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Dispersion of surface waves in nonlocal dielectric fluids

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THE THEORY of nonlocal electromagnetic fluids developed in our recent work [1] is employed to investigate the propagation of electromechanical surface waves in a dielectric fluid. The solutions of the field equations and boundary conditions based on the linear nonlocal theory are found to lead to a decoupling of the waves into Rayleigh type mechanical waves and Zenneck type electromagnetic waves. It is found that dispersion does occur for the consequence of incorporating viscous and nonlocal effects. Assuming the nonlocal magnetic effects to be negligible, a dispersion relation involving the nonlocal electric effects is derived. The electromagnetic wave also is found to be dispersive only if the nonlocal effects are incorporated.

Teorię nielokalnych cieczy elektromagnetycznych, rozwiniętą w poprzedniej pracy Autorów [1], zastosowano do badania propagacji powierzchniowych fal elektromagnetycznych w cieczy dielektrycznej. Stwierdzono, że rozwiązania równań pola i warunków brzegowych oparte na liniowej teorii nielokalnej prowadzą do rozprzężenia fal na fale mechaniczne typu Rayleigha i fale elektromagnetyczne typu Zennecka. Stwierdzono, że rozproszenie istotnie występuje w przypadku mechanicznych fal powierzchniowych w cieczach dielektrycznych jako wynik uwzględnienia efektów lepkich i nielokalnych. Przy założeniu, że nielokalne efekty magnetyczne można pominąć, wprowadzono związek dyspersyjny uwzględniający nielokalne efekty elektryczne. Stwierdzono, że również fala elektromagnetyczna ulega dyspersji w przypadku uwzględnienia jedynie efektów nielokalnych.

Теория нелокальных электромагнитных жидкостей, развитая в предыдущей работе авторов [1], применена для исследования распространения поверхностных электромагнитных волн в диэлекрической жидкости. Констатировано, что решения уравнений поля и граничных условий, опирающиеся на линейную нелокальную теорию, приводят к распряжению волн на механические волны типа Рэлея и электромагнитные волны типа Зеннека. Констатировано, что рассеяние существенным образом выступает в случае механических поверхностных волн в диэлекрических жидкостях, как результат учета вязких и нелокальных эффектов. При предложении, что нелокальные магнитные эффекты пренебрежимо малы, выведено дисперсионное соотношение, учитывающее нелокальные электрические эффекты. Констатировано, что также электромагнитная волна подлежит дисперсии в случае учета только нелокальных эффектов.

1. Introduction

THE PHENOMENON of dispersion of surface waves in elastic solids is experimentally well documented [2] and theoretically well supported by lattice dynamical studies [3]. When a surface wave propagates through a medium, for example, a seismic wave propagating in the earth's crust which consists of materials existing both in solid and fluid states, a variety of complicated phenomena occur depending on the nature of the material response to the wave. The most outstanding of these phenomena is the scattering of the waves into several trains of waves each propagating with its own frequency and wave length. The frequency of these waves is dependent on their wave length in a nonlinear manner. Thus waves with different wave lengths will propagate with different phase velocities. It is this dependence of frequency on the wave number (reciprocal of the wave length) that is termed as wave dispersion. The study of dispersion of surface waves in materials is of great importance since it contributes to a proper understanding of the internal make-up of the material as well as the nature of the long range cohesive forces that are responsible in binding together the various internal substructures of the material.

Since one is almost always confronted with the problem of deducing phenomenalogical properties of a material from its internal substructures, it becomes at once clear that one has to develop a suitable continuum approach to explain wave dispersion. It is well known, for example, that classical elasticity predicts a constant phase velocity for all wave lengths for plane longitudinal waves propagating in an isotropic elastic solid while the experiments show that the phase velocity depends on the wave length. Thus the classical continuum mechanics fails to predict wave dispersion. The main source of this difficulty stems from the fact that classical continuum mechanics does not have a mechanism to take into account the internal long-range cohesive forces in a medium. Nonlocal continuum mechanics as developed by ERINGEN [4, 5] provides the necessary mechanism to take into account at a local material point (or local substructure) of a body the influences due to *all* the other substructres within the body both near and distant (long-range or nonlocal effects) with respect to the local material point. Using his concept of nonlocality, ERINGEN [6] successfully predicted the wave dispersion phenomenon in elastic solids obtaining both the optic and acoustic modes of wave propagation thus overcoming the discrepancy of the classical continuum theories. His results agree remarkably well with those of experiments as well as lattice dynamical results.

All real materials, regardless of their internal constitution, are dispersive in character to varying extents with respect to waves propagating through them. These materials range in their internal structures, from solids with a very high degree of molecular ordering and structure to fluids with random distribution of molecules and practically devoid of a structure. In between these two extremities, there exists a well-known class of real materials, such a liquid crystals, high polymers, colloidal suspensions, animal blood, gels, emulsions, and thick oils such as lubricating oils with a certain degree of molecular ordering and structure, yet possessing fluid properties such as viscosity and capacity to flow. In particulas, there also exist fluids with an electromechanical constitution. The dispersive nature of electromechanical surface waves in dielectric fluids is of special importance since such materials are utilized as insulators in energy transducers. In power generating processes the machinery involved is such that fluid insulators (or dielectric fluids) are preferred to solid insulators since the former can readily deform and adapt themselves to the shape of the machinery. Also, dielectric fluids are used as cooling media in nuclear reactors. Similarly, wave dispersion in dielectric fluids is of major concern owing to their involvement in natural oceanic and seismic disturbances as well as underground explosions.

Utilizing the nonlocal continuum theory of electromagnetic fluids we have recently developed [1], the dispersive character of electromechanical surface waves in a dielectric fluid is undertaken in the present paper. The application of the linear nonlocal theory results in the decoupling of the surface waves in the medium with electromechanical constitution into a purely mechanical wave and an electromagnetic wave. Analyzing the mechanical surface waves, we obtain a dispersion relation incorporating nonlocal effects and viscous influences. In the absence of any experimental data for dispersion in real fluids, it is to be realized that the results obtained for fluids for which no lattice structure is possible cannot be meaningfully compared with the results available only for solids with lattice structure. Our results must therefore await future experimental work for comparison. Furthermore, due to the lack of experimental data, the dispersion curve for the electromagnetic surface wave incorporating both nonlocal electric and magnetic effects is not obtained. Ignoring nonlocal magnetic effects, however, a dispersion relation is derived, which incorporates the nonlocal electric effects. Using a quantum-mechanical description for comparison, this nonlocal dispersive relation yields a deterministic form for the nonlocal electric material coefficient. Furthermore, in the high-frequency limit the nonlocal result is shown to be in agreement with the classical electromagnetic frequency wave number relationship.

