Arch. Mech., 33, 5, pp. 763-771, Warszawa 1981

Nonlinear micropolar continuum model of a composite reinforced by elements of finite rigidity
Part II. Stability at compression

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The model of elastic composite proposed in the first part of the paper is appied to the analysis of internal stability of a layer cut out from the body in the direction transversal to the fibers and compressed normally to the middle surface. The stability is investigated using the method of superposition of small periodic strains on finite uniform deformation. In the limiting case of inextensible reinforcement elements relations are obtained, which enable the prediction of the instability point on the basis of the formulae expressing the phase and group velocities of transversal waves in terms of the applied load.

Zaproponowany w pierwszej części pracy model momentowy kompozytu sprężystego wykorzystuje się do badania stateczności wewnętrznej warstwy materiału wyciętego poprzecznie do kierunku zbrojenia i ściskanej prostopadle do płaszczyzny środkowej. Stateczność bada się metodą nałożenia małych periodycznych odkształceń na duże jednorodne. Dla granicznego przypadku niewydłużalnych elementów zbrojenia otrzymuje się zależności pozwalające przewidzieć punkt utraty stateczności na podstawie zależności prędkości fazowej i grupowej fal poprzecznych od przyłożonego obciążenia.

Предложенная в первой части работы моментная модель композитного материала используется для исследования внутренней устойчивости слоя с поперечным направлением армировки, сжимаемого перпендикулярно серединной плоскоти. Устойчивость исследуется посредством наложения малых периодических деформаций на ка конечные однородные деформации. Для предельного случая нерастяжимой армировки получены зависимости, позволяющие предвидеть значение критической нагрузки по характеру зависимости фазовой и групповой скоростей поперечных волн от приложенного сжимающего усилия.

1. Introduction

In Part I of this paper [1] we have derived the equations of motion and equilibrium of a composite material reinforced by a single family of elements characterized by finite stiffness; the equations have the form

$$T^{\nu\mu}_{;\nu} + \varrho f^{\mu} = \varrho w^{\mu}, \tag{1.1}$$

the stress tensor $T^{\nu\mu}$ being, in general, not symmetric. The tensor is determined from the constitutive relation

$$T^{\nu\mu} = \varrho\tau^{(\nu\mu)} - \left\{2\varrho\left[\frac{\partial w}{\partial J_6}\varkappa^{\mu} - \frac{\partial w}{\partial J_7}(\varkappa_{\alpha}a_{\beta}B^{\alpha\beta}a^{\mu} - \varkappa_{\beta}B^{\beta\mu})\right]a^{\nu}a^{\gamma}\right\}_{,\gamma}, \tag{1.2}$$

in which $\mathbf{B} = \mathbf{F}\mathbf{F}^T$; in the reference frame of convective material coordinates the contravariant components of tensor $\mathbf{B}$ are equal to the components of the metric tensor in the undeformed state. $\varkappa^\mu$ denote the components of the vector of curvature of the reinforcement elements, a^ν are components of the unit vector tangent to the fibers. $\frac{\partial w}{\partial J_6}$ and $\frac{\partial w}{\partial J_7}$ denote the respective derivatives of the elastic energy density function w with respect to $\boldsymbol{\varkappa}\cdot\boldsymbol{\varkappa}$ and $\boldsymbol{\varkappa}\mathbf{B}\boldsymbol{\varkappa}$, and the symmetric tensor $\varrho\tau^{(\nu\mu)}$ is reduced, in the case of reinforcement of vanishing stiffness, to the classical stress tensor. The complete expression for $\tau^{\nu\mu}$ will be derived in this paper; other notations have been explained in [1].

The relations derived in [1] will now be applied to the analysis of stability of the material reinforced (in the undeformed state) by rectilinear and parallel elements. To this end we shall consider the field of infinitesimal strains superposed on uniform finite deformations.

