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# Existence and uniqueness of solutions of a two-dimensional BGK equation

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THE THEOREM of existence and uniqueness of solutions of a linear two-dimesional BGK equation written in integral form is proved; linearization is performed in the neighbourhood of the solution of a nonlinear Couette flow. The proof is based on the theorem of contraction mappings in a certain Hibert space. Boundary conditions are prescribed along the lines $y = 0$ and $y = y_0$, and the nonlinear Coutte flow is assumed to have a solution.

W pracy udowodniono twierdzenie o istnieniu i jednoznaczności dla całkowej postaci liniowego dwuwymiarowego równania BGK, przy czym linearyzację przeprowadzono wokół rozwiązania nieliniowego przepływu Couette'a. Dla dowodu stosowano zasadę odwzorowań zwężających w pewnej przestrzeni Hilberta. Przyjęto, że warunki brzegowe są dane na liniach $y = 0$ i $y = y_0$ oraz że dla nieliniowego przepływu Couete'a rozwiązanie istnieje.

В работе доказана теорема существования и единственности решения для интегрального вида линейного двумерного управлнения БГК, причем линеаризация проведена вокруг решения нелинейного течения Куэтта. Для доказательства применен принцип сживающих отображений в некотором пространстве Гильберта. Принято, что граничные условия заданы на линиях $y = 0$ и $y = y_0$, а также, что для нелинейного течения Куэтта существует решение.

## Introduction

THE PRESENT state of development of the theory of existence and uniqueness of solutions of kinetic equations is not satisfactory. Research in this field deals mainly with the linear Boltzmann equation and the linear BGK equation. In this paper the problem of existence and uniqueness is discussed in the case of a linearized, two-dimensional and stationary BGK equation under simple geometric conditions. Linearization is performed not in the neighbourhood of an absolutely Maxwellian function but at a Couette flow between parallel planes. Boundary conditions for the Couette flow are disturbed at one of the walls. Such problems arise, for example, in the case of flow past a wavy wall.

## 1. Linearization of the BGK equation and the integral form

Let us consider the two dimensional BGK equation written in dimensionless variables:

$$c_1 \frac{\partial f}{\partial x} + c_2 \frac{\partial f}{\partial y} = -An(f - f_0) \tag{1.1}$$

with the boundary conditions prescribed along the lines $y = 0$ and $y = y_0$. Let

$$(1.2) \qquad f_0 = n\pi^{-\frac{3}{2}}\left(\frac{T_0}{T_1}\right)^{\frac{3}{2}} e^{-\frac{T_0}{T}(\mathbf{c}-\mathbf{u})^2}$$

be a locally Maxwellian function with parameters $n$, $\mathbf{u}$, $T$ corresponding to the distribution function $f(\mathbf{c}, x, y)$. Let us write $f$ in the form

$$(1.3) \qquad f = F+\varphi.$$

Here $F$ satisfies the one-dimensional BGK equation

$$(1.4) \qquad c_2\frac{\partial F}{\partial y} = -AN(F-F_0)$$

with diffusion conditions at the walls which describe the Couette flow between parallel walls; $F(\mathbf{c}, y)$ is the distribution function corresponding to that flow.

The function

$$(1.5) \qquad F_0 = N\pi^{-\frac{3}{2}}\left(\frac{T_0}{\mathcal{T}}\right)^{\frac{3}{2}} e^{-\frac{T_0}{\mathcal{T}}(\mathbf{c}-\mathbf{U})^2}$$

is a locally Maxwellian function with parameters $N$, $\mathbf{U}$, $\mathcal{T}$ corresponding to $F$,

$$(1.6) \qquad \begin{aligned} n(x, y) &= N(1+\nu), \\ \mathbf{u}(x, y) &= \mathbf{U}+\mathbf{v}, \\ T(x, y) &= \mathcal{T}(1+\tau), \end{aligned}$$

$\varphi(\mathbf{c}, x, y)$ is the perturbation function.