2. Nomenclature and notation

Since the present paper is a direct sequel to our previous work [1], we retain the same nomenclature and notation in the present work also. For full details of the derivation of the linear nonlocal theory of electromagnetic fluids we refer the reader to our previous work [1]. We collect here, for convenience, the following nomenclature and notations for further reference:

ϱ — mass density, π — thermodynamic pressure, q — free charge density, c — speed of light in vacuum, $\mathbf{u}$ — displacement vector, $\mathbf{v}$ — velocity vector, $\mathbf{f}$ — total body force density, $\mathbf{t}_k$ — stress vector, $\mathbf{g}$ — electromagnetic momentum density, $\mathbf{E}$ — electric field, $\mathbf{H}$ — magnetic field, $\mathbf{D}$ — electric displacement vector, $\mathbf{B}$ — magnetic induction vector, $\mathbf{M}$ — magnetization vector, $\mathbf{P}$ — polarization vector, $\mathbf{J}^f$ — free current density vector, E–M fluids—electromagnetic fluids, ψ — free energy density, and

$$\text{(2.1)}\qquad \begin{aligned} &\underline{\mathscr{E}} = \mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}, \quad \underline{\mathscr{H}} = \mathbf{H}-\frac{1}{c}\mathbf{v}\times\mathbf{D}, \quad \underline{\mathscr{D}} = \mathbf{D}+\frac{1}{c}\mathbf{v}\times\mathbf{H},\\ &\underline{\mathscr{B}} = \mathbf{B}-\frac{1}{c}\mathbf{v}\times\mathbf{E}, \quad \underline{\mathscr{P}} = \mathbf{P}-\frac{1}{c}\mathbf{v}\times\mathbf{M}, \quad \underline{\mathscr{M}} = \mathbf{M}+\frac{1}{c}\mathbf{v}\times\mathbf{P}\\ &\underline{\mathscr{J}} = \mathbf{J}^f - q\mathbf{v}. \end{aligned}$$

Furthermore, we use Cartesian tensor notation throughout, x_k, $k = 1, 2, 3$, being the rectangular Cartesian coordinates of a spatial point. A subscript comma shall denote partial differentation and a superposed dot shall denote material time-rate, for example,

$$\text{(2.2)}\qquad \mathbf{v}_{,k} = \frac{\partial\mathbf{v}(\mathbf{x}, t)}{\partial x_k}; \quad \dot{q} = \frac{Dq}{Dt} = \frac{\partial q}{\partial t} + q_{,k}v^k.$$

We use the usual summation convention over repeated indices. Moreover, for any differentiable vector field $A = A(\mathbf{x}, t)$, we define the convective derivative

$$\text{(2.3)}\qquad \mathbf{A}^* = \frac{\partial\mathbf{A}}{\partial t} + \check{\mathbf{A}},$$

where

$$\text{(2.4)}\qquad \check{\mathbf{A}} = (\nabla\cdot\mathbf{A})v + \nabla + (\mathbf{A}\times\mathbf{v}).$$

3. Basic equations of linear theory of nonlocal *E-M* fluids

The balance laws, the constitutive equations and the full set of field equations governing the flow of *E–M* fluids having no discontinuities within the material volume v with bounding surface $\mathcal{S}$, and for which mass production and heat conduction are negligible, have been derived in our paper [1]. Here we give only the pertinent constitutive equations and field equations of the theory for purposes of quick reference, and we refer the reader for full details to reference [1]. The stress constitutive equation obtained in [1] is

$$t_{kl} = (-\pi+\lambda_v d_{rr})\delta_{kl}+2\mu_v d_{kl}+\int_v [\sigma'+\lambda'_v d'_{rr})\delta_{kl}+2\mu'_v d'_{kl}]dv', \tag{3.1}$$

where

$$\begin{aligned} &\pi = -\partial\psi/\partial\varrho^{-1}, \quad \sigma' = \delta\psi/\delta r'^{-1}, \quad d_{kl} = \frac{1}{2}(v_{lk}+v_{lk}),\\ &r'^{-1} = \varrho'^{-1}-\varrho^{-1}, \quad dv' = dv(\boldsymbol{\lambda}), \qquad d'_{kl} = \frac{1}{2}(v'_{k,l}+v'_{l,k}), \end{aligned} \tag{3.2}$$

δ_{kl} — Kronecker delta, ϱ — mass density at a *local point* $\mathbf{x}$ (Eulerian frame) in the body with material volume v; ϱ' — mass density at a *nonlocal point* $\mathbf{x}'$ other than $\mathbf{x}$ in the same body; δ/δ represents Fréchet differentiation and denotes functional gradients. $\boldsymbol{\lambda}$ is any arbitrary vector in the function (Hilbert) space chosen for purposes of defining the Fréchet derivative and the functional gradients [1]; λ_v and μ_v are, respectively, the classical dilatational and shear viscosity coefficients and λ'_v and μ'_v which depend on $\|\mathbf{x}-\mathbf{x}'\|$ for homogeneous fluids are, respectively, the nonlocal dilatational and shear viscosity coefficients. Note that in the linear constitutive theory the stress constitutive equation becomes uncoupled from the electromagnetic constitutive equations [1]. The constitutive equations for the electromagnetic response functions $\underline{\mathcal{E}}$ and $\underline{\mathcal{H}}$ are given by [1]

$$\underline{\mathcal{E}} = \varrho[s_3\underline{\mathcal{D}}+s_4\dot{\underline{\mathcal{D}}}+\int_v \underline{\mathcal{A}}'dv'], \tag{3.3}$$

$$\underline{\mathcal{H}} = \varrho[b_1\underline{\mathcal{B}}+\underline{\mathcal{A}}+\int_v \mathbf{r}'dv'], \tag{3.4}$$

where

$$\begin{aligned} &\underline{\mathcal{A}} = b_2\dot{\underline{\mathcal{B}}}+b_3\underline{\mathcal{D}}+b_4\dot{\underline{\mathcal{D}}}, \quad \underline{\mathcal{A}}' = s'_3\underline{\mathcal{D}}'+s_4\dot{\underline{\mathcal{D}}}',\\ &\mathbf{r}' = b'_1\underline{\mathcal{B}}'+b'_2\dot{\underline{\mathcal{B}}}'+b'_3\underline{\mathcal{D}}'+b'_4\dot{\underline{\mathcal{D}}}'+b'_5\boldsymbol{\beta}', \end{aligned} \tag{3.5}$$

and s_3, s_4, b_i are the *E—M* material coefficients, while s'_3, s'_4 and b'_i are the nonlocal *E—M* material coefficients and are functions of $\|\mathbf{x}'-\mathbf{x}\|$ for homogeneous materials, and the nonlocal mechanical measure $\boldsymbol{\beta}'$ is given by [1]:

$$\boldsymbol{\beta}' = \nabla'\mathbf{v}'\cdot(\mathbf{x}'-\mathbf{x})+\mathbf{v}'-\mathbf{v}. \tag{3.6}$$

The fields $\underline{\mathcal{B}}$ and $\underline{\mathcal{D}}$ are as defined in Eq. (2.1). ∇' denotes the spatial gradient with respect to the nonlocal variable $\mathbf{x}'$; $\mathbf{v}' = \mathbf{v}(\mathbf{x}', t)$. $\underline{\mathcal{B}}'$, $\underline{\mathcal{D}}'$, $\dot{\underline{\mathcal{B}}}'$ and $\dot{\underline{\mathcal{D}}}'$ are the *nonlocal* fields which are functions of $\mathbf{x}'$ and t and have corresponding physical interpretations as their unprimed counterparts. It is to be noted that $\mathbf{x}$ represents the local point of interest in the body (in the Eulerian frame), at which the influences of all other points $\mathbf{x}'$ of the body are sought in the nonlocal theory. Unless otherwise stated, the primes over the various physical quantities denote that they are functions of the *nonlocal* variables such as $\mathbf{x}'$ while the unprimed ones are functions of the *local* variables such as $\mathbf{x}$.

