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Stress distribution in a transversely isotropic solid containing a penny-shaped crack

M. DAHAN (PALAISEAU)

AN AXIALLY symmetric stress distribution inside an infinite transversely isotropic elastic solid containing a penny-shaped crack is treated. It is assumed that the load is concentrated on one internal disc located at a finite distance from the crack. An analytical solution is presented for the displacement and stress distribution and for the stress intensity factors. Closed form solutions are given for the case of a point force as well as curves of numerical results, showing the influence of this type of anisotropy.

Rozważono przypadek osiowo-symetrycznego rozkładu naprężenia wewnątrz nieskończonego, poprzecznie izotropowego ciała sprężystego zawierającego szczelinę kołową. Założono, że obciążenie rozłożone jest na powierzchni koła znajdującego się w skończonej odległości od szczeliny. Przedstawiono rozwiązanie analityczne dla rozkładu naprężeń i przemieszczeń oraz dla współczynników intensywności naprężenia. Rozwiązania zamknięte otrzymano dla przypadku obciążenia skupionego, jak również przedstawiono wyznaczone numerycznie wykresy obrazujące wpływ przyjętego rodzaju anizotropii.

Рассмотрен случай осесимметричного распределения напряжения внутри безконечного, поперечно изотропного упругого тела, содержавшего круговую щель. Предположено, что нагрузка распределена на поверхности круга, находящегося в конечном расстоянии от щели. Представлено аналитическое решение для распределения напряжений и перемещений, а также для коэффициентов интенсивности напряжения. Замкнутые решения получены для случая сосредоточенной нагрузки, как тоже представлены, определенные численно, диаграммы, образующие влияние принятого рода анизотропии.

Notations

a_{ij}	elastic coefficients of the anisotropic medium,
$a, b, c, d,$	constants of the material (2.6),
A_i, B_i, C_i, D_i	amplitude functions,
f	constant of the material (4.2),
g_1, g_2	constants of the material (3.5),
h	distance crack-loading,
I, J, K	integrals defined in (3.19), (3.21) and (3.27)
k_1, k_2, k_3	stress intensity factors,
J_0, J_1	Bessel functions,
m_1, m_2	constants of the material (4.2),
$p(r)$	body force,
$p^H(m)$	Hankel transform of $p(r)$,
p_1, p_2, q_1, q_2	constants of the material (3.5),
q	constant of the material (4.2),
r, θ, z	cylindrical polar coordinates,
r_0	radius of the crack,
s_1, s_2	constants of the material (2.14),
u_r, u_z	components of displacement,
w	maximum width of the crack,

$\varepsilon_{ij}, \sigma_{ij}$ stress and strain components,
Ω_i partition of the space (2.13)
φ potential function of the Love type.

1. Introduction

THE STRESS distribution inside an infinite isotropic elastic solid containing a penny-shaped crack opened by pressure applied directly over its surface in a symmetric or asymmetric fashion was first considered by SNEDDON [1] and, subsequently, by GREEN and ZERNA [2] and COLLINS [3].

When two symmetric body forces were concentrated on two surfaces situated at the same finite distance from the crack, the stress intensity factor was calculated by SNEDDON and TWEED [4] for an isotropic material and by DAHAN [5] for a transversely isotropic one. This paper examines the behaviour of a crack embedded in an infinite medium with transverse isotropy and deformed by an asymmetrical loading which is concentrated on one disc located at a finite distance from the crack.

2. Formulation of the problem

We consider an infinite elastic solid containing a penny-shaped crack of radius r_0 given by $z = 0$ $(0 \leqslant r \leqslant r_0)$, where (r, θ, z) are cylindrical polar coordinates. The center and axis of the crack are respectively the origin and z-axis. We assume that the medium is characterized by transverse isotropy, with respect to the z-axis. Regarding the opening of the crack by an axially symmetric loading concentrated at an interior disc of the infinite solid, located on the plane $z = h$ (cf. Fig. 1), we denote $(u_r, 0, u_z)$ the components of the displace-

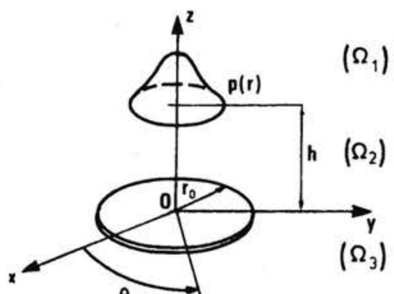

FIG. 1. Diagram of the problem.

ment field, $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz})$ the nonzero components of the stress tensor and $(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz})$ the components of the strain tensor. Then the strain-stress relations for the transversely isotropic solid can be defined by

$$
\begin{aligned}
\varepsilon_{rr} &= a_{11}\sigma_{rr}+a_{12}\sigma_{\theta\theta}+a_{13}\sigma_{zz},\\
\varepsilon_{\theta\theta} &= a_{12}\sigma_{rr}+a_{11}\sigma_{\theta\theta}+a_{13}\sigma_{zz},\\
\varepsilon_{zz} &= a_{13}\sigma_{rr}+a_{13}\sigma_{\theta\theta}+a_{33}\sigma_{zz},\\
\varepsilon_{rz} &= a_{44}\sigma_{rz},
\end{aligned}
\tag{2.1}
$$

where $a_{11}, a_{12}, a_{13}, a_{33}, a_{44}$ are the five independent elastic coefficients of the medium.