Let us assume that $|\varphi| \ll 1$. Then Eq. (1.1) may be linearized in the neighbourhood of the solution $F$, and the following equation is obtained for the function $\varphi$:

$$(1.7) \qquad c_1\frac{\partial\varphi}{\partial x}+c_2\frac{\partial\varphi}{\partial y} = L\varphi \equiv -AN\varphi+H\varphi,$$

where

$$(1.8) \qquad L\varphi = -AN\left\{\varphi-F_0\left[\nu+2\frac{T_0}{\mathcal{T}}\mathbf{v}(\mathbf{c}-\mathbf{U})+\left(\frac{T_0}{\mathcal{T}}(\mathbf{c}-\mathbf{U})^2-\frac{3}{2}\right)\tau\right]+(F-F_0)\nu\right\}.$$

The quantities $\nu$, $\mathbf{v}$, $\tau$ are expressed in terms of $\varphi$ in the following manner:

$$(1.9) \qquad \begin{aligned} \nu &= \frac{1}{N}\int\varphi d\mathbf{c}, \\ \mathbf{v} &= \frac{1}{N}\int(\mathbf{c}-\mathbf{U})\varphi d\mathbf{c}, \\ \tau &= \frac{2}{3N}\int\left[\frac{T_0}{\mathcal{T}}(\mathbf{c}-\mathbf{U})^2-\frac{3}{2}\right]\varphi d\mathbf{c}. \end{aligned}$$

The boundary conditions for Eq. (1.7) have the form

$$(1.10) \qquad \begin{aligned} \varphi(\mathbf{c}, x, y_0) &= 0, \quad \text{for} \quad c_2<0, \\ \varphi(\mathbf{c}, x, 0) &= Y(\mathbf{c}, x)\pi^{-\frac{3}{2}}e^{-c^2}, \quad \text{for} \quad c_2>0, \end{aligned}$$

$Y(\mathbf{c}, x)$ being the function known from the solution of Eq. (1.4) and the boundary conditions for Eq. (1.1).

Equation (1.7) with the conditions (1.10) may be reduced to the integral form

$$\varphi = \varphi_0 + BH\varphi, \tag{1.11}$$

where $\varphi_0$ is the solution of the equation

$$c_1 \frac{\partial \varphi_0}{\partial x} + c_2 \frac{\partial \varphi_0}{\partial y} + AN\varphi_0 = 0 \tag{1.12}$$

under the boundary conditions (1.10), $B$ is an operator inverse to the operator $c_1 \partial/\partial x + c_2 \partial/\partial y + AN$ under homogeneous boundary conditions, i.e.

$$Bg = \begin{cases} \dfrac{1}{c_2} \displaystyle\int_0^y g\left(\mathbf{c}, x - \frac{c_1}{c_2}(y-z), z\right) e^{-\frac{A}{c_2}\int_z^y N(s)ds} dz & \text{for} \quad c_2 > 0, \\ \dfrac{1}{c_2} \displaystyle\int_{y_0}^y g\left(\mathbf{c}, x - \frac{c_1}{c_2}(y-z), z\right) e^{-\frac{A}{c_2}\int_z^y N(s)ds} dz & \text{for} \quad c_2 < 0. \end{cases} \tag{1.13}$$

## 2. Existence and uniqueness theorems

Let us assume Eq. (1.4) to have a unique solution $F(\mathbf{c}, y)$ positive and bounded together with the hydrodynamic moments $N(y)$, $\mathbf{U}(y)$, $\mathcal{T}(y)$. Let us now pass to the problem of existence and uniqueness of the solution of Eq. (1.11). Principal concepts of the proof are similar to those used in the proof of existence and uniqueness of the solution of the linearized Boltzmann equation [2].