The full set of field equations governing the flow of nonlocal E—M fluids is [1]:

$$\frac{\partial\varrho}{\partial t}+(\varrho v_k)_{,k}=0, \tag{3.7}$$

$$\begin{aligned}(-\pi+{}_Mh)_{,l}+(\lambda_v+\mu_v)v_{k,kl}+\mu_v v_{l,kk}+\varrho[s_3(\mathscr{D}_k\mathscr{D}_l)_{,k}+s_4(\dot{\mathscr{D}}_k\mathscr{D}_l)_{,k}+(r_k\mathscr{B}_l)_{,k}+\tau_{kl,k}]\\ +\int_v\{(\lambda_v'+\mu_v')v_{k',k'l'}+\mu_v'v_{l'k'k'}+\varrho[\mathscr{D}_l(s_3'\mathscr{D}_{k,k'}'+s_4'\dot{\mathscr{D}}_{k,k'}')+\mathscr{D}_{l,k}(s_3'\mathscr{D}_k'+s_4'\dot{\mathscr{D}}_k')\\ +\mathscr{B}_{l,k},r_k']\}dv'+\oint_{\mathscr{S}}s_{kl}'da_k'+\varrho(f_k-\dot{v}_k-\dot{g}_k)-\varrho\hat{f}_k=0,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\nabla^2\underline{\mathscr{D}}-\alpha^2\frac{\partial^2\underline{\mathscr{D}}}{\partial t^2}=\nabla(q+\hat{q})+(\dot{s}_4/s_3)\nabla\times\nabla\times\dot{\underline{\mathscr{D}}}+(1/\varrho cs_3)\nabla\times(\check{\underline{\mathscr{B}}}-\hat{\mathbf{b}})\\ +(1/\varrho s_3)\nabla\times\nabla\times\int_v\underline{\mathscr{A}}'dv'-c\alpha^2\frac{\partial}{\partial t}\left[\nabla\times\underline{\mathscr{A}}+\nabla\times\int_v\mathbf{r}'dv'-\frac{1}{c}(\check{\underline{\mathscr{D}}}+\underline{\mathscr{J}}+\hat{\underline{\mathscr{J}}})\right],\end{aligned} \tag{3.9}$$

$$\begin{aligned}\nabla^2\underline{\mathscr{B}}-\alpha^2\frac{\partial^2\underline{\mathscr{B}}}{\partial t^2}=\nabla\hat{m}+(1/\varrho b_1)\nabla\times\nabla\times\underline{\mathscr{A}}-(1/\varrho cb_1)\nabla\times(\check{\underline{\mathscr{D}}}+\check{\underline{\mathscr{J}}}+\hat{\underline{\mathscr{J}}})\\ -(1/\varrho b_1)\nabla\times\nabla\times\int_v\mathbf{r}'dv'+c\alpha^2\frac{\partial}{\partial t}\left[\varrho s_4\nabla\times\dot{\underline{\mathscr{D}}}+\nabla\times\int_v\underline{\mathscr{A}}'dv'+\frac{1}{c}(\check{\underline{\mathscr{B}}}-\hat{\mathbf{b}})\right],\end{aligned} \tag{3.10}$$

where da_k' is the nonlocal surface element, and

$$\begin{aligned}s_{kl}'&=\sigma'\delta_{kl}+\lambda_v'v_{r,r'}\delta_{kl}+\mu_v'(v_{k,l'}+v_{l,k'})+\varrho(\mathscr{D}_ls_3'\mathscr{D}_k'+\mathscr{D}_ls_4'\dot{\mathscr{D}}_k'+\mathscr{B}_lr_k'),\\ \alpha^2&=1/(\varrho c)^2b_1s_3,\qquad {}_Mh=\frac{1}{2}(\underline{\mathscr{E}}\cdot\mathbf{D}+\underline{\mathscr{H}}\cdot\mathbf{B}),\\ r_k&=b_1\mathscr{B}_k+b_2\dot{\mathscr{B}}_k+b_3\mathscr{D}_k+b_4\dot{\mathscr{D}}_k\\ \tau_{kl}&=\frac{1}{c}[\mathscr{H}_k(\mathbf{v}\times\mathbf{E})_l-\mathscr{E}_k(\mathbf{v}\times\mathbf{H})_l].\end{aligned} \tag{3.11}$$

Moreover, the superposed carats (^) over the various quantities in Eqs. (3.8)–(3.10) denote the following localization residuals at the point $\mathbf{x}$ (for example, $\hat{\mathbf{f}}$ denotes the nonlocal body force, say, due to chemical reactions) induced by all other points $\mathbf{x}'$ of the body:

$\hat{\mathbf{f}}$ — nonlocal body force, $\hat{q}$ — nonlocal charge, $\hat{\mathbf{b}}$ — nonlocal magnetic induction, $\hat{\underline{\mathscr{J}}}$ — nonlocal conduction current, $\hat{m}$ — nonlocal magnetic pole strength.

These localization residuals must satisfy the following relations [5]:

$$\begin{aligned}\int_v\varrho\hat{\mathbf{f}}dv=\mathbf{0},\qquad\int_v(\hat{\mathbf{b}},\hat{\underline{\mathscr{J}}})\cdot d\mathbf{a}=0,\\ \hat{\underline{\mathscr{J}}}=\hat{\mathbf{J}}^f-\hat{q}\mathbf{v},\qquad\frac{\partial\hat{m}}{\partial t}+\nabla\cdot(\hat{m}\mathbf{v})=\nabla\cdot\hat{\mathbf{b}},\end{aligned} \tag{3.12}$$

where v is as before the material volume, S is a surface within v, and $\mathbf{J}^f$ — nonlocal free current.

4. Basic assumptions of the problem

Before formulating the problem we specialize the field equations (3.7)–(3.10) by introducing the following assumptions. The fluid is assumed to be an incompressible, homogeneous, isotropic and dielectric material. We restrict our attention to the case of dielectric, conducting materials and to the nonrelativistic case where $v^2/c^2 \ll 1$, v being the local material speed and c is the speed of light in vacuum. Following GROT [7] we make the following assumptions:

$$\text{(4.1)}\qquad \begin{aligned} &\text{(i)} && \text{Nonconducting assumption:} && \underline{\mathscr{J}} = 0,\\ &\text{(ii)} && \text{Dielectric assumption:} && \mathbf{M} = \frac{1}{c}\mathbf{P}\times\mathbf{v},\\ &\text{(iii)} && \text{Nonrelativistic assumption:} && \mathbf{g} = \mathbf{0}. \end{aligned}$$

The nonconducting assumption states that there can be no current flow in a dielectric medium. For this to be true we should have no free charges; consequently, we take $q = 0$. For nonconduction at all parts of the body the localization residuals, $\hat{\underline{\mathscr{J}}} = \mathbf{0}$ and $\hat{q} = 0$.

The dielectric assumption states that the dielectric material has no magnetic moment when viewed in a frame moving with the material. Thus the magnetic pole strength induced on a material point by all of the other material points in the body is zero. Hence we take $\hat{m} = 0$, which through Eqs. $(3.12)_{2,4}$ implies that $\hat{\mathbf{b}} = \mathbf{0}$.