For axisymmetric problems, the governing equations are completed by the following equilibrium equations:

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0, \tag{2.2}$$
$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=0$$

and the conditions of compatibility:

$$(a_{11}-a_{12})(\sigma_{rr}-\sigma_{\theta\theta})-r\frac{\partial}{\partial r}(a_{12}\sigma_{rr}+a_{11}\sigma_{\theta\theta}+a_{13}\sigma_{zz})=0, \tag{2.3}$$
$$\frac{\partial^2}{\partial z^2}(a_{11}\sigma_{rr}+a_{12}\sigma_{\theta\theta}+a_{13}\sigma_{zz})+\frac{\partial^2}{\partial r^2}(a_{13}\sigma_{rr}+a_{13}\sigma_{\theta\theta}+a_{33}\sigma_{zz})-a_{44}\frac{\partial^2\sigma_{rz}}{\partial r\partial z}=0.$$

Using a potential function of the Love type [6, 7] for the representation of the stress and displacement fields such that

$$\begin{aligned}
\sigma_{rr} &= -\frac{\partial}{\partial z}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{b}{r}\frac{\partial\varphi}{\partial z}+a\frac{\partial^2\varphi}{\partial z^2}\right),\\
\sigma_{\theta\theta} &= -\frac{\partial}{\partial z}\left(b\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+a\frac{\partial^2\varphi}{\partial z^2}\right),\\
\sigma_{zz} &= \frac{\partial}{\partial z}\left(c\frac{\partial^2\varphi}{\partial r^2}+\frac{c}{r}\frac{\partial\varphi}{\partial r}+d\frac{\partial^2\varphi}{\partial z^2}\right),\\
\sigma_{rz} &= \frac{\partial}{\partial r}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+a\frac{\partial^2\varphi}{\partial z^2}\right),\\
u_r &= -(1-b)(a_{11}-a_{12})\frac{\partial^2\varphi}{\partial r\partial z},\\
u_z &= a_{44}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+(a_{33}d-2a_{13}a)\frac{\partial^2\varphi}{\partial z^2},
\end{aligned} \tag{2.4}$$

the three equations $(2.2)_1$ and (2.3) of elastostatics in the absence of body forces are identically satisfied. Equation $(2.2)_2$ is equivalent to

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2\varphi+(a+c)\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\varphi+d\frac{\partial^4\varphi}{\partial z^4}=0. \tag{2.5}$$

The constant a, b, c, d are defined by

$$\begin{aligned}
a &= a_{13}(a_{11}-a_{12})/(a_{11}a_{33}-a_{13}^2),\\
b &= [a_{13}(a_{13}+a_{44})-a_{12}a_{33}]/(a_{11}a_{33}-a_{13}^2),\\
c &= [a_{13}(a_{11}-a_{12})+a_{11}a_{44}]/(a_{11}a_{33}-a_{13}^2),\\
d &= (a_{11}^2-a_{12}^2)/(a_{11}a_{33}-a_{13}^2).
\end{aligned} \tag{2.6}$$

The solution of Eq. (2.5) has to satisfy the boundary conditions as follows: If $p(r)$ is the loading on the plane $z = h$, the presence of body forces implies the continuity of displacements u_r, u_z and shear stress σ_{rz} and the discontinuity of normal stress σ_{zz} following the condition

$$\lim_{\varepsilon\to 0^+} [\sigma_{zz}(r, h-\varepsilon)-\sigma_{zz}(r, h+\varepsilon)] = p(r), \tag{2.7}$$

where p is an arbitrary function so defined for $r \geqslant 0$ that the Hankel transform of order zero p^H exists.

Over the plane $z = 0$, the stresses and displacements are continuous in the exterior of the crack ($r_0 < r < \infty$); the boundary of which is stress-free, i.e. the stresses σ_{zz} and σ_{rz} take prescribed values over the surface of the crack so that

$$\begin{aligned} \sigma_{zz}(r, 0) &= 0, \\ \sigma_{rz}(r, 0) &= 0, \end{aligned} \qquad 0 \leqslant r < r_0. \tag{2.8}$$

For the remaining boundary conditions, it is assumed that the components of stress and displacement vanish as $(r^2+z^2)^{1/2}\to\infty$.

In order to solve Eq. (2.5), we introduce the Hankel transform of order zero, $\mathscr{H}_0$, defined by

$$\mathscr{H}_0[\varphi(r, z)] \equiv \varphi^H(m, z) = \int_0^\infty r\varphi(r, z)\mathrm{J}_0(mr)dr, \tag{2.9}$$

where J_0 is the Bessel function of zero order of the first kind.