LEMMA 1. Let $\varphi$ belong to the space of real variables, $\mathbf{c}, x, y$ be square-integrable over all variables, and assume zero values on the set $y = 0$, $y = y_0$; let it also be a periodic function. The integration with respect to $x$ is performed along the segment equal to the period, and with respect to $y$—in the interval $(0, y_0)$. Assume that all the integrals and derivatives written below exist. Then the inequality

$$\int \left(c_1 \frac{\partial \varphi}{\partial x} + c_2 \frac{\partial \varphi}{\partial y}\right)^2 \alpha(\mathbf{c}, y) d\mathbf{c}dxdy \geqslant \gamma y_0^{-2} \int c_2^2 \alpha(\mathbf{c}, y) d\mathbf{c}dxdy \tag{2.1}$$

holds true, $\alpha(\mathbf{c}, y)$ satisfying the condition

$$0 \leqslant \gamma\alpha(\mathbf{c}, y) \leqslant \tilde{\alpha}(\mathbf{c}) \leqslant \alpha(\mathbf{c}, y). \tag{2.2}$$

Here $\gamma > 0$ is a certain constant, and $\tilde{\alpha}(\mathbf{c})$ depends solely on the velocity $\mathbf{c}$.

Consider the equation

$$c_1 \frac{\partial \varphi}{\partial x} + c_2 \frac{\partial \varphi}{\partial y} = g(\mathbf{c}, x, y) \tag{2.3}$$

with the homogeneous boundary conditions. If a periodic solution exists, then $g$ must also be a periodic function with the same period. The solution may be represented by the formula

$$\varphi = \begin{cases} \int_0^y \frac{1}{c_2} g(\beta, z)dz & \text{for} \quad c_2 > 0, \\ -\int_y^{y_0} \frac{1}{c_2} g(\beta, z)dz & \text{for} \quad c_2 < 0, \end{cases} \tag{2.4}$$

where $\beta = x-(c_1/c_2)(y-z)$. Consequently, applying the Cauchy–Buniakovski inequality, we obtain

$$\int c_2^2 \tilde{\alpha} \varphi^2 d\mathbf{c} dx dy = \int_{c_2>0} c_2^2 \tilde{\alpha}(\mathbf{c}) \left( \int_0^y \frac{1}{c_2} g(\beta, z)dz \right)^2 d\mathbf{c} dx dy \tag{2.5}$$

$$+ \int_{c_2<0} \tilde{\alpha}(\mathbf{c}) c_2^2 \left( \int_y^{y_0} g(\beta, z)dz \right)^2 d\mathbf{c} dx dy \leqslant \int c_2^2 \alpha(\mathbf{c}) \left( \int_0^{y_0} \left| \frac{1}{c_2} g(\beta, z) \right| dz \right)^2 d\mathbf{c}\, dx dy$$

$$\leqslant \int c_2^2 \tilde{\alpha}(\mathbf{c}) y_0 \int_0^{y_0} \frac{1}{c_2^2} g^2(\beta, z) dz d\mathbf{c} dx dy.$$

From the periodicity of $g$ it follows that

$$\int g(\beta, z)dx = \int g(x, z)dx \tag{2.6}$$

for the integration interval equal to the period.

Changing the order of integration at the right-hand side of Eq. (2.5) and using Eq. (2.6), we obtain the inequality

$$\int c_2^2 \tilde{\alpha}(\mathbf{c}) \varphi^2 d\mathbf{c} dx dy \leqslant \int \tilde{\alpha}(\mathbf{c}) y_0 \int_0^{y_0} g^2(x, z) dx dz dy d\mathbf{c} = \int \tilde{\alpha}(\mathbf{c}) y_0^2 g^2(x, y) d\mathbf{c} dx dy \tag{2.7}$$

which, combined with Eq. (2.2), yields the inequality (2.1).