The assumption that $\mathbf{g} = \mathbf{0}$ requires a careful understanding. GROT and ERINGEN [8] have shown that, basing on the relativistic principle, the total energy-momentum tensor is symmetric. This is equivalent to assuming that the balance law of moment of momentum is valid in every inertial frame. From this principle one can deduce the equivalence of momentum flow and energy flux following TRUESDELL and TOUPIN [9]:

$$\text{(4.2)}\qquad \mathbf{g} = (\mathbf{q}/c^2)-(\mathbf{t}\cdot\mathbf{v}/c^2)+[\varrho(e-c^2)\mathbf{v}/c^2],$$

where ϱe — total energy density, $\mathbf{q}$ — total energy flux vector, $\mathbf{t}$ — stress tensor, $\mathbf{g}$ — electromagnetic momentum density, $\mathbf{v}$ — velocity vector, and c — speed of light in vacuum. Since we have already assumed the heat conduction $\mathbf{Q}-\mathbf{q}-\underline{\mathscr{E}}\times\underline{\mathscr{H}} = \mathbf{0}$, see [1] and in view of the Lorentz approximation $v^2/c^2 \ll 1$, the assumption $\mathbf{g} = \mathbf{0}$ is equivalent to assuming

$$\text{(4.3)}\qquad (\operatorname{tr}\mathbf{t}^2)^{1/2} \leqslant \varrho c^2, \quad |\varrho(e-c^2)| \leqslant \varrho c^2,$$

where the notation tr stands for the trace operator.

In the classical E—M theory, the constitutive equations for a linear, homogeneous isotropic medium are given by

$$\text{(4.4)}\qquad \mathbf{B} = \mu\mathbf{H}, \quad \mathbf{D} = \varepsilon\mathbf{E},$$

where μ and ε are, respectively, the magnetic permeability and electric permittivity of the medium. Following along these lines we are motivated to simplify our linear constitutive relations of the nonlocal theory for the fields $\underline{\mathscr{E}}$ and $\underline{\mathscr{H}}$ by restricting our attention to the class of dielectric fluids whose electromagnetic constitution permits Eqs. (3.3) and (3.4) to reduce to the form

$$\text{(4.5)}\qquad \underline{\mathscr{E}} = \varepsilon^{-1}\underline{\mathscr{D}}+\int_v \varepsilon^*\underline{\mathscr{D}}'dv', \quad \underline{\mathscr{H}} = \mu^{-1}\underline{\mathscr{B}}+\int_v \mu^*\underline{\mathscr{B}}'dv',$$

with

$$\varepsilon^{-1} = \varrho s_3, \quad \mu^{-1} = \varrho b_1, \quad \varepsilon^* = \varrho s_3', \quad \mu^* = \varrho b_1', \tag{4.6}$$

so that the classical theory is included as a special case of our nonlocal theory.

Furthermore, since we are concerned with wave phenomena, we find it advantageous to express our basic equations in terms of the displacement vector $\mathbf{u}$ where $\dot{\mathbf{u}} = \mathbf{v}$, as opposed to the velocity vector. Since we are considering a completely linearized set of field equations, we note further that

$$\dot{\mathbf{u}} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}, \quad \dot{\mathbf{u}}' = \frac{\partial \mathbf{u}(\mathbf{x}', t)}{\partial t}. \tag{4.7}$$

Utilizing Eqs. (4.1), (4.5)–(4.7), the linearized field equations (3.7)–(3.10) governing the flow of nonlocal E—M fluids with the incompressibility condition ($\lambda_v = 0$, $\lambda_v' = 0$) for which the thermodynamic pressure π is replaced by an undetermined fluid pressure p (which can be determined by solving the field equations and boundary conditions of the problem), in the absence of all body forces take the form

$$\dot{u}_{k,k} = 0, \tag{4.8}$$

$$t_{kl,k} = \varrho \ddot{u}_l, \tag{4.9}$$

$$D_{k,ll} - \alpha^2 \ddot{D}_k = \int_v (\varepsilon'_{,lk} D'_l - \varepsilon'_{,ll} D'_k) dv' - c\alpha^2 e_{klm} \int_v \mu'_{,l} \ddot{B}'_m dv', \tag{4.10}$$

$$B_{k,ll} - \alpha^2 \ddot{B}_k = \int_v (\mu'_{,lk} B'_l - \mu'_{,ll} B'_k) dv' + c\alpha^2 e_{klm} \int_v \varepsilon'_{,l} \dot{D}'_m dv', \tag{4.11}$$

where e_{klm} is the permutation tensor and

$$\alpha^2 = \varepsilon\mu/c^2, \quad \varepsilon' = \varepsilon\varepsilon^*, \quad \mu' = \mu\mu^*, \tag{4.12}$$

and t_{kl} is given by the constitutive equation (3.1) which can be written in the form

$$t_{kl} = -p\delta_{kl} + \mu_v(\dot{u}_{k,l} + \dot{u}_{l,k}) + \int_v \{\sigma'\delta_{kl} + \mu_v'(\dot{u}'_{k,l} + \dot{u}_{l,k})\} dv', \tag{4.13}$$

with δ_{kl} as the Kronecker delta and p — total fluid pressure. Thus a decoupling of the electromechanical surface wave occurs, allowing the study of the mechanical surface wave (Rayleigh type) and the E—M surface wave (Zenneck type) to be undertaken separately.

5. Formulation of the problem

We consider an incompressible dielectric fluid in a half-space covered by a rectangular Cartesian coordinate system, $x_k, k = 1, 2, 3$. The half-space occupies the region $x_1 \in (-\infty, \infty)$, $x_2 \geqslant 0$ with $x_2 = 0$ as the free surface of the fluid. We consider plane surface disturbances propagating in the x_1 direction and the resulting surface waves are assumed to be confined to a very thin layer near $x_2 = 0$. As in the classical treatment of the Rayleigh surface wave in an elastic medium, the problem will be considered as a two-dimensional one in the domain $x_1 \in (-\infty, \infty)$, $x_2 \in (0, \infty)$, everything being uniform in the x_3 direction. For the geometry under consideration the Zenneck wave may be described by only two componets D_1 and D_2 of the field **D** and one component of B_3 of the field **B** (see reference [10]).

ERINGEN [4] has shown for the mechanical case that the expression for s'_{kl} in Eq. $(3.11)_1$ incorporates surface effects such as surface tension, surface stresses, surface viscosities, etc., at the material surface. He has also demonstrated that an analysis of σ' which appears in Eq. $(3.11)_1$ leads to the fact that it represents the surface tension density. Utilizing ERINGEN's results [4] and, in particular, assuming no continuous surface effects (for example, physico-chemical), σ' may be expressed in a thin layer near $x_2 = 0$ in our problem as

$$\sigma' = \varrho^{-1}\tau'(u'_{2,22}+u'_{1,22}), \tag{5.1}$$

where $\mathbf{u}'$ is the displacement vector at the nonlocal point $\mathbf{x}'$ and τ' is given by

$$\tau' = \tau'(x_1-x'_1, x_2-x'_2, t) = \varrho'\partial F'_1/\partial\varrho'^{-1}, \tag{5.2}$$

in terms of the nonlocal function F'_1 representing the jump suffered by the free energy density across a surface of discontinuity, which when evaluated at $\mathbf{x}_2 = 0$ and substituted in Eq. (5.1) yields the value of σ' (representing the surface tension density) at the free surface $x_2 = 0$. ϱ and ϱ' are, respectively, the mass densities at $\mathbf{x}$ and $\mathbf{x}'$. The derivations of the expressions (5.1) and (5.2) are an immediate consequence of ERINGEN's Eqs. (13.4)–(13.8) derived in reference [4] and hence will not be repeated here. Since the problem becomes two-dimensional in the domains $x_1 \in (-\infty, \infty)$, $x_2 \in (0, \infty)$ and everything being uniform in the x_3 direction, the volume integrals in Eqs. (4.10), (4.11) and (4.13) reduce to surface integrals over x'_1 and x'_2 in their ranges.