Thus the solution of Eq. (2.5) can be represented by

$$\varphi(r, z) = \mathscr{H}_0^{-1}[\varphi^H(m, z)] = \int_0^\infty m\varphi^H(m, z)\mathrm{J}_0(mr)dm, \tag{2.10}$$

where φ^H is the solution of the differential equation

$$d\frac{\mathrm{d}^4\varphi^H}{\mathrm{d}z^4} - (a+c)m^2\frac{\mathrm{d}^2\varphi^H}{\mathrm{d}z^2} + m^4\varphi^H = 0. \tag{2.11}$$

We find for the potential function

$$\begin{gathered} \varphi(r, z) = \int_0^\infty [A_i(m)e^{+s_1mz} + B_i(m)e^{+s_2mz} + C_i(m)e^{-s_1mz} + D_i(m)e^{-s_2mz}]\mathrm{J}_0(mr)mdm, \\ (r, z)\in\Omega_i, \quad i = 1, 2, 3, \end{gathered} \tag{2.12}$$

where A_i, B_i, C_i, D_i, defined on each Ω_i, are amplitude functions to be determined from the boundary conditions and Ω_i are three parts composing the whole space as follows:

$$\begin{aligned} \Omega_1 &= \{(r, z): r\in\mathscr{R}_+,\ h < z < +\infty\}, \\ \Omega_2 &= \{(r, z): r\in\mathscr{R}_+,\ 0 < z < h\}, \\ \Omega_3 &= \{r, z): r\in\mathscr{R}_+,\ -\infty < z < 0\}. \end{aligned} \tag{2.13}$$

The constants s_1 and s_2 depend only on the coefficients defined in Eq. (2.6) and are characteristics of the material. We have

$$\begin{aligned} s_1 &= \left[\left(a+c+\sqrt{(a+c)^2-4d}\right)/2d\right]^{1/2}, \\ s_2 &= \left[\left(a+c-\sqrt{(a+c)^2-4d}\right)/2d\right]^{1/2}. \end{aligned} \tag{2.14}$$

3. Solution of the problem

In order to solve the problem completely, i.e. to determine the function φ continuous by parts on each Ω_i, we have to calculate the twelve functions A_i, B_i, C_i, D_i. To this end we take into account the boundary conditions on the planes $z = h$ and $z = 0$ and the conditions at infinity.

The displacement components vanish at a large distance $(r^2+z^2)^{1/2}$ from the origin so that

$$\begin{aligned} A_1(m) &= B_1(m) = 0, \\ C_3(m) &= D_3(m) = 0. \end{aligned} \tag{3.1}$$

3.1. Boundary conditions on the plane z = h

The presence of body forces yields four algebraic relations connecting the amplitudes. Typically these are derived as follows. From the condition that u_r is continuous on $z = h$, we obtain

$$\int_0^\infty [s_1(A_2 e^{ms_1h} - C_2 e^{-ms_1h}) + s_2(B_2 e^{ms_2h} - D_2 e^{-ms_2h}) + s_1 C_1 e^{-ms_1h} + s_2 D_1 e^{-ms_2h}] \times m^3 J_1(mr) dm = 0 \quad (\text{all } r), \tag{3.2}$$

and since this is true for all r, the Hankel inversion theorem implies the vanishing of the integrand, i.e.

$$s_1(A_2 e^{ms_1h} - C_2 e^{-ms_1h}) + s_2(B_2 e^{ms_2h} - D_2 e^{-ms_2h}) + s_1 C_1 e^{-ms_1h} + s_2 D_1 e^{-ms_2h} = 0. \tag{3.3}$$

Similarly, the continuity of u_z and σ_{rz} on $z = h$ and the condition (2.7) yield

$$\begin{aligned} &q_1(A_2 e^{ms_1h} + C_2 e^{-ms_1h}) + q_2(B_2 e^{ms_2h} + D_2 e^{-ms_2h}) - q_1 C_1 e^{-m_1sh} + q_2 D_1 e^{-ms_2h} = 0, \\ &p_1(A_2 e^{ms_1h} + C_2 e^{-ms_1h}) + p_2(B_2 e^{ms_2h} + D_2 e^{-ms_2h}) - p_1 C_1 e^{-ms_1h} - p_2 D_1 e^{-ms_2h} = 0, \\ &s_1 g_1(-A_2 e^{ms_1h} + C_2 e^{-ms_1h}) + s_2 g_2(-B_2 e^{ms_2h} + D_2 e^{-ms_2h}) - s_1 g_1 C_1 e^{-ms_2h} \\ &\qquad - s_2 g_2 D_1 e^{-ms_2h} = \frac{1}{m^3} p^H(m). \end{aligned} \tag{3.4}$$

with the notations

$$\begin{aligned} q_i &= s_i^2(a_{33}d - 2a_{13}a) - a_{44}, \\ p_i &= 1 - as_i^2, \\ g_i &= c - ds_i^2, \qquad i = 1, 2. \end{aligned} \tag{3.5}$$

From Eqs. (3.3) and (3.4), we get

$$\begin{aligned} A_2(m) &= e^{-ms_1h} p^H(m)/[2m^3 ds_1(s_1^2 - s_2^2)], \\ B_2(m) &= e^{-ms_2h} p^H(m)/[2m^3 ds_2(s_2^2 - s_1^2)], \\ C_1(m) &= e^{+ms_1h} p^H(m)/[2m^3 ds_1(s_1^2 - s_2^2)] + C_2(m), \\ D_1(m) &= e^{+ms_2h} p^H(m)/[2m^3 ds_2(s_2^2 - s_1^2)] + D_2(m). \end{aligned} \tag{3.6}$$