Let us introduce the space $\mathscr{H}_1$ of square-integrable functions with respect to $\mathbf{c}$, $x$, $y$, with the weight $F_0^{-1}$ and the scalar product

$$((g, h)) = \int F_0^{-1} g h d\mathbf{c} dx dy. \tag{2.8}$$

LEMMA 2. If the function $Bg$ satisfies the requirements of Lemma 1 and the functions used below belong to $\mathscr{H}_1$, then there exists a positive constant $\eta$ such that the equality

$$\varrho = F_0^{-1} \sqrt{(AN)^2 + \eta^2 c_2^2 y_0^{-2}} \tag{2.9}$$

implies

$$((\varrho F_0^2 Bg, Bg)) \leqslant ((\varrho^{-1} g, g)). \tag{2.10}$$

Proof. By assuming

$$Bg = \varphi \tag{2.11}$$

the inequality (2.10) may be written in the form

$$((\varrho F_0^2 \varphi, \varphi)) \leqslant \left( \left( \varrho^{-1} \left[ c_1 \frac{\partial \varphi}{\partial x} + c_2 \frac{\partial \varphi}{\partial y} + AN\varphi \right], \left[ c_1 \frac{\partial \varphi}{\partial x} + c_2 \frac{\partial \varphi}{\partial y} + AN\varphi \right] \right) \right) \tag{2.12}$$

where $\varphi$ satisfies homogeneous boundary conditions. The right-hand side of the inequality (2.12) may be rewritten as

$$(2.13)\quad \left(\left(\varrho^{-1}\left[\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}+AN\varphi\right],\left[\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}+AN\varphi\right]\right)\right)$$
$$= \int \varrho^{-1}F_0^{-1}\left[\left(\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\right)^2+2\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\varphi\cdot AN+(AN\varphi)^2\right]d\mathbf{c}d\mathbf{x}.$$

For the sake of brevity we denote $\mathbf{x} = (x, y)$

$$(2.14)\quad \int (\varrho F_0)^{-1}AN\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\cdot\varphi d\mathbf{c}d\mathbf{x} = \int \mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\varphi d\mathbf{c}d\mathbf{x}$$
$$+\int (AN-\varrho F_0)\varrho^{-1}F_0^{-1}\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\varphi d\mathbf{c}d\mathbf{x}.$$

Since $(AN-\varrho F_0)^2 < |(AN)^2-\varrho^2F_0^2| = \eta^2c_2^2y_0^{-2}$, the Cauchy-Buniakovski inequality may be used to verify that

$$(2.15)\quad \left|\int (AN-\varrho F_0)\varrho^{-1}F_0^{-1}\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\varphi d\mathbf{c}d\mathbf{x}\right|$$
$$< \frac{\eta}{y_0}\left(\int c_2^2\varrho^{-1}F_0^{-1}\varphi^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}\left(\int \varrho^{-1}F_0^{-1}\left(\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\right)^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}$$
$$= \frac{\eta}{y_0}\left((\varrho^{-1}c_2^2\varphi,\varphi)\right)^{\frac{1}{2}}\left(\left(\varrho^{-1}\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}},\ \mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\right)\right)^{\frac{1}{2}}.$$

Let us observe that

$$(2.16)\quad \int \mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\varphi d\mathbf{c}d\mathbf{x} = \int_0^{\frac{1}{2}}\int \mathbf{c}\frac{\partial\varphi^2}{\partial\mathbf{x}}d\mathbf{c}dxdydz.$$

Function $\varphi$ assumes zero values on the planes $y = 0$ and $y = y_0$. Since it is a periodic function in $x$ and does not depend on $z$, the Ostrogradski formula yields

$$(2.17)\quad \int \mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\varphi d\mathbf{c}d\mathbf{x} = 0.$$

The following estimate is now made in Eq. (2.13):

$$(2.18)\quad \left(\left(\varrho^{-1}\left[\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}+AN\varphi\right],\left[\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}+AN\varphi\right]\right)\right)$$
$$\geqslant \left[\left(\int \varrho^{-1}F_0^{-1}\left(\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\right)^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}-2\frac{\eta}{y_0}\left(\int c_2^2\varrho^{-1}F_0^{-1}\varphi^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}\right]$$
$$\cdot\left(\int \varrho^{-1}F_0^{-1}\left(\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}\right)^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}+\int \varrho^{-1}F_0^{-1}(AN)^2\varphi^2 d\mathbf{c}d\mathbf{x}.$$

Let us substitute $\alpha(\mathbf{c}, y) = \varrho^{-1}F_0^{-1} = [(AN)^2+\eta^2c_2^2y_0^{-2}]^{-1/2}$.