Let us first reduce the expressions for $\varkappa^\alpha$ to a more convenient form than that derived in [1]. From the definition we have

$$\varkappa^\alpha = a^\alpha_{,\gamma}a^\gamma = \left(\frac{A^\alpha}{\sqrt{g_{\mu\nu}A^\mu A^\nu}}\right)_{,\gamma}\frac{A^\gamma}{\sqrt{g_{\varphi\theta}A^\varphi A^\theta}}, \tag{1.3}$$

where A^α are components of the vector tangent to the fibers in the actual base and remain independent of the deformation. Equation (1.3) is transformed by differentiation to yield

$$\varkappa^\alpha = \frac{A^\nu A^\gamma A^\delta_{,\alpha}A^\varphi\varepsilon_{\delta\varphi\varrho}\varepsilon^{\varrho\alpha u}}{(g_{\varphi\theta}A^\varphi A^\theta)^2}\, g_{\mu\nu}. \tag{1.4}$$

In the case of a material which is (in the undeformed state) reinforced by a family of rectilinear fibers, the components $\mathring{\varkappa}^\alpha = 0$ vanish, the index o referring to the initial state; on remembering that $\mathring{A}^\alpha = A^\alpha$, we obtain

$$\frac{A^\nu A^\gamma A^\delta_{;\gamma}A^\varphi\mathring{\varepsilon}_{\delta\varphi\varrho}\mathring{\varepsilon}^{\varrho\alpha\mu}}{(\mathring{g}_\theta A^\varphi A^\theta)^2}\,\mathring{g}_{\mu\nu} = 0, \tag{1.5}$$

the semicolon denoting the covariant differentiation in the base $\mathring{\mathbf{g}}_\alpha$. The vector with the components $A^\gamma A^\delta_{;\gamma}A^\varphi\mathring{\varepsilon}_{\delta\varphi\varrho}$ is orthogonal to the vector $\mathring{\mathbf{A}}$ since

$$A^\gamma A^\delta_{;\gamma}A^\varphi\mathring{\varepsilon}_{\delta\varphi\varrho}A^\varrho \equiv 0, \tag{1.6}$$

and thus the vanishing of their vector product (1.5) implies

$$A^\gamma A^\delta_{;\gamma}A^\varphi\mathring{\varepsilon}_{\gamma\varphi\varrho} = 0. \tag{1.7}$$

The values $\mathring{\varepsilon}_{\delta\varphi\varrho}$ and $\varepsilon_{\delta\varphi\varrho}$ are related to each other,

$$\varepsilon_{\delta\varphi\varrho} = \mathring{\varepsilon}_{\delta\varphi\varrho}\sqrt{\frac{\det g_{\alpha\beta}}{\det\mathring{g}_{\alpha\beta}}}, \tag{1.8}$$

and thus by writing down the expression for the covariant derivative, Eq. (1.7) may be put in the form

$$\frac{\partial A^\delta}{\partial u^\gamma}A^\gamma A^\varphi\varepsilon_{\delta\varphi\varrho} = -A^\sigma\mathring{\Gamma}^\delta_{\sigma\gamma}A^\gamma A^\varphi\varepsilon_{\delta\varphi\varrho}. \tag{1.9}$$

Consequently, after writing the necessary expression for the covariant derivative in Eq. (1.4) and using the relation (1.9) in order to eliminate the term containing the partial derivative we obtain

$$\varkappa^{\alpha} = a^{\nu}a^{\gamma}a^{\varphi}a^{\sigma}\varepsilon_{\delta\varphi\varrho}\varepsilon^{\varrho\alpha\mu}g_{\mu\nu}K^{\delta}_{\sigma\gamma}, \tag{1.10}$$

where

$$K^{\delta}_{\sigma\gamma} \equiv \Gamma^{\delta}_{\sigma\gamma} - \mathring{\Gamma}^{\delta}_{\sigma\gamma}. \quad (^1) \tag{1.11}$$

