{u_z|_\infty-u_{z0}}{u_{z1}},\quad\frac{P_r|_\infty-P_{r0}}{P_{r1}},\right.$$

$$(A.6)\qquad \left.\frac{P_z|_\infty-P_{z0}}{P_{z1}},\quad\frac{\phi|_\infty-\phi_0}{\phi_1},\quad\frac{\psi|_\infty-\psi_0}{\psi_1}\right)=(0,0,0,0,0).$$

As

$$(A.7)\qquad (u_r|_\infty,u_z|_\infty,P_r|_\infty,P_z|_\infty,\phi_\infty,\psi|_\infty)=(0,0,0,0,0),$$

it implies that

$$(A.8)\qquad (u_{r0},u_{z0},P_{r0},P_{z0},\phi_0,\psi_0)=(0,0,0,0,0),$$

$$(ii)\qquad \tau_{zz}^*(r^*,0)=c_{12}\left(\frac{\partial u_r^*}{\partial r^*}+\frac{u_r^*}{r^*}\right)\boxed{\frac{z_0}{r_0}}+c\frac{\partial u_z^*}{\partial z^*}\boxed{\frac{u_{z1}}{u_{r1}}}+\left\{d_{12}\left(\frac{\partial P_r^*}{\partial r^*}\right.\right.$$

$$\left.\left.+\frac{P_r^*}{r^*}\right)\boxed{\frac{z_0}{r_0}}+d\frac{\partial P_z^*}{\partial z^*}\boxed{\frac{P_{z1}}{P_{r1}}}\right\}\left(\frac{P_{r1}}{u_{r1}}\right)=-F\frac{\delta(r^*)}{r^*}\boxed{\frac{z_0}{r_0}\frac{1}{u_{r1}r_0}},$$

where we have used the result $\delta(r_0r^*)=\dfrac{\delta(r^*)}{r_0}$

$$(A.10)\qquad (iii)\qquad \tau_{zr}^*(r^*,0)=0=\left(\frac{P_{r1}}{u_{r1}}\right)\left\{d_{44}\left(\frac{\partial P_r^*}{\partial z^*}+\frac{\partial P_z^*}{\partial r^*}\boxed{\frac{P_{z1}}{P_{r1}}\frac{z_0}{r_0}}\right)\right\}$$

$$+c_{44}\left(\frac{\partial u_r^*}{\partial z^*}+\frac{\partial u_z^*}{\partial r^*}\frac{u_{z1}}{u_{r1}}\boxed{\frac{z_0}{r_0}}\right),$$

$$(A.11)\qquad (iv)\qquad E_{zr}^*(r^*,0)=0=\left(\frac{P_{r1}}{u_{r1}}\right)\left\{b_{44}\left(\frac{\partial P_r^*}{\partial z^*}+\frac{\partial P_z^*}{\partial r^*}\right)\boxed{\frac{P_{z1}}{P_{r1}}\frac{z_0}{r_0}}\right.$$

$$\left.+b_{77}\left(\frac{\partial P_r^*}{\partial r^*}-\frac{\partial P_z^*}{\partial r^*}\frac{P_{z1}}{P_{r1}}\right)\boxed{\frac{z_0}{r_0}}\right\}+d_{44}\left(\frac{\partial u_r^*}{\partial z^*}+\frac{\partial u_z^*}{\partial r^*}\boxed{\frac{u_{z1}}{u_{r1}}\frac{z_0}{r_0}}\right),$$

$$(A.12)\qquad (v)\qquad E_{zz}^*(r^*,0)=0=\frac{P_{r1}}{u_{r1}}\left\{b_{12}\left(\frac{\partial P_r^*}{\partial r^*}+\frac{P_r^*}{r^*}\right)\boxed{\frac{z_0}{r_0}}+b\frac{\partial P_z^*}{\partial z^*}\boxed{\frac{P_{z1}}{P_{r1}}}\right\}$$

$$+d_{12}\left(\frac{\partial u_r^*}{\partial r^*}+\frac{u_r^*}{r^*}\right)\boxed{\frac{z_0}{r_0}}+d\frac{\partial u_z^*}{\partial z^*}\boxed{\frac{u_{z1}}{u_{r1}}},$$

$$(A.13)\qquad (vi)\qquad 0=\frac{\partial\phi^*}{\partial z^*}-\varepsilon_0^{-1*}\left(1+\left(\frac{z^*}{r^*}\right)^2\left(\frac{z_0}{r_0}\right)^2\right)^{-1}P_z^*\boxed{\frac{P_{z1}}{\phi_1}\frac{z_0}{r_0^2}}.$$

Appendix B

Evaluation for the potential function Φ:

Substituting the expressions for $P(\eta)$ and $Q(\eta)$ from Eqs. (4.20) and (4.21) into Eq. (4.5), the equation for Φ assumes the form

$$(1+\eta^2)\frac{d^2\Phi}{d\eta^2}+5\eta\frac{d\Phi}{d\eta}+4\Phi = \varepsilon_0^{*-1}\left[\frac{1}{(1+\eta^2)^2}A^*+\frac{2\eta}{(1+\eta^2)^{5/2}}A_6\right]. \tag{B.1}$$

For the homogeneous part of the Eq. (B.1), one solution, by inspection, is found to be

$$\Phi_1(\eta) = \frac{\eta}{(1+\eta^2)^{3/2}}. \tag{B.2}$$

The second solution is generated by the method of variation of parameters and is found as

$$\Phi_2(\eta) = \frac{\eta}{(1+\eta^2)^{3/2}}V(\eta). \tag{B.3}$$

Substituting Eq. (A.3) into the homogeneous part of Eq. (A.1), one finds that

$$\frac{\eta}{\sqrt{\eta^2+1}}\frac{d^2V}{d\eta^2}+\frac{2+\eta^2}{(1+\eta^2)^{3/2}}\frac{dV}{d\eta} = 0. \tag{B.4}$$

Elementary integration leads to

$$V(\eta) = \sinh^{-1}\eta-\frac{\sqrt{1+\eta^2}}{\eta}. \tag{B.5}$$

The particular integral is found to be

$$\Phi_e(\eta) = \varepsilon_0^{*-1}\left[\frac{A^*}{2(\eta^2+1)}+\frac{A_6}{3}\frac{\eta}{(\eta^2+1)^{3/2}}\log(\eta^2+1)\right]. \tag{B.6}$$

The complete solution of Eq. (A.1) is given by

$$\Phi(\eta) = \frac{\eta}{(\eta^2+1)^{3/2}}\left[B_1+B_2\left(\sinh^{-1}\eta-\frac{\sqrt{\eta^2+1}}{\eta}\right)+\varepsilon_0^{*-1}\frac{A_6}{3}\log(\eta^2+1)\right]+\frac{\varepsilon_0^{*--1}A^*}{2(\eta^2+1)}. \tag{B.5}$$

Since

$$\phi(r,z) = \frac{1}{r^2}\Phi(\eta)\to 0 \quad \text{for} \quad r=\varrho_0, \quad z\to\infty,$$

the constant B_2 in Eq. (A.7) equals zero.

Thus, the form for the function $\Phi(\eta)$ is given by

$$\Phi(\eta) = \frac{\eta}{(\eta^2+1)^{3/2}}\left[B_1+\frac{\varepsilon_0^{*-1}}{3}A_6\log(\eta^2+1)\right]+\frac{\varepsilon_0^{*-1}A^*}{2(\eta^2+1)}.$$

Appendix C

The integral boundary condition:

$$\int_0^{2\pi}\int_0^{\infty} r\tau_{zz}(r,z)\,dr\,d\theta = -2\pi F. \tag{C.1}$$

Substituting the expression for τ_{zz} from Eq. (4.25), one finds

$$\begin{aligned}\int_0^{\infty} r\tau_{zz}(r,z)\,dr &= \frac{2(cd_{12}-dc_{12})}{c}A^*I_1+\frac{c_{44}}{c}\left[-c_{12}I_2+3(c-c_{44})I_3\right]A_5\\ &+\frac{3}{2}\left[\left[(-2d_{12}+d_{44})+\frac{c_{12}}{c}(2d+d_{44})\right]I_3+\left[\left\{3d_{12}-\frac{c_{12}}{c}(3d_{12}+7d_{44})\right\}\right]I_2\right]\\ &\times A_6+\left(\frac{c_{12}}{c}-1\right)I_2A,\end{aligned} \tag{C.2}$$

where I_1, I_2, I_3 are elementary integrals which are easily evaluated and are given by

$$I_1 = \int_0^{\infty}\frac{z^2r}{(z^2+r^2)^2}\,dr = \frac{1}{2}, \tag{C.3}$$

$$I_2 = \int_0^{\infty}\frac{zr}{(z^2+r^2)^{3/2}}\,dr = 1, \tag{C.4}$$

$$I_3 = \int_0^{\infty}\frac{zr^3}{(z^2+r^2)^{5/2}}\,dr = \frac{2}{3}. \tag{C.5}$$

The integral boundary condition (C.1) becomes

$$c_{44}A_5+\frac{2dc_{44}-cd_{44}}{c}A_6-2\frac{c_{44}}{c}A_7+c\frac{d_{12}-dc_{12}}{c}A^* = -F, \tag{C.6}$$

which is Eq. (4.34).

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