This function satisfies the inequality (2.2) provided $\tilde{\alpha}(\mathbf{c}) = [(AN_{\max})^2+\eta^2 c_2^2 y_0^{-2}]^{-1/2}$ and the constant $\gamma$ is chosen properly. Applying the inequality (2.1) to the right-hand side of (2.18), we obtain

$$(2.19)\quad \left(\left(\varrho^{-1}\left[\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}+AN\varphi\right],\left[\mathbf{c}\frac{\partial\varphi}{\partial\mathbf{x}}+AN\varphi\right]\right)\right)$$
$$\geqslant\left[\frac{\sqrt{\gamma}}{y_0}\left(\int c_2^2\varrho^{-1}F_0^{-1}\varphi^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}-2\frac{\eta}{y_0}\left(\int c_2^2\varrho^{-1}F_0^{-1}\varphi^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}\right]$$
$$\frac{\sqrt{\gamma}}{y_0}\left(\int c_2^2\varrho^{-1}F_0^{-1}\varphi^2 d\mathbf{c}d\mathbf{x}\right)^{\frac{1}{2}}+\int\varrho^{-1}F_0^{-1}(AN)^2\varphi^2 d\mathbf{c}d\mathbf{x}$$
$$=\int\left[(AN)^2+\frac{\gamma-2\eta\gamma^{\frac{1}{2}}}{y_0^2}c_2^2\right]\varrho^{-1}F_0^{-1}\varphi^2 d\mathbf{c}d\mathbf{x}\geqslant((\varrho F_0^2\varphi,\varphi)).$$

The last inequality in Eq. (2.19) holds true if $\eta > 0$ is selected so as to satisfy the condition $\gamma-2\eta\gamma^{1/2} > \eta^2$, what is always possible.

Let $\mathfrak{h}$ denote the Hilbert space of real functions of $\mathbf{c}$, square-integrable and having the scalar product

$$(2.20)\qquad (g,\varphi)=\int F_0^{-1}g\varphi d\mathbf{c}.$$

The functions

$$(2.21)\qquad \begin{aligned}\Psi_1 &= 2F_0-F,\\ \Psi_2 &= F_0(c_1-U_1),\\ \Psi_3 &= F_0(c_2-U_2),\\ \Psi_4 &= F_0\left[(\mathbf{c}-\mathbf{U})^2\frac{T_0}{\mathcal{T}}-\frac{3}{2}\right]\end{aligned}$$

are orthogonal in $\mathfrak{h}$. The following equalities hold true for the functions $\Psi_i$:

$$(2.22)\qquad L\Psi_i\equiv 0,\quad i=1,2,3,4,$$

$$(2.23)\qquad (\Psi_i, L\varphi)=0.$$

Operator $L$ may then be written in the form

$$(2.24)\qquad L\varphi=(-ANI+H)\varphi=-AN(I-P)\varphi,$$

$I$ being the identity operator, and $P$ — the projection operator onto the subspace spanned on the functions $\psi_i$. From the general properties of the pojection operators [3] it follows that $L$ and $H$ are self-adjoint operators in $\mathfrak{h}$, $L$ being a nonpositive and $H$ — a nonnegative operator.

LEMMA 3. If the functions $\varphi$ and $\varrho^{1/2}\varphi$ belong to $\mathfrak{h}$, then there exists such a positive constant $q<1$ that the equation

$$(2.25)\qquad (\varrho^{-1}H\varphi, H\varphi)\leqslant q^2(\varrho F_0^2\varphi,\varphi)$$

is satisfied.