2. Equations for infinitesimal strains imposed of finite homogeneous deformation

Without reducing the generality of our approach, let us confine our considerations to homogeneous deformations along the principal axes of tensor **B**, and assume the initial system of material coordinates to be an orthonormal Cartesian frame of reference. The position vector of the material particle $\mathbf{R}(u^{\alpha})$ is transformed to the form

$$\mathbf{R}(u^{\alpha}) = \hat{\mathbf{R}}(u^{\alpha}) + \varepsilon\mathbf{R}'(u^{\alpha}), \tag{2.1}$$

where the primed symbols are referred to the small deformations ($\varepsilon \ll 1$); moreover,

$$\begin{gathered} \mathring{g}^{\alpha\beta} = \hat{B}^{\alpha\beta} = \delta^{\alpha\beta}, \quad \mathring{g}_{\alpha\beta} = \overset{-1}{\hat{B}}_{\alpha\beta} = \delta_{\alpha\beta}, \\ \hat{g}_{11} = \lambda_1^2, \quad \hat{g}_{22} = \lambda_2^2, \quad \hat{g}_{33} = \lambda_3^2, \\ \hat{g}_{12} = \hat{g}_{13} = \hat{g}_{23} = 0, \end{gathered} \tag{2.2}$$

whence it follows that

$$\begin{gathered} B^{\alpha\beta} = \hat{B}^{\alpha\beta}, \quad \overset{-1}{B}_{\alpha\beta} = \overset{-1}{\hat{B}}_{\alpha\beta}, \\ \mathring{\Gamma}^{\alpha}_{\beta\gamma} = 0, \quad \hat{\Gamma}^{\alpha}_{\beta\gamma} = 0. \end{gathered} \tag{2.3}$$

In the following considerations the nonlinear terms in ε will be disregarded in the expressions for the metric and stress tensors. From the assumptions (2.2) and (2.1) it follows that

$$\mathbf{g}_{\alpha} = \frac{\partial\mathbf{R}}{\partial u^{\alpha}} = \frac{\partial\hat{\mathbf{R}}}{\partial u^{\alpha}} + \varepsilon\frac{\partial\mathbf{R}'}{\partial u^{\alpha}} = \hat{\mathbf{g}}^{\alpha} + \varepsilon\mathbf{g}'_{\alpha}. \tag{2.4}$$

This immediately yields the following relations for the primed magnitudes, i.e. those referring to the superposed small deformations:

$$g'_{\alpha\beta} = \hat{\mathbf{g}}_{\alpha}\cdot\mathbf{g}'_{\beta} + \mathbf{g}'_{\alpha}\cdot\hat{\mathbf{g}}_{\beta}. \tag{2.5}$$

(1) Expression (1.11) is valid in convective coordinates only; the fact that the magnitudes $K^{\delta}_{\sigma\gamma}$ form inthe base $\mathbf{g}_{\alpha}$ the components of a certain tensor of rank 3 follows immediately from the transformation properties of the Christoffel symbols. It can be easily verified that in an arbitrary base $K^{\delta}_{\sigma\gamma} = \frac{1}{2}(\overset{-1}{B}_{\varphi\gamma,\sigma} - \overset{-1}{B}_{\varphi\sigma,\gamma} - \overset{-1}{B}_{\sigma\gamma,\varphi})B^{\varphi\delta}$ and, moreover, in convective material coordinates the interesting relation $\overline{\dot{K}^{\delta}_{\sigma\gamma}} = V^{\delta}_{,\sigma\gamma}$ holds true.

The relation

$$\mathbf{g}_\alpha \cdot \mathbf{g}^\beta = \delta^\beta_\alpha \tag{2.6}$$

yields

$$\mathbf{g}'^\alpha = -(\mathbf{g}'_\beta \cdot \hat{\mathbf{g}}^\alpha)\hat{\mathbf{g}}^\beta \tag{2.7}$$

and

$$g'^{\gamma\mu} = -g'_{\alpha\beta}\hat{g}^{\alpha\mu}\hat{g}^{\beta\gamma}. \tag{2.8}$$

Due to Eqs. (2.3), the Christoffel symbols are written in the form

$$\Gamma^\alpha_{\beta\gamma} = \varepsilon\Gamma'^\alpha_{\beta\gamma} = \varepsilon\frac{\partial \mathbf{g}'^\beta}{\partial u^\gamma}\hat{\mathbf{g}}^\alpha. \tag{2.9}$$