3.2. Boundary conditions on the plane $z = 0$

From the conditions (2.8) that $\sigma_{zz} = 0$ for $z = 0\pm, 0 \leqslant r < r_0$, while for $r \geqslant r_0$, σ_{zz} is continuous on $z = 0$, it follows that σ_{zz} is continuous on $z = 0$ for all r. The argument for the continuity of σ_{rz} for all r is similar to that of σ_{zz}. We have two supplementary relations for determining the amplitude functions giving

$$(3.7)\quad \begin{aligned} A_3(m) &= A_2(m)+\frac{s_1+s_2}{s_1-s_2}C_2(m)+\frac{2s_1}{s_1-s_2}\cdot\frac{p_2}{p_1}D_2(m),\\ B_3(m) &= B_2(m)+\frac{s_2+s_1}{s_2-s_1}D_2(m)+\frac{2s_2}{s_2-s_1}\cdot\frac{p_1}{p_2}C_2(m).\end{aligned}$$

We have to determine the functions C_2 and D_2. The remaining boundary conditions on $z = 0$ are valid either for $0 \leqslant r < r_0$ or for $r \geqslant r_0$ and necessarily give integral equations. From the continuity of the displacements u_r and u_z over the surface for $r \geqslant r_0$, there result respectively:

$$(3.8)\quad \begin{aligned} &\int\limits_0^\infty [s_2p_1C_2(m)+s_1p_2D_2(m)]m^3\mathrm{J}_0(mr)\mathrm{d}m = 0,\\ &\int\limits_0^\infty [p_1C_2(m)+p_2D_2(m)]m^3\mathrm{J}_1(mr)\mathrm{d}m = 0,\end{aligned}\qquad r_0 \leqslant r < +\infty,$$

while the conditions (2.8) lead respectively to

$$(3.9)\quad \begin{aligned} &\int\limits_0^\infty \left[s_1g_1C_2(m)+s_2g_2D_2(m)-(g_1\mathrm{e}^{-ms_1h}-g_2\mathrm{e}^{-ms_2h})\frac{p^H(m)}{2m^3d(s_1^2-s_2^2)}\right]m^4\mathrm{J}_0(mr)\mathrm{d}m = 0,\\ &\int\limits_0^\infty \Bigg[p_1C_2(m)+p_2D_2(m)\\ &\qquad +(p_1s_2\mathrm{e}^{-ms_1h}-p_2s_1\mathrm{e}^{-ms_2h}\frac{p^H(m)}{2m^3\sqrt{d}(s_1^2-s_2^2)}\Bigg]m^4\mathrm{J}_1(mr)\mathrm{d}m = 0,\quad 0 \leqslant r < r_0.\end{aligned}$$

We have obtained two coupled pairs of dual integral equations for the remaining two amplitudes C_2 and D_2.

3.3. Solution of the integral equations

Equations equivalent to Eqs. (3.8) and (3.9), but expressed entirely in terms of J_0 rather than J_0 and J_1, may be obtained as follows. First, using the relation $\mathrm{J}_0 = -\mathrm{J}_1$, Eq. $(3.9)_2$ may be written as

$$(3.10)\quad \frac{\mathrm{d}}{\mathrm{d}r}\int\limits_0^\infty\left[\mathsf{A}(m)+(p_1s_2\mathrm{e}^{-ms_1h}-p_2s_1\mathrm{e}^{-ms_2h})\frac{p^H(m)}{2m^3\sqrt{d}(s_1^2-s_2^2)}\right]m^3\mathrm{J}_0(mr)\mathrm{d}m = 0,$$

where

$$(3.11)\quad \mathsf{A}(m) = p_1C_2(m)+p_2D_2(m),$$

and after integration

$$(3.12)\quad \int\limits_0^\infty \left[\mathsf{A}(m)+(p_1 s_2 e^{-ms_1h}-p_2 s_1 e^{-ms_2h})\frac{p^H(m)}{2m^3\sqrt{d}(s_1^2-s_2^2)}\right] m^3 J_0(mr)\,dm = C$$

$$(0 \leqslant r < r_0),$$

where C is an unknown constant to be determined later. To express Eq. $(3.8)_2$ in a similar form, we differentiate with respect to r and make use of Bessel's differential equation for the zero-order function. We obtain

$$(3.13)\quad \int\limits_0^\infty \mathsf{A}(m)m^4 J_0(mr)\,dm = 0, \quad r_0 \leqslant r < +\infty.$$

However, while Eq. $(3.8)_2$ implies Eq. (3.13), the converse is not necessarily true. If the analysis is reversed, Eq. (3.13) implies

$$(3.14)\quad \int\limits_0^\infty \mathsf{A}(m)m^3 J_1(mr)\,dm = \frac{C'}{r}, \quad r_0 \leqslant r < +\infty,$$

where C' is a constant of integration. Equation $(3.8)_2$ imposes the condition $C' = 0$ and provides a means for determining C.