Proof. Since $H$ is a self-adjoint operator in $\mathfrak{h}$, the operator $\check{H} = \varrho^{-1/2}H\varrho^{-1/2}$ is also a self-adjoint operator in $\mathfrak{h}$,

$$(g, Hh) = (Hg, h) \Rightarrow (g, \check{H}h) = (g, \varrho^{-\frac{1}{2}}H\varrho^{-\frac{1}{2}}h)$$
$$= (\varrho^{-\frac{1}{2}}g, H\varrho^{-\frac{1}{2}}h) = (H\varrho^{-\frac{1}{2}}g, \varrho^{-\frac{1}{2}}h) = (\varrho^{-\frac{1}{2}}H\varrho^{-\frac{1}{2}}g, h) = (\check{H}g, h).$$

Hence, from the general properties of the norm of self-adjoint operators in $\mathfrak{h}$ it follows [3]

$$(2.26)\qquad \frac{(\varrho^{-1}H\varphi, H\varphi)}{(\varrho F_0^2\varphi, \varphi)} = \frac{||\check{H}\cdot\varrho^{\frac{1}{2}}\varphi||^2}{||\varrho^{\frac{1}{2}}F_0\varphi||^2} \leqslant \sup_{\varrho^{\frac{1}{2}}\varphi\in\mathfrak{h}} \frac{|(\varrho^{\frac{1}{2}}\varphi, \check{H}\varrho^{\frac{1}{2}}\varphi)|^2}{||\varrho^{\frac{1}{2}}\varphi||^2||\varrho^{\frac{1}{2}}F_0\varphi||^2} = \sup_{\varrho^{\frac{1}{2}}\varphi\in\mathfrak{h}} \frac{(\varphi, H\varphi)^2}{(\varphi,\varrho\varphi)(\varphi, \varrho F_0^2\varphi)}$$

The Cauchy-Buniakovski inequality yields

$$(2.27)\qquad (\varphi, \varrho\varphi)(\varphi, \varrho F_0^2\varphi) = \left(\int \frac{\varrho}{F_0}\varphi^2 d\mathbf{c}\right)\left(\int \varrho F_0\varphi^2 d\mathbf{c}\right) \geqslant \left(\int \varrho\varphi^2 d\mathbf{c}\right)^2 = (\varphi, \varrho F_0\varphi)^2$$

and so from Eqs. (2.26) and (2.27) it follows that

$$(2.28)\qquad \frac{(\varrho^{-1}H\varphi, H\varphi)}{(\varrho F_0^2\varphi, \varphi)} \geqslant \sup_{\varrho^{\frac{1}{2}}\varphi\in\mathfrak{h}} \frac{(\varphi, H\varphi)^2}{(\varphi, \varrho F_0\varphi)^2}.$$

The inequality (2.28) indicates that in order to prove Lemma 3 it is sufficient to show that

$$(2.29)\qquad (\varphi, H\varphi) \leqslant q(\varphi, \varrho F_0\varphi).$$

Owing to the fact that $(\varphi, H\varphi) = (\varphi F_0^{1/2}\varrho^{1/2}, G\varphi F_0^{1/2}\varrho^{1/2})$ and $(\varphi, \varrho F_0\varphi) = (\varphi F_0^{1/2}\varrho^{1/2}, \varphi F_0^{1/2}\varrho^{1/2})$ the problem is reduced to the demonstration that the spectrum of $G$ is bounded by a constant less than unity, [4].

The following notation is introduced here:

$$(2.30)\qquad G = F_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}HF_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}} = ANF_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}PF_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}};$$

$G$ is a compact operator [4] since $P$ is the operator of projection onto the four-dimensional space, and hence $G$ has a discrete spectrum only.