From Eqs. (1.10) and (1.11) we obtain

$$\varkappa^\alpha = \varepsilon\varkappa'^\alpha = \varepsilon\hat{a}^\nu\hat{a}^\gamma\hat{a}^\varphi\hat{a}^\sigma\hat{\varepsilon}_{\delta\varphi\varrho}\hat{\varepsilon}^{\varrho\alpha\mu}\hat{g}_{\mu\nu}\Gamma'^\delta_{\sigma\gamma},$$

and hence we obtain for the divergence of the second right-hand term in Eq. (1.2) denoted in [1] by the symbol $S^{\gamma\nu\mu}_{,\gamma\nu}$ the following expression:

$$S^{\gamma\nu\mu}_{,\gamma\nu} = \varepsilon\frac{\partial S'^{\gamma\nu\mu}}{\partial u^\gamma\partial u^\nu} = 2\varepsilon\hat{\varrho}\left[\frac{\partial w}{\partial J_6}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}\frac{\partial^2\varkappa'^\mu}{\partial u^\gamma\partial u^\nu} - \frac{\partial w}{\partial J_7}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}\frac{\partial^2\varkappa'^\theta}{\partial u^\gamma\partial u^\nu}\right.$$
$$\left.\times(\hat{g}_{\theta\alpha}B^{\alpha\beta}\hat{a}^\mu\hat{a}_\beta - \hat{g}_{\theta\alpha}B^{\alpha\mu})\right]\hat{a}^\gamma\hat{a}^\nu. \tag{2.10}$$

The convariant derivative appearing in Eq. (2.10) is reduced to the partial derivative owing to the fact that both the $S^{\gamma\nu\mu}$ and the Christoffel symbols are of the order of ε; in the case of the covariant derivative $[\varrho\tau^{(\mu\nu)}]_{,\nu}$ the situation is different:

$$[\varrho\tau^{(\mu\nu)}]_{,\nu} = \frac{\partial}{\partial u^\nu}(\varrho\tau^{(\mu\nu)}) + \varepsilon\varrho(\tau^{(\mu\alpha)}\Gamma'^\nu_{\alpha\nu} + \tau^{(\alpha\nu)}\Gamma'^\mu_{\alpha\nu})$$
$$= \varepsilon\left(\hat{\varrho}\frac{\partial}{\partial u^\nu}\tau'^{(\mu\nu)} + \frac{\partial\varrho'}{\partial u^\nu}\hat{\tau}^{(\mu\alpha)} + \hat{\varrho}\hat{\tau}^{(\mu\alpha)}\Gamma'^\nu_{\alpha\nu} + \hat{\varrho}\hat{\tau}^{(\alpha\nu)}\Gamma'^\mu_{\alpha\nu}\right), \tag{2.11}$$

use being made of the proprety $\frac{\partial}{\partial u^\alpha}(\hat{\varrho}\hat{\tau}^{(\mu\nu)}) = 0$.

In order to determine ϱ', let us use the formula

$$\left(\frac{\varrho_0}{\varrho}\right)^2 = \frac{\det g_{\alpha\beta}}{\det\mathring{g}_{\alpha\beta}}. \tag{2.12}$$

It can be easily proved that, with an accuracy up to the higher order terms,

$$\det g_{\alpha\beta} = \det\hat{g}_{\alpha\beta}(1+\varepsilon g'_{\mu\nu}\hat{g}^{\mu\nu}), \tag{2.13}$$