In the absence of external body forces, the field equations governing the Rayleigh type surface waves thus take the form

$$\begin{aligned} \dot{u}_{1,1}+\dot{u}_{2,2} &= 0, \\ t_{11,1}+t_{21,1}-\varrho\ddot{u}_1 &= 0, \\ t_{12,1}+t_{22,2}-\varrho\ddot{u}_2 &= 0, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} t_{11} &= -p+2\mu_v\dot{u}_{1,1}+\int_0^{\infty}\int_{\infty}^{-\infty}[\varrho^{-1}\tau'(u'_{2,22}+u'_{1,12})+2\mu'_v\dot{u}'_{1,1}]dx'_1dx'_2, \\ t_{12} = t_{21} &= \mu_v(\dot{u}_{1,2}+\dot{u}_{2,1})+\int_0^{\infty}\int_{\infty}^{-\infty}[\mu'_v(\dot{u}'_{1,2}+\dot{u}'_{2,1})dx'_1dx'_2, \\ t_{22} &= -p+2\mu_v\dot{u}_{2,2}+\int_0^{\infty}\int_{\infty}^{-\infty}[\varrho^{-1}\tau'(u'_{2,22}+u'_{1,12})+2\mu'_v\dot{u}'_{2,2}]dx'_1dx'_2. \end{aligned} \tag{5.4}$$

The boundary conditions to be satisfied are: that on the surface, $x_2 = 0$, the surface traction must vanish and as $x_2 \to \infty$, the fields must vanish, that is,

$$\begin{aligned} t_{22} = 0 = t_{21}, &\quad \text{at} \quad x_2 = 0, \\ u_k \to 0 &\quad \text{as} \quad x_2 \to \infty. \end{aligned} \tag{5.5}$$

The field equations for the Zenneck type waves may be written as

(5.6)
$$D_{1,11}+D_{1,22}-a^2\ddot{D}_1 = \int_0^\infty \int_{-\infty}^\infty (\varepsilon'_{,12}D'_2-\varepsilon'_{,22}D'_1)dx'_1dx'_2-c\alpha^2\int_0^\infty\int_{-\infty}^\infty \mu'_{,2}\dot{B}'_3dx'_1dx'_2,$$
$$D_{2,11}+D_{2,22}-a^2\ddot{D}_2 = \int_0^\infty \int_{-\infty}^\infty (\varepsilon'_{,12}D'_1-\varepsilon'_{,11}D'_2)dx'_1dx'_2+c\alpha^2\int_0^\infty\int_{-\infty}^\infty \mu'_{,1}\dot{B}'_3dx'_1dx'_2,$$
$$B_{3,11}+B_{3,22}-\alpha^2\ddot{B}_{23} = -\int_0^\infty \int_{-\infty}^\infty (\mu'_{,11}+\mu'_{,22})B'_3dx'_1dx'_2+c\alpha^2\int_0^\infty\int_{-\infty}^\infty (\varepsilon'_{,1}\dot{D}'_2-\varepsilon'_{,2}\dot{D}'_1)dx'_1dx'_2.$$

Since all surface tractions must vanish at the free surface and since the dielectric medium is nonconducting, the boundary conditions for the E—M waves are

(5.7)
$$D_1 = 0 = D_2, \quad \partial B_3/\partial x_2 = 0 = \partial B_3/\partial x_1 \quad \text{at} \quad x_2 = 0,$$
$$D_1 = 0 = D_2, \quad B_3 = 0 \quad \text{as} \quad x_2 \to \infty.$$

6. Solution for the Rayleigh type waves

We consider a solution field in the form of the Fourier integrals:

(6.1)
$$u_r(x_1, x_2, t) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty \bar{u}_r(\xi, x_2, \omega)e^{-i(\xi x_1+\omega t)}d\xi d\omega, \quad r = 1, 2;$$
$$p(x_1, x_2, t) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty \bar{p}(\xi, x_2, \omega)e^{-i(\xi x_1+\omega t)}d\xi d\omega.$$

In Eq. (6.1) ξ and ω, representing the wave number and the frequency, respectively, of the surface waves, can in general be taken as complex numbers (see ERINGEN [11]). In that case if these quantities were to have any physical identification at all, then only their real parts, denoted by $\mathrm{Re}(\xi)$ and $\mathrm{Re}(\omega)$ become relevant. Substituting Eq. (6.1) into Eq. (5.3) yields

$$-i\xi\bar{u}_1+\bar{u}_{2,2} = 0,$$
$$-i\xi\bar{t}_{11}+\bar{t}_{21,2}+\varrho\omega^2\bar{u}_1 = 0,$$
$$-i\xi\bar{t}_{12}+\bar{t}_{22,2}+\varrho\omega^2\bar{u}_2 = 0,$$

where $\bar{t}_{11}$, $\bar{t}_{21}$ and t_{22} are obtained from Eqs. (5.4) and (6.1), and are given by

(6.3)
$$\bar{t}_{11} = -\bar{p}-2\omega\xi\bar{u}_1+\int_0^\infty [\varrho^{-1}\bar{\tau}'(\bar{u}'_{2,22}-i\xi\bar{u}'_{1,2})-2\omega\xi\bar{\mu}'_v\bar{u}'_1]dx'_2,$$
$$\bar{t}_{12} = \bar{t}_{21} = -\omega\mu_v(i\bar{u}_{1,2}+\xi\bar{u}_2)+\int_0^\infty (-\omega\bar{\mu}'_v)(i\bar{u}'_{1,2}+\xi\bar{u}'_2)dx'_2,$$
$$\bar{t}_{22} = -\bar{p}-2i\omega\mu_v\bar{u}_{2,2}+\int_0^\infty [\varrho^{-1}\bar{\tau}'(\bar{u}'_{2,22}-i\xi\bar{u}'_{1,2})-2i\omega\bar{\mu}'_v\bar{u}'_{2,2}]dx'_2,$$

where a superposed bar on letters indicates the Fourier transformation. The nonlocal material coefficients μ_v' and τ' are expected to change very sharply as we move from the surface, $x_2 = 0$, to within the medium. Since they obey the axiom of attenuating neighborhoods [12, 9], they must die out rapidly as $\|x'-x\| \to \infty$. We idealize this situation mathematically by considering the behavior of these nonlocal coefficients to be δ-functions in the x_2-variable, that is

$$\bar{\tau}' = \bar{\tau}(\xi)\delta(|x_2-x_2'|), \quad \bar{\mu}_v' = \bar{\mu}_v(\xi)\delta(|x_2-x_2'|). \tag{6.4}$$

Using Eq. (6.4) in Eq. (6.3) and substituting the result into Eq. (6.2), the field equations take the following form:

$$\begin{aligned} &-i\xi\bar{u}_1+\bar{u}_{2,2} = 0,\\ &i\xi\bar{p}-i\xi\varrho^{-1}\bar{\tau}(\bar{u}_{2,22}-i\xi\bar{u}_{1,2})+2i\omega\xi^2(\mu_v+\bar{\mu}_v)\bar{u}_1-i\omega\bar{u}_{1,22}(\mu_v+\bar{\mu}_v)\\ &\qquad -\omega\xi\bar{u}_{2,2}(\mu_v+\bar{\mu}_v)+\varrho\omega^2\bar{u}_1 = 0,\\ &-\bar{p}_{,2}+\varrho^{-1}\bar{\tau}(\bar{u}_{2,222}-i\xi\bar{u}_{1,22})+i\omega\xi^2(\mu_v+\bar{\mu}_v)\bar{u}_2-2i\omega(\mu_v+\bar{\mu}_v)\bar{u}_{2,22}\\ &\qquad -\omega\xi\bar{u}_{1,2}(\mu_v+\mu_v)+\varrho\omega^2\bar{u}_2 = 0. \end{aligned} \tag{6.5}$$