The final integral equations for solution are Eqs. $(3.8)_1$, $(3.9)_1$, (3.12), and (3.13). The last two equations are dual integral equations of the unknown function A. If we introduce a supplementary function χ such that

$$(3.15)\quad \mathsf{A}(m) = \frac{1}{m^3}\int\limits_0^{r_0} \cos(mt)\chi(t)\,dt,$$

the condition (3.13) is identically verified. The condition (3.12) can be reduced to an Abel integral equation:

$$(3.16)\quad \int\limits_0^r \chi(t)(r^2-t^2)^{-1/2}dt = C - \int\limits_0^\infty (p_1 s_2 e^{-ms_1h}-p_2 s_1 e^{-ms_2h})\frac{p^H(m)}{2\sqrt{d}(s_1^2-s_2^2)} J_0(mr)\,dm.$$

The following solution for which is

$$(3.17)\quad \chi(t) = \frac{2}{\pi}\left[C - \int\limits_0^\infty (p_1 s_2 e^{-ms_1h}-p_2 s_1 e^{-ms_2h})\frac{p^H(m)}{2\sqrt{d}(s_1^2-s_2^2)}\cos(mt)\,dm\right].$$

Using Eq. (3.15), we get

$$(3.18)\quad \mathsf{A}(m) = \frac{2C}{\pi m^4}\sin(mr_0) - \frac{J(m)}{m^3(s_1+s_2)},$$

where

$$(3.19)\quad J(m) = \frac{1}{\pi\sqrt{d}(s_1-s_2)}\int\limits_0^{r_0}\left[\int\limits_0^\infty (s_2 p_1 e^{-\alpha s_1 h}-s_1 p_2 e^{-\alpha s_2 h})p^H(\alpha)\cos(\alpha t)\,d\alpha\right]\cos(mt)\,dt.$$

It remains to impose the condition (3.14) with $C' = 0$ for determining C. Finally we obtain

$$\mathsf{A}(m) = [I(m)-J(m)]/m^3(s_1+s_2), \tag{3.20}$$

where $I(m)$ is given by

$$I(m) = \frac{1}{\pi\sqrt{d}(s_1-s_2)}\frac{\sin(mr_0)}{mr_0}\int\limits^{\infty}(s_2p_1e^{-\alpha s_1h}-s_1p_2e^{-\alpha s_2h})p^H(\alpha)\sin(\alpha r_0)\frac{d\alpha}{\alpha} \tag{3.21}$$

Inserting $\mathsf{A}(m)$ into Eqs. $(3.8)_1$–$(3.9)_1$ and re-arranging, we obtain two dual integral equations of the function C_2:

$$\int\limits_0^{\infty}\left[(s_1-s_2)p_1C_2(m)-s_1\mathsf{A}(m)+(g_1e^{-ms_1h}-g_2e^{-ms_2h})\frac{p^H(m)}{2m^3d^{3/2}(s_1^2-s_2^2)}\right]m^4J_0(mr)dm = 0$$

(3.22) $(0 \leqslant r < r_0)$,

$$\int\limits_0^{\infty}[(s_1-s_2)p_1C_2(m)-s_1\mathsf{A}(m)]m^3J_0(mr)dm = 0 \quad (r_0 \leqslant r < +\infty).$$

We introduce a new function ξ such that

$$(s_1-s_2)p_1C_2(m)-s_1\mathsf{A}(m) = \frac{1}{m^3}\int\limits_0^{r_0}\xi(t)\sin(mt)dt, \tag{3.23}$$

with $\xi(0) = 0$. Thus the condition $(3.22)_2$ is identically satisfied and Eq. $(3.22)_1$ gives an Abel integral equation:

$$\int\limits_0^{r}\xi'(t)(r^2-t^2)^{-1/2}dt = -\int\limits_0^{\infty}(g_1e^{-ms_1h}-g_2e^{-ms_2h})\frac{p(m^H)}{2d^{3/2}(s_1^2-s_2^2)}mJ_0(mr)dm, \tag{3.24}$$

the solution for which can be written as

$$\xi(t) = \frac{-1}{\pi d^{3/2}(s_1^2-s_2^2)}\int\limits_0^{\infty}(g_1e^{-m_1sh}-g_2e^{-ms_2h})p^H(m)\sin(mt)dm. \tag{3.25}$$

Substituting from Eqs. (3.25) into Eqs. (3.23) and (3.20), we obtain finally

$$\begin{aligned} C_2(m) &= \frac{1}{m^3p_1(s_1^2-s_2^2)}[s_1I(m)-s_1J(m)-K(m)],\\ D_2(m) &= \frac{1}{m^3p_2(s_2^2-s_1^2)}[s_2I(m)-s_2J(m)-K(m)], \end{aligned} \tag{3.26}$$

where the functions I, J, K depend on the loading and the geometrical parameters following the notations (3.19), (3.21) and the relation

$$K(m) = \frac{1}{\pi d^{3/2}(s_1-s_2)}\int\limits_0^{r_0}\left[\int\limits_0^{\infty}(g_1e^{-\alpha s_1h}-g_2e^{-\alpha s_2h})p^H(\alpha)\sin(\alpha t)d\alpha\right]\sin(mt)dt. \tag{3.27}$$

The displacement and stress distribution can be directly calculated from Eq. (2.4) by using the potential function φ defined by Eq. (2.12) on each part Ω_i on which the amplitude functions are given by Eqs. (3.1), (3.6), (3.7) and (3.26).