The eigenfunctions $\varphi_i$ of the operator $G$ satisfy the equation

$$(2.31)\qquad \lambda_i\varphi_i = G\varphi_i$$

and so, due to the fact that the projection operator I–P is positive, we obtain

$$(2.32)\qquad \lambda_i(\varphi_i, \varphi_i) = (\varphi_i, G\varphi_i) = (\varphi_i, F_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}ANPF_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}\varphi_i)$$
$$= (\varphi_i, -F_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}AN(I-P)F_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}\varphi_i + ANF_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}IF_0^{-\frac{1}{2}}\varrho^{-\frac{1}{2}}\varphi_i) \leqslant (\varphi_i, ANF_0^{-1}\varrho^{-1}\varphi_i),$$

whence

$$(2.33)\qquad \lambda_i \leqslant \frac{(\varphi_i, AN\varrho^{-1}F_0^{-1}\varphi_i)}{(\varphi_i, \varphi_i)} < 1,$$

since $AN\varrho^{-1}F_0^{-1} < 1$ for $c_2 \neq 0$, i.e. on the set of nonzero measure.

Let $\mathscr{H}$ denote the space of square-integrable functions of variables $\mathbf{c}$, $\mathbf{x}$ with the weight $\varrho_1 = \sqrt{(AN)^2+\eta^2 c_2^2 y_0^{-2}}$, and the scalar product and norm are expressed by

$$(h, g)_{\mathscr{H}} = \int \varrho_1 hg d\mathbf{c} d\mathbf{x}, \qquad ||h||^2_{\mathscr{H}} = (h, h)_{\mathscr{H}}. \tag{2.34}$$

THEOREM. *In the space $\mathscr{H}$ there exists a unique solution of Eq.* (1.11) *provided $\varphi_0 \in \mathscr{H}$ and $\varphi_0 \in \mathfrak{h}$.*

Proof. From the Lemmas 2 and 3 it follows that

$$||BH\varphi||^2_{\mathscr{H}} = (BH\varphi, BH\varphi)_{\mathscr{H}} = ((\varrho F_0^2 BH\varphi, BH\varphi)) \leqslant ((\varrho^{-1}H\varphi, H\varphi)) \leqslant q^2((\varrho F_0^2\varphi, \varphi))_{\mathscr{H}} = q^2||\varphi||_{\mathscr{H}}. \tag{2.35}$$

This means that operator $BH$ is in the space $\mathscr{H}$ a contraction operator and the necessary proof follows immediately from the contraction mapping principle, [4].

## 3. Final remarks

The assumption of the particular form (1.10) of the boundary conditions is immaterial. The proof is not changed if the boundary conditions are expressed by arbitrary functions of a corresponding class and given explicitly on the lines $y = 0$ and $y = y_0$.

Let us remark that the operator $B$ defined in $L^2$ is defined explicitly by Eq. (1.13). From its form it follows that the corresponding derivatives appearing in the proofs of Lemmas 1 and 2 exist almost everywhere. However, the solution $\varphi$ which was proved to exist in space $\mathscr{H}$ does not have to be differentiable, i.e. it is not known whether it fulfills the original Eq. (1.7) in addition to its integral form. It may be stated that for the obtained solution the directional derivative exists almost everywhere, what means that the original equation is fulfilled in the generalized sense.

In the proof of existence and uniqueness it was assumed that $\varphi_0 \in \mathfrak{h}$. This implies that the solution $\varphi$ cannot be an arbitrary element of the space $\mathscr{H}$ but only an element of the subspace of $\mathscr{H}$ which is defined by the product of sets of functions belonging to $\mathscr{H}$ and to $\mathfrak{h}$. It is very essential since only in such a subspace operator $H$ may be treated as self-adjoint. Simultaneously, the solution belonging to that subspace represents the conditions of existence of hydrodynamic moments since then $|\varphi| < AF_0^{-1/2}$, $A$ being a certain constant. This conclusion allows for a clear physical interpretation of the function $\varphi$.

The procedure used here in proving the theorem confirms the possibility of applying the method of consecutive approximations, what is of importance for practical purposes.

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