Now letting

$$k = 1/(\mu_v+\bar{\mu}_v), \tag{6.6}$$

we may write Eq. (6.5) as

$$\begin{aligned} &-i\xi\bar{u}_1+\bar{u}_{2,2} = 0,\\ &i\xi k\bar{p}-i\xi\varrho^{-1}\bar{\tau}k(\bar{u}_{2,22}-i\xi\bar{u}_{1,2})+2i\omega\xi^2\bar{u}_1-i\omega\bar{u}_{1,22}-\omega\xi\bar{u}_{2,2}+\varrho\omega^2k\bar{u}_1 = 0,\\ &-k\bar{p}_{,2}+\varrho^{-1}\bar{\tau}k(\bar{u}_{2,222}-i\xi\bar{u}_{1,22})+i\omega\xi^2\bar{u}_2-2i\omega\bar{u}_{2,22}-\omega\xi\bar{u}_{1,2}+\varrho\omega^2k\bar{u}_2 = 0. \end{aligned} \tag{6.7}$$

Since Rayleigh surface waves are assumed to decay exponentially as they penetrate the medium, we have

$$\begin{aligned} u_r(\xi, x_2, \omega) &= \bar{U}_r(\xi, \omega)e^{-ax_2}, \quad r = 1, 2;\\ \bar{p}(\xi, x_2, \omega) &= \bar{P}(\xi, \omega)e^{-ax_2}, \end{aligned} \tag{6.8}$$

where the real part of the rate of amplitude attenuation, namely, $\mathrm{Re}(a) > 0$. Substituting Eq. (6.8) into Eq. (6.7) yields

$$\begin{aligned} &i\xi\bar{U}_1+a\bar{U}_2 = 0,\\ &i\xi k\bar{P}-\left[i\omega(a^2-2\xi^2)-\varrho\omega^2k-\xi^2\frac{ka}{\varrho}\bar{\tau}\right]\bar{U}_1+\left[\omega\xi a-i\xi a^2\frac{k}{\varrho}\bar{\tau}\right]\bar{U}_2 = 0,\\ &ak\bar{P}+\left[\omega\xi a-i\xi a^2\frac{k}{\varrho}\bar{\tau}\right]U_1-\left[i\omega(2a^2-\xi^2)-\varrho\omega^2k+a^3\frac{k}{\varrho}\bar{\tau}\right]\bar{U}_2 = 0, \end{aligned} \tag{6.9}$$

which has nontrivial solutions for $\bar{U}_1$, $\bar{U}_2$ and $\bar{P}$ if and only if the roots for a^2 in the determinantal equation formed by the coefficients of $\bar{P}$, $\bar{U}_1$ and $\bar{U}_2$ in Eq. (6.9) are given by

$$a_1^2 = \xi^2, \quad a_2^2 = \xi^2-ik\varrho\omega. \tag{6.10}$$

Thus we may write the general solution of Eq. (6.9) in the form

$$\bar{u}_1 = e^{-a_1x_2}\bar{U}_{11}+e^{-a_2x_2}\bar{U}_{12},$$
$$(6.11)\qquad \bar{u}_2 = \gamma_{21}e^{-a_1x_2}\bar{U}_{11}+\gamma_{22}e^{-a_2x_2}\bar{U}_{12},$$
$$\bar{P} = \gamma_{31}e^{a_1x_2}\bar{U}_{11}+\gamma_{32}e^{a_2x_2}\bar{U}_{12},$$

where

$$(6.12)\qquad \gamma_{2j} = -i\xi/a_j, \quad j = 1, 2, \quad \gamma_{31} = i\varrho\omega^2/\xi, \quad \gamma_{32} = 0.$$

Using Eq. (6.11) the boundary conditions $t_{22} = 0 = t_{21}$ can be satisfied if and only if

$$(6.13)\qquad \begin{aligned}(2\xi^2/a_1)\bar{u}_{11}+[(2\xi^2-i\varrho\omega k)/a_2]\bar{U}_{12} &= 0,\\ (2\xi^2-i\varrho\omega k)\bar{U}_{11}+2\xi^2\bar{U}_{12} &= 0,\end{aligned}$$

which has nontrivial solutions if and only if the following frequency equation is satisfied:

$$(6.14)\qquad \gamma^3-\frac{3}{2}\gamma^2+\frac{1}{2}\gamma-\frac{1}{16} = 0,$$

where

$$(6.15)\qquad \gamma = \xi^2/i\varrho\omega k.$$

Noting that $k = 1/(\mu_v+\bar{\mu}_v)$ from Eq. (6.6), Eq. (6.15) can be expressed as

$$(6.16)\qquad \omega = (\xi^2/i\gamma)(\mu_v/\varrho)(1+\bar{\mu}_v(\xi)/\mu_v).$$

The cubic equation (6.14) in γ yields two roots that lead to the following dispersion relation:

$$(6.17)\qquad \omega = K(\mu_v/\varrho)|\xi|^2(1+\bar{\mu}_v(\xi)/\mu_v),$$

where

$$(6.18)\qquad K = 3.087.$$

Thus the above analysis shows that the surface waves in nonlocal viscous fluids are definitely *dispersive*.

The dispersion relation (6.17) can also be expressed in terms of the phase velocity c_p as

$$(6.19)\qquad c_p = \frac{\omega}{|\xi|} = K\nu|\xi|(1+\bar{\mu}_v(\xi)/\mu_v),$$

where $\nu = \mu_v/\varrho$ is the kinematic viscosity, and which gives the surface wave velocity for the nonlocal viscous fluid. Letting

$$(6.20)\qquad c_R = K\nu|\xi|,$$

where c_R is analogous to the classical Rayleigh type surface wave velocity for the viscous fluid, we may express the dispersion relation (6.17) in the form

$$(6.21)\qquad \frac{c_p}{c_R} = \frac{\omega}{K\nu|\xi|^2} = 1+(\bar{\mu}_v(\xi)/\mu_v).$$

It is to be noted that when the nonlocal material coefficient $\bar{\mu}_v(\xi)$ vanishes, c_p reduces to the classical value c_R. Thus nonlocality which enters the frequency relation clearly predicts that the surface waves in nonlocal viscous fluids are dispersive.