4. Calculations

4.1. Stresses and displacements on the crack plane

As an illustration, we calculate the displacement u_z on the plane $z = 0$. Using the results of Sect. 3, we find this displacement on the crack for an arbitrary loading:

$$(4.1)\qquad u_z(r,0_\pm) = \frac{1}{2f\sqrt{d}(s_1^2-s_2^2)}\int_0^\infty (s_2 m_1 e^{-ms_1h} - s_1 m_2 e^{-ms_2h}) p^H(m) J_0(mr)\,dm$$

$$+\frac{s_1p_2q_1-s_2p_1q_2}{2f(s_1^2-s_2^2)(s_1-s_2)r_0}\int_0^\infty (s_2p_1 e^{-ms_1h}-s_1p_2 e^{-ms_2h})p^H(m)\sin(mr_0)\frac{dm}{m}$$

$$\mp\frac{q}{\pi d(s_1^2-s_2^2)}\int_r^{r_0}\left[\int_0^\infty (g_1 e^{-ms_1h}-g_2 e^{-ms_2h})p^H(m)\sin(mt)\,dm\right](t^2-r^2)^{-1/2}dt,$$

with the supplementary notations

$$f = (d-ac)/\sqrt{d},$$

$$(4.2)\qquad q = (a_{11}-a_{12})(1-b)(s_1+s_2)/f,$$

$$m_i = fq_i-(s_1p_2q_1-s_2p_1q_2)\sqrt{d}p_i/(s_1-s_2),\qquad i = 1,2.$$

The upper and lower signs correspond to the faces $z = 0_+$ and $z = 0_-$ of the crack, respectively.

The boundary of the crack is stress-free. For the region $r \geqslant r_0$, we calculate the norma, stress and the shear stress. We obtain

$$(4.3)\qquad \sigma_{zz}(r,0) = \frac{-1}{2d(s_1^2-s_2^2)}\int_0^\infty (g_1 e^{-ms_1h}-g_2 e^{-ms_2h})p^H(m)J_0(mr)m\,dm$$

$$-\frac{(r^2-r_0^2)^{-1/2}}{\pi d(s_1^2-s_2^2)}\int_0^\infty (g_1 e^{-ms_1h}-g_2 e^{-ms_2h})p^H(m)\sin(mr_0)\,dm$$

$$+\frac{1}{\pi d(s_1^2-s_2^2)}\int_0^{r_0}\left[\int_0^\infty (g_1 e^{-ms_1h}-g_2 e^{-ms_2h})p^H(m)\cos(mt)m\,dm\right](r^2-t^2)^{-1/2}dt,$$

$$\sigma_{rz}(r,0) = \frac{1}{2\sqrt{\bar{d}}(s_1^2-s_2^2)}\int_0^\infty (s_2p_1e^{-ms_1h}-s_1p_2e^{-ms_2h})p^H(m)J_1(mr)m\,dm \tag{4.3 [cont.]}$$

$$+\frac{r_0(r^2-r_0^2)^{-1/2}}{\pi\sqrt{\bar{d}}(s_1^2-s_2^2)r}\int_0^\infty (s_2p_1e^{-ms_1h}-s_1p_2e^{-ms_2h})\left[\frac{\sin(mr_0)}{mr_0}-\cos(mr_0)\right]p^H(m)\,dm$$

$$-\frac{1}{\pi\sqrt{\bar{d}}(s_1^2-s_2^2)r}\int_0^{r_0}\left[\int_0^\infty (s_2p_1e^{-ms_1h}-s_1p_2e^{-ms_2h})p^H(m)\sin(mt)m\,dm\right]\times t(r^2-t^2)^{-1/2}dt.$$

4.2. Stress intensity factors

For further discussions of interest in Fracture Mechanics, we can calculate the different stress intensity factors defined by the limits

$$\begin{aligned} k_1 &= \lim_{r\to r_0^+}[2\pi(r-r_0)]^{1/2}\sigma_{zz}(r,0),\\ k_2 &= \lim_{r\to r_0^+}[2\pi(r-r_0)]^{1/2}\sigma_{rz}(r,0),\\ k_3 &= \lim_{r\to r_0^+}[2\pi(r-r_0)]^{1/2}\sigma_{\theta z}(r,0) = 0.\end{aligned} \tag{4.4}$$

From these definitions and the relations (4.3), we deduce

$$\begin{aligned} k_1 &= \frac{-1}{\sqrt{\pi r_0}d(s_1^2-s_2^2)}\int_0^\infty (g_1e^{-ms_1h}-g_2e^{-ms_2h})p^H(m)\sin(mr_0)\,dm,\\ k_2 &= \frac{1}{\sqrt{\pi r_0\bar{d}}(s_1^2-s_2^2)}\int_0^\infty (s_2p_1e^{-ms_1h}-s_1p_2e^{-ms_2h})\left[\frac{\sin(mr_0)}{mr_0}-\cos(mr_0)\right]p^H(m)\,dm.\end{aligned} \tag{4.5}$$

This solution makes it possible to calculate the stress intensity factors in the case of two loads p and p' applied on each side of the crack at the distances h and h'. From the principle of superposition we have the new factors by using for each k_i the expressions (4.5):

$$\begin{aligned} k_1 &= k_1(p,h)+k_1(p',h'),\\ k_2 &= k_2(p,h)-k_2(p',h').\end{aligned} \tag{4.6}$$

For equal and symmetrically spaced loads, we remark that the factor k_1 is double the value given by Eq. (4.5) and the factor k_2 is zero. These expressions are in good agreement with the results obtained in a previous paper [5].