and hence

$$\varrho' = -\frac{1}{2}\hat{\varrho}g'_{\mu\nu}g^{\mu\nu}_{12}. \tag{2.14}$$

Let us now calculate $\tau'^{(\mu\nu)}$, according to [1],

$$\tau^{\mu\nu} = 2\left[\sum_{K=1}^{5} \frac{\partial w}{\partial J_k} \frac{\partial J_k(A^\alpha, B^{\varphi\theta}, g^{\mu\nu})}{\partial g_{\mu\nu}} + \frac{\partial W}{\partial J_6}(\varkappa^\mu \varkappa^\nu - 2\varkappa^2 a^\mu a^\nu) + 2\frac{\partial w}{\partial J_7}(\varkappa_\alpha \varkappa_\beta B^{\alpha\beta} a^\mu a^\nu + a_\alpha \varkappa_\beta B^{\alpha\beta} \varkappa^\mu a^\nu - \varkappa_\beta B^{\nu\beta}\varkappa^\mu)\right] = \hat{\tau}^{\mu\nu} + \varepsilon\tau'^{\mu\nu}, \tag{2.15}$$

where

$$\hat{\tau}^{\mu\nu} = \sum_{K=1}^{5} \frac{\partial w}{\partial J_k} \frac{\partial J_k}{\partial g_{\mu\nu}}\bigg|_{g=\hat{g}}. \tag{2.16}$$

The first and second derivatives with respect to the invariants are assumed to be bounded; it is then found that the coefficients of $\frac{\partial w}{\partial J_6}$ and $\frac{\partial w}{\partial J_7}$ are of the order of ε^2 (the invariants J_6, J_7 are of the same order) what implies

$$\varepsilon\tau'^{\mu\nu} = 2\varepsilon\left[\sum_{K=1}^{5}\left(\frac{\partial w}{\partial J_K}\right)' \frac{\partial J_K}{\partial g_{\mu\nu}} + \sum_{K=1}^{5} \frac{\partial w}{\partial J_K}\left(\frac{\partial J_K}{\partial g_{\mu\nu}}\right)'\right]\bigg|_{g=\hat{g}}, \tag{2.17}$$

$$\left(\frac{\partial J_1}{\partial g_{\mu\nu}}\right)' = \left(\frac{\partial J_4}{\partial g_{\mu\nu}}\right)' = 0, \quad \left(\frac{\partial J_2}{\partial g_{\mu\nu}}\right)' = J_1' B^{\mu\nu} - B^{\mu\alpha}B^{\nu\beta}g'_{\alpha\beta},$$

$$\left(\frac{\partial J_3}{\partial g_{\mu\nu}}\right)' = (J_3' g^{\mu\nu} + J_3 g'^{\mu\nu})|_{g=\hat{g}}, \quad \left(\frac{\partial J_5}{\partial g^{\mu\nu}}\right)' = 2A^\nu B^{\nu\beta} A^\gamma g'_{\gamma\beta}.$$

In the case of $\left(\frac{\partial w}{\partial J_k}\right)'$ $(k_1 = 1, \ldots, 5)$, we obtain

$$\left(\frac{\partial w}{\partial J_K}\right)' = \sum_{L=1}^{5} \frac{\partial w}{\partial J_K \partial J_L}\bigg|_{g=\hat{g}} J_L', \tag{2.18}$$

where

$$J_K' = \frac{\partial J_K(g_{\alpha\beta}A^\mu, \varkappa^\nu, \mathring{g}_{\varphi\theta})}{\partial g_{\alpha\beta}}\bigg|_{g=\hat{g}} g'^{\alpha\beta}$$

and finally

$$[\varrho\tau^{(\mu\nu)}]_{,\nu} = \varepsilon\hat{\varrho}\left[\frac{\partial}{\partial u^\nu}\tau'^{(\mu\nu)} + \hat{\tau}^{(\alpha\nu)}\Gamma^{\mu}_{\alpha\nu}\right]. \tag{2.19}$$

It is easily observed (cf. Eqs. (2.15)–(1.28)) that

$$\tau'^{(\mu\nu)} = 2\frac{\partial^2 w}{\partial g_{\mu\nu}\partial g_{\alpha\beta}}\bigg|_{g=\hat{g}} g'_{\alpha\beta}, \quad \hat{\tau}^{(\alpha\nu)} = 2\frac{\partial w}{\partial g_{\alpha\nu}}\bigg|_{g=\hat{g}}. \tag{2.20}$$