7. Solution for Zenneck type waves

We again assume a solution field in the form of Fourier integrals:

$$(7.1)\qquad \begin{aligned} D_r(x_1, x_2, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{D}_r(\xi, x_2, \omega)e^{-i(\xi x_1+\omega t)}d\xi d\omega; \quad r = 1, 2, \\ B_3(x_1, x_2, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{B}_3(\xi, x_2, \omega)e^{-i(\xi x_1+\omega t)}d\xi d\omega. \end{aligned}$$

Substituting Eq. (7.1) into the field equations (5.6) yields

$$(7.2)\qquad \begin{aligned} (\alpha^2\omega^2-\xi^2)\bar{D}_1+\bar{D}_{1,22} &= -\int_0^{\infty}(i\xi\bar{\varepsilon}'_{,2}\bar{D}'_2+\bar{\varepsilon}'_{,22}\bar{D}'_1)dx'_2+ic\omega\alpha^2\int_0^{\infty}\bar{\mu}'_{,2}\bar{B}'_3dx'_2, \\ (\alpha^2\omega^2-\xi^2)\bar{D}_2+\bar{D}_{2,22} &= -\int_0^{\infty}(i\xi\bar{\varepsilon}'_{,2}\bar{D}'_1-\xi^2\bar{\varepsilon}'\bar{D}'_2)dx'_2-c\omega\alpha^2\xi\int_0^{\infty}\bar{\mu}'\bar{B}'_3dx'_2, \\ (\alpha^2\omega^2-\xi^2)\bar{B}_3+\bar{B}_{3,22} &= \int_0^{\infty}(\xi^2\bar{\mu}'-\bar{\mu}'_{,22})\bar{B}'_3dx'_2+ic\omega\alpha^2\int_0^{\infty}(i\xi\bar{\varepsilon}'\bar{D}'_2+\bar{\varepsilon}'_{,2}\bar{D}'_1)dx'_2. \end{aligned}$$

Assuming that the nonlocal *E—M* material coefficients attenuate rapidly as we move into the medium $x_2 > 0$, in accordance with the axiom of attenuating neighborhoods [1] we idealize, as before, this situation mathematically by considering their behavior to be δ-functions in the x_2-variable. Thus we have

$$(7.3)\qquad \bar{\varepsilon}' = \bar{\varepsilon}(\xi)\delta(|x_2-x'_2|), \quad \bar{\mu} = \bar{\mu}(\xi)\delta(|x'_2-x|).$$

Since the delta function has compact support, using Eq. (7.3) in Eq. (7.2) yields

$$(7.4)\qquad \begin{aligned} (n+1)\bar{D}_{1,22}+m^2n\bar{D}_1-i\xi\bar{D}_{2,2}+ic\omega\alpha^2(n/l)B_{3,2} &= 0, \\ -i\xi\bar{D}_{1,2}+n\bar{D}_{2,22}+(m^2n-\xi^2)\bar{D}_2-c\omega\alpha^2\xi(n/l)\bar{B}_{3,2} &= 0, \\ ic\omega\alpha^2(l/n)\bar{D}_{1,2}+c\omega\xi\alpha^2(l/n)\bar{D}_2+(l+1)\bar{B}_{3,22}+(m^2l-\xi^2)\bar{B}_3 &= 0, \end{aligned}$$

where

$$(7.5)\qquad n = \bar{\varepsilon}^{-1}, \quad l = \bar{\mu}^{-1}, \quad m^2 = \alpha^2\omega^2-\xi^2.$$

Since Zenneck type waves die out exponentially as they penetrate the medium [1], we have

$$(7.6)\qquad \begin{aligned} \bar{D}_r(\xi, x_2\, \omega) &= \bar{d}_r(\xi, \omega)e^{-bx_2}, \quad r = 1, 2; \\ \bar{B}_3(\xi, x_2, \omega) &= \bar{h}_3(\xi, \omega)e^{-bx_2}, \end{aligned}$$

where the real part of the rate of amplitude attenuation, $\text{Re}(b) > 0$. Using Eq. (7.6) in Eq. (7.4) gives

$$\begin{aligned}
&[b^2(n+1)+m^2n]\bar{d}_1 + i\xi b\bar{d}_2 - ic\omega\alpha^2 b(n/l)\bar{h}_3 = 0,\\
(7.7)\qquad &i\xi b\bar{d}_1 + [n(b^2+m^2)-\xi_2]\bar{d}_2 + c\omega\alpha_2\xi(n/l)\bar{h}_3 = 0,\\
&-ic\omega a^2 b\bar{d}_1 + c\omega\alpha^2\xi\bar{d}_2 + (n/l)[l(b^2+m^2)+b^2-\xi^2]\bar{h}_3 = 0.
\end{aligned}$$

A nontrivial solution set for $\bar{d}_1$, $\bar{d}_2$ and $\bar{h}_3$ for the linear algebraic system (7.7) exists for values of b satisfying the following determinantal equation formed by the coefficients of $\bar{d}_1$, $\bar{d}_2$ and $\bar{h}_3$ in Eq. (7.7):

$$(7.8)\qquad \begin{vmatrix} [b^2(n+1)+m^2n] & i\xi b & -ic\omega\alpha^2 b \\ i\xi b & [n(b^2+m^2)-\xi^2] & c\omega\alpha^2\xi \\ -ic\omega\alpha^2 b & c\omega\alpha^2\xi & [l(b^2+m^2)+b^2-\xi^2] \end{vmatrix} = 0.$$

Equation (7.8) results in a sixth degree algebraic equation in b with nonvanishing coefficients which are functions of the E—M material parameters n, l and m. Of these three parameters, only m is a known function of ξ given by Eq. $(7.5)_3$. But the determination of the nature of the dependence of n and l on ξ requires experimental data to be fitted into our theoretical formulation. The nature of determination of these material parameters n and l is, *in principle*, no way different from that of the determination, for instance, of the classical shear viscosity coefficient by using a Couette rotational viscometer and fitting the experimental data with the expression for the torques on the cylinders of the viscometer. For a complete and objective determination of the roots for b from the sixth degree algebraic equation in b given by Eq. (7.8) one must resort to numerical means. However, to follow such a procedure the dependence of n and l on ξ must be known. At present, due to the lack of the experimental data this dependence cannot be determined for the given material, namely, lubricating oil. Once the values for b are known, one can determine the dispersion relation for the Zenneck type wave by following an identical procedure as given in Sect. 6.

In order to reveal the dispersive character introduced through the nonlocal considerations, we will examine the situation where the nonlocal magnetic effects in the x_1-direction are small in comparison with the nonlocal electric effects. Thus we shall set $\bar{\mu} \equiv 0$, which reduces Eq. (7.8) to

$$(7.9)\qquad \begin{vmatrix} b^2+n(m^2+b^2) & i\xi b & 0 \\ i\xi b & n(b^2+m^2)-\xi^2 & 0 \\ -ic\omega\alpha^2 b & c\omega\alpha^2\xi & n(m^2+b^2) \end{vmatrix} = 0.$$

Solving Eq. (7.9) yields two possible values for b^2:

$$(7.10)\qquad b_1^2 = -m^2, \quad b_2^2 = (\xi^2-nm^2)(n+1).$$

Thus the general solution to Eq. (7.7) may be written in the form

$$\begin{aligned}
&\bar{d}_1 = e^{-b_1x_2}\bar{F}_{11} + e^{-b_2x_2}\bar{F}_{12},\\
(7.11)\qquad &\bar{d}_2 = \alpha_{21}e^{-b_1x_2}\bar{F}_{11} + \alpha_{22}e^{-b_2x_2}\bar{F}_{12},\\
&\bar{h}_3 = \alpha_{31}e^{-b_1x_2}\bar{F}_{11} + \alpha_{32}e^{-b_2x_2}\bar{F}_{12},
\end{aligned}$$

where

$$\alpha_{21} = ib_1/\xi, \quad \alpha_{22} = i\xi/b_2, \tag{7.12}$$
$$\alpha_{31} \text{ arbitrary}, \quad \alpha_{32} = -ic\omega\alpha^2/b_2.$$