When the loads are applied directly on the crack's surface, we have $h = 0$ and we deduce for the last case

$$\begin{aligned} k_1 &= \frac{2}{\sqrt{\pi r_0}}\int_0^\infty p^H(m)\sin(mr_0)\,dm,\\ k_2 &= k_3 = 0.\end{aligned} \tag{4.7}$$

It is important to remark that the result depends on the loading $p(r)$ applied on the crack only through its Hankel transform $p^H(m)$. Thus the elastic coefficients of the material do not appear in the formula (4.7) and therefore every result already existing for the stress intensity factors in an isotropic medium can be readily generalized to a transversely isotropic medium when the loads are applied on the crack's surfaces. This is surely not true when h is different from zero.

5. Special loading: point force

The expressions (4.3) and (4.5) of the stresses and stress intensity factors give closed form results for the usual loading geometries (uniform loads over a disc, concentrated ring load, point force, ...). Specially, for a force of magnitude P acting at the point $(0, 0, h)$, we obtain the following results, utilizing $p^H(m) = P/2\pi$:

5.1. Vertical displacement on the crack plane

On the crack surface $(0 \leqslant r \leqslant r_0)$:

$$(5.1)\quad u_z(r, 0\pm) = \frac{P}{2\pi(s_1^2-s_2^2)}\left\{\frac{1}{2f\sqrt{d}}\left[s_2 m_1(r^2+s_1^2h^2)^{-1/2} - s_1 m_2(r^2+s_2^2h^2)^{-1/2}\right]\right.$$
$$+\frac{s_1p_2q_1-s_2p_1q_2}{2f(s_1-s_2)r_0}\left[s_2p_1\,\mathrm{arctg}\,(r_0/s_1h) - s_1p_2\,\mathrm{arctg}\,(r_0/s_2h)\right]$$
$$\left.\mp\frac{q}{\pi d}\left[q_1(r^2+s_1^2h^2)^{-1/2}\,\mathrm{arctg}\sqrt{\frac{r_0^2-r^2}{r^2+s_1^2h^2}} - g_2(r^2+s_2^2h^2)^{-1/2}\,\mathrm{arctg}\sqrt{\frac{r_0^2-r^2}{r^2+s_2^2h^2}}\right]\right\}.$$

In the exterior of the crack $(r_0 \leqslant r < \infty)$:

$$(5.2)\quad u_z(r, 0) = \frac{P}{2\pi(s_1^2-s_2^2)}\left\{\frac{1}{2\sqrt{d}}\left[s_2q_1(r^2+s_1^2h^2)^{-1/2} - s_1q_2(r^2+s_2^2h^2)^{-1/2}\right]\right.$$
$$+\frac{s_1p_2q_1-s_2p_1q_2}{\pi f(s_1-s_2)r_0}\arcsin\frac{r_0}{r}\left[s_2p_1\,\mathrm{arctg}\,(r_0/s_1h) - s_1p_2\,\mathrm{arctg}\,(r_0/s_2h)\right]$$
$$\left.+\frac{s_1p_2q_1-s_2p_1q_2}{2\pi f\sqrt{d}(s_1-s_2)}\sum_{i=1}^{2}\frac{(-1)^ip_i}{s_i(r^2+s_i^2h^2)^{1/2}}\left[\frac{\pi}{2}+\mathrm{arctg}\,\frac{r_0^2(r^2+s_i^2h^2)-s_i^2h^2(r^2-r_0^2)}{2r_0s_ih(r^2+s_i^2h^2)^{1/2}(r^2-r_0^2)^{1/2}}\right]\right\}.$$

The maximum width w of the crack is at $r = 0$ and is given by

$$(5.3)\quad \begin{aligned} w &= u_z(0, 0_+) - u_z(0, 0_-) \\ &= \frac{-Pq}{\pi^2 d(s_1^2-s_2^2)}\left[\frac{q_1}{s_1h}\,\mathrm{arctg}\,\frac{r_0}{s_1h} - \frac{g_2}{s_2h}\,\mathrm{arctg}\,\frac{r_0}{s_2h}\right]. \end{aligned}$$

In order to illustrate the variation of u_{z+} and u_{z-} over the crack, these quantities have been calculated for various values of h. The results are plotted in Figs. 2 and 3. [In these figures there are curves for two anisotropic materials — thallium and cadmium — and an isotropic one having $\nu = 0.25$].

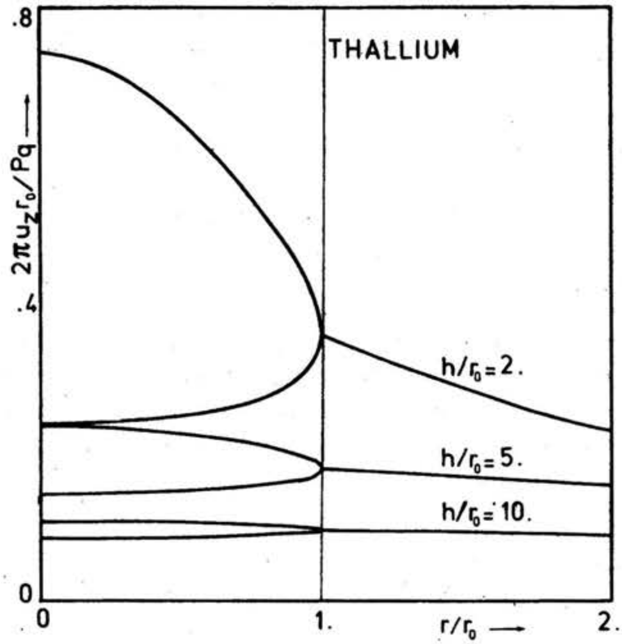

FIG. 2. Crack's opening as a function of the load's distance for thallium: $a_{11} = 1.04 \cdot 10^{-4}$ SI, $a_{12} = -8.1 \cdot 10^{-9}$, $a_{13} = -1.2 \cdot 10^{-9}$, $a_{33} = 3.25 \cdot 10^{-9}$, $a_{44} = 1.38 \cdot 10^{-8}$.