Introducing the notations

$$\mathbf{R}'\cdot\hat{\mathbf{g}}^\alpha \equiv r^\alpha; \quad \mathbf{R}'\cdot\hat{\mathbf{g}}_\beta = (\mathbf{R}'\cdot\hat{\mathbf{g}}^\alpha)\hat{g}_{\alpha\beta} \equiv r_\beta \tag{2.21}$$

and taking into account Eqs. (2.9) and (2.10) we obtain the final form of equations of motion (in the absence of body forces):

$$(2.22)\quad \frac{\partial^2 r_\xi}{\partial u^\alpha \partial u^\nu}\left(4\frac{\partial^2 w}{\partial g_{\mu\nu}\partial g_{\alpha\xi}}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}+2\frac{\partial w}{\partial g_{\alpha\nu}}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}\hat{g}^{\mu\xi}\right)-\frac{\partial^4 r_\xi}{\partial u^\sigma \partial u^\varkappa \partial u^\alpha \partial u^\nu}\hat{a}^\sigma \hat{a}^\varkappa \hat{a}^\alpha \hat{a}^\nu$$
$$\times 2\left[\frac{\partial w}{\partial J_6}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}(\hat{g}^{\mu\xi}-\hat{a}^\xi\hat{a}^\mu)+\frac{\partial w}{\partial J_7}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}(\hat{g}^{\varphi\xi}-\hat{a}^\xi\hat{a}^\varphi)B_{\varphi\varrho}(g^{\varrho\mu}-a^\mu a^\varrho)\right]=\ddot{r}^\mu.$$

3. Uniform deformation with the principal axis of the strain tensor directed parallel to the fibers; plane transversal wave

The problem is obtained by assuming

$$(3.1)\qquad \mathbf{A} = \mathring{\mathbf{g}}_3$$

that is

$$(3.2)\qquad \hat{a}^3 = \frac{1}{\lambda_3},\quad \hat{a}^2 = 0,\quad \hat{a}^1 = 0,$$
$$\hat{a}_3 = \lambda_3,\quad \hat{a}_2 = 0,\quad \hat{a}_1 = 0.$$

Equation (2.22) is then simplified to the form

$$(3.3)\quad \frac{\partial^2 r_\xi}{\partial u^\alpha \partial u^\nu}\left(4\frac{\partial^2 w}{\partial g_{\mu\nu}\partial g_{\alpha\xi}}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}+2\frac{\partial w}{\partial g_{\alpha\nu}}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}\hat{g}^{\mu\xi}\right)$$
$$-\frac{2}{\lambda_3^4}\frac{\partial^4 r_\xi}{(\partial u^3)^4}\left[\frac{\partial w}{\partial J_6}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}(\hat{g}^{\mu\xi}-\hat{a}^\xi\hat{a}^\mu)+\frac{\partial w}{\partial J_7}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}(\hat{g}^{\varphi\xi}-\hat{a}^\xi\hat{a}^\varphi)\delta_{\varphi\varrho}(\hat{g}^{\varrho\mu}-\hat{a}^\mu\hat{a}^\varrho)\right]=\ddot{r}^\mu.$$

Following the assumptions $r^1 = 0$ and $\frac{\partial r^\alpha}{\partial u^1} = \frac{\partial r^\alpha}{\partial u^2} = 0$, representing a plane wave propagating in the direction of $\hat{\mathbf{g}}_3$, we obtain

$$\ddot{r}^1 = 0,$$

$$(3.4)\quad \ddot{r}^2 = \left\{\frac{\partial^2 r_2}{\partial u^3 \partial u^3}\left(4\frac{\partial^2 w}{\partial g_{23}\partial g_{23}}+2\frac{\partial w}{\partial g_{33}}\frac{1}{\lambda_2^2}\right)+\frac{\partial^2 r_3}{\partial u^3\partial u^3}4\frac{\partial^2 w}{\partial g_{23}\partial g_{33}}\right.$$
$$\left.-\frac{2}{\lambda_3^4}\frac{\partial^4 r_2}{(\partial u^3)^4}\left(\frac{1}{\lambda_2^2}\frac{\partial w}{\partial J_6}+\frac{1}{\lambda_2^4}\frac{\partial w}{\partial J_7}\right)\right\}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}},$$