Under Eq. (6.1) the boundary conditions at $x_2 = 0$, given by Eq. (5.7), become

$$\bar{d}_1 = \bar{d}_2 = 0 = \bar{h}_3. \tag{7.13}$$

Using Eq. (7.11), Eq. (7.13) can be satisfied if and only if

$$\begin{aligned} \bar{F}_{11}+\bar{F}_{12} &= 0,\\ \alpha_{21}\bar{F}_{11}+\alpha_{22}\bar{F}_{12} &= 0,\\ \alpha_{31}\bar{F}_{11}+\alpha_{32}\bar{F}_{12} &= 0, \end{aligned} \tag{7.14}$$

which has nontrivial solutions if

$$\alpha_{22} = \alpha_{21}, \quad \alpha_{31} = -\alpha_{32}\bar{F}_{12}/\bar{F}_{11}. \tag{7.15}$$

Using Eqs. (7.12) and (7.10), Eq. $(7.15)_1$ yields the following frequency equation which must be satisfied:

$$nm^4-\xi m^2-(n+1)\xi^4 = 0, \tag{7.16}$$

where m^2 is given by Eq. (7.5). Recalling that $n = \overline{\varepsilon^{-1}}(\xi)$ and $\alpha^2 = \varepsilon\mu/c^2$, Eq. (7.16) yields the following dispersion relation:

$$\omega(\xi) = c\xi[(2+\bar{\varepsilon}(\xi))/\varepsilon\mu]^{1/2}. \tag{7.17}$$

The index of refraction, k_{ne}, for nonlocal electromagnetic waves may be defined by

$$k_{ne} = \frac{c\xi}{\omega} = k\big(2+\bar{\varepsilon}(\xi)\big)^{-1/2}, \tag{7.18}$$

where $k = (\varepsilon\mu)^{1/2}$ is the classical expression for the index of refraction. In terms of the phase velocity for nonlocal electromagnetic waves, c_{ne}, and the classical expression for the phase velocity, $c_c = ck^{-1}$, we have

$$\frac{c_{ne}}{c_c} = \big(2+\bar{\varepsilon}(\xi)\big)^{1/2}. \tag{7.19}$$

Hence it is clear that nonlocal electric interactions do affect the phase velocity of the electromagnetic waves.

In classical electromagnetic theory the dispersive character of the propagating waves is brought out by relating the dielectric constant to the frequency, i.e. $\varepsilon = \varepsilon(\omega)$, through microscopic considerations. By ignoring the magnetic force effects and assuming the bound charges e to be harmonically bound, a simple model relating ω to ε is given by (see JACKSON [12], p. 285)

$$\varepsilon(\omega) = 1+F, \tag{7.20}$$

where

$$F = \frac{Ne^2}{m}\sum_j f_j(\omega_j^2-\omega^2-i\omega\gamma_j)^{-1} \tag{7.21}$$

In Eq. (7.21) N is the number of molecules per unit volume, m is the mass of each charge e, f_j are the numbers of electrons per molecule with binding frequencies ω_j and damping constants γ_j, and the "oscillator strengths f_j" satisfy the sum rule

$$\sum_j f_j = Z, \tag{7.22}$$

Z being the number of electrons per molecule. "With suitable quantum-mechanical definitions of f_j, γ_j, and ω_j Eq. (7.20) is an accurate description of the atomic contribution to the dielectric constant" ([12], p. 285). Using Eq. (7.17) and $k = (\varepsilon\mu)^{1/2}$, we find that

$$\varepsilon(\omega) = 2c^2\xi^2/\big(\mu\omega^2 - c^2\xi^2\bar{\varepsilon}^*(\xi)\big), \tag{7.23}$$

where $\bar{\varepsilon}^*(\xi)$ is the Fourier transform of ε^* under Eq. (6.1).

Requiring Eq. (7.23) to coincide with the dispersive relation from the microscopic considerations (7.20) yields

$$\bar{\varepsilon}^* = (\mu/2)(k/k_{ne})^2 - 2/(1+F). \tag{7.24}$$

Thus, once f_j, ω_j, and γ_j are determined from quantum mechanical considerations, one can, at least in principle, determine the nonlocal electric material coefficient.

In the high-frequency range, where ω is far above the resonant (binding) frequencies ω_j, the dielectric constant (7.20) takes on the simple form

$$\varepsilon(\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2}, \tag{7.25}$$

where

$$\omega_p = NZe^2/m \tag{7.26}$$

is the plasma frequency of the medium. Furthermore, the wave number varies with frequency as for a mode in a wave guide with cut-off frequency ω_p. In such a situation the following dispersion relation is obtained ([12], p. 344):

$$\omega = c\xi \Big/ \left[\varepsilon\mu\left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)\right]^{1/2}. \tag{7.27}$$

Comparing Eq. (7.17) with Eq. (7.27) yields the following expression for $\bar{\varepsilon}$:

$$\bar{\varepsilon} = \left(2\left(\frac{\omega_p}{\omega}\right)^2 - 1\right) \Big/ \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right). \tag{7.28}$$

For dielectric media Eq. (7.25) is valid only when $\omega \gg \omega_p$, so that ε is then close to unity although slightly less. Thus, in the limit $\omega_p/\omega \to 0$ we have, from Eq. (7.28),

$$\bar{\varepsilon} \simeq -1. \tag{7.29}$$

Substituting Eq. (7.29) into Eq. (7.17) we recover the classical frequency-wave number relationship

$$\omega = c\xi/(\varepsilon\mu)^{1/2} = ck^{-1}\xi. \tag{7.30}$$

Thus, in the high-frequency limit the nonlocal interactions reduce to the classical results.

It is important to note that if we ignore all the nonlocal effects, that is, examine the problem in the classical setting, then no dispersion relation can be obtained. In other words, excluding nonlocal effects reverts the E—M field equations back to the classical hyperbolic system. Thus the dispersive character of the Zenneck type waves is brought out only by the inclusion of the nonlocal effects.

References

1. B. M. MCCAY, M. N. L. NARASIMHAN, *Theory of nonlocal electromagnetic fluids*, Arch. Mech., **33**, 3, 1981.
2. J. L. YARNELL, J. L. WARREN and S. H. KOENIG, *Lattice dynamics* (Edited by R. F. WALLIS), Pergamon Press, New York 1965.
3. L. BRILLOUIN, *Wave propagation in periodic structures*, Dover 1953.
4. A. C. ERINGEN, *Continuum physics*, 4, (Edited by A. C. ERINGEN) Academic Press, New York 1976.
5. A. C. ERINGEN. J. Math. Physics, **14**, 733, 1973.
6. A. C. ERINGEN, Lett. Applied Engng. Sci., **1**, 129, 1973.
7. R. A. GROT, *Continuum physics*, 3 (Edited by A. C. Eringen), Academic Press, New York 1976.
8. R. A. GROT and A. C. ERINGEN, Int. J. Engng. Sci., **4**, 611, 1966; Int. J. Engng., Sci., 4, 638, 1966.
9. C. A. TRUESDELL and R. A. TOUPIN, *The classical field theories*, in: Handbuch Der Physik, (Edited by S. FLÜGGE), 3, Part I, Springer-Verlag, New York 1960.
10. H. M. BARLOW and J. BROWN, *Radio surface waves*, Clarendon Press, Oxford 1962.
11. A. C. ERINGEN, *Mechanics of continua*, Wiley, New York 1967.
12. A. C. ERINGEN, Lett. in Applied Engng. Sci., **1**, 11, 1973.
13. J. D. JACKSON, *Classical electrodynamics*, 2nd ed., John Wiley and Sons, New York 1975.

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