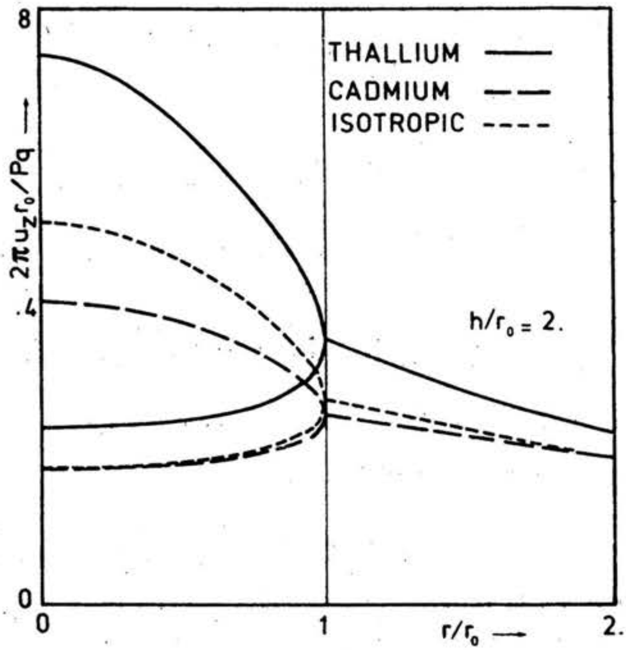

FIG. 3. Crack's opening for various materials. (Cadmium: $a_{11} = 1.22 \cdot 10^{-9}$ SI, $a_{12} = -1.15 \cdot 10^{-10}$, $a_{13} = -8.7 \cdot 10^{-10}$, $a_{33} = 3.34 \cdot 10^{-9}$, $a_{44} = 5.01 \cdot 10^{-9}$).

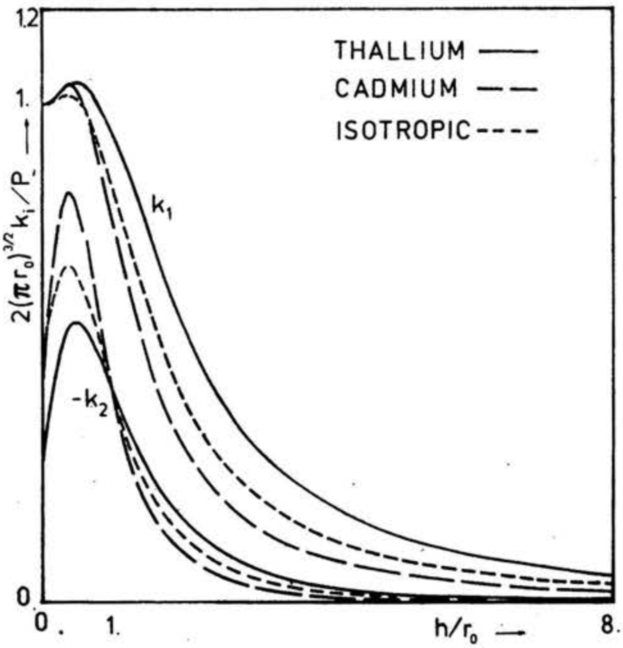

FIG. 4. Variations of the stress intensity factors k_1 and k_2.

5.2. Stress intensity factors

$$
(5.4)\quad
\begin{aligned}
k_1 &= \frac{P}{2(\pi r_0)^{3/2} d(s_1^2 - s_2^2)} \left[-g_1(1+s_1^2 h^2/r_0^2)^{-1} + g_2(1+s_2^2 h^2/r_0^2)^{-1}\right],\\
k_2 &= \frac{P}{2(\pi r_0)^{3/2} d(s_1^2 - s_2^2)} \Bigg[\sqrt{d}\, s_2 p_1 \operatorname{arctg}(r_0/s_1 h) - \sqrt{d}\, s_1 p_2 \operatorname{arctg}(r_0/s_2 h)\\
&\qquad - \frac{p_1 h/r_0}{1+s_1^2 h^2/r_0^2} + \frac{p_2 h/r_0}{1+s_2^2 h^2/r_0^2}\Bigg].
\end{aligned}
$$

Putting $s_1 = 1 + i\varepsilon$ and $s_2 = 1 - i\varepsilon$ in the relation (5.4) and letting ε approach zero, we get the solution for the isotropic case given by KASSIR and SIH [8]. Figure 4 shows the variation of the factors as a function of h/r_0.

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LABORATOIRE DE MECANIQUE DES SOLIDES
ECOLE POLYTECHNIQUE, PALAISEAU, FRANCE.

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