$$\ddot{r}^3 = \left\{\frac{\partial^2 r_3}{\partial u^2 \partial u^3}\left(4\frac{\partial^2 w}{\partial g_{33}\partial g_{33}}+2\frac{\partial w}{\partial g_{33}}\frac{1}{\lambda_3^2}\right)+\frac{\partial^2 r_2}{\partial u^3\partial u^3}4\frac{\partial^2 w}{\partial g_{33}\partial g_{23}}\right\}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}.$$

Differentiation with respect to the natural space coordinate z equal to the distance measured along fibers in the direction of $\hat{\mathbf{g}}_3$ (denoted by primes) yields

$$(3.5)\quad \ddot{W} = \left\{W''\left(4\frac{\partial^2 w}{\partial g_{33}\partial g_{33}}\lambda_3^4+2\frac{\partial w}{\partial g_{33}}\lambda_3^2\right)+V''\lambda_2\lambda_3^3 4\frac{\partial^2 w}{\partial g_{33}\partial g_{23}}\right\}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}},$$

(3.5) [cont.]
$$\ddot{V} = \left\{ V''\left(4\frac{\partial^2 w}{\partial g_{23}\partial g_{23}}\lambda_3^2\lambda_2^2+2\frac{\partial w}{\partial g_{33}}\lambda_3^2\right)+W''\lambda_2\lambda_3^3 4\frac{\partial^2 w}{\partial g_{33}\partial g_{23}} -2V^{(\mathrm{IV})}\left(\frac{1}{\lambda_2}\frac{\partial w}{\partial J_6}+\frac{1}{\lambda_2^3}\frac{\partial w}{\partial J_7}\right)\right\}\Bigg|_{\mathbf{g}=\hat{\mathbf{g}}}.$$

Here V, W are the physical components of displacements in the respective directions $\hat{\mathbf{g}}_3$ and $\hat{\mathbf{g}}_2$: $W \equiv r^3/\lambda_3$; $V \equiv r^2/\lambda_2$. It is easily verified that if $\hat{g}_{23} = 0$, if the vectors $\hat{\mathbf{g}}$ and $\mathbf{B}$ are coaxial and the vector $\mathbf{A}$ is parallel to one of the principal directions of $\mathbf{B}$ (and hence of $\hat{\mathbf{g}}$), then, for each K and L the following equalities are satisfied:

$$\frac{\partial J_L}{\partial g_{23}}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}} = 0, \qquad \frac{\partial^2 J_K}{\partial g_{23}\partial g_{33}}\bigg|_{\mathbf{g}=\hat{\mathbf{g}}} = 0. \tag{3.6}$$

Equations (3.5) are then decoupled to yield

$$\begin{aligned} \ddot{W} &= W''\left(4\frac{\partial^2 w}{\partial g_{33}\partial g_{33}}\lambda_3^4+2\frac{\partial w}{\partial g_{33}}\lambda_3^2\right)\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}, \\ \ddot{V} &= V''\left(4\frac{\partial^2 w}{\partial g_{23}\partial g_{23}}\lambda_3^2\lambda_2^2+2\frac{\partial w}{\partial g_{33}}\lambda_3^2\right)\bigg|_{\mathbf{g}=\hat{\mathbf{g}}} -2V^{(\mathrm{IV})}\left(\frac{\partial w}{\partial J_6}\frac{1}{\lambda_2}+\frac{\partial w}{\partial J_7}\frac{1}{\lambda_2^3}\right)\bigg|_{\mathbf{g}=\hat{\mathbf{g}}}. \end{aligned} \tag{3.7}$$

The solutions may now be sought for in the form of a transversal wave:

$$\begin{aligned} V &= V_0 e^{i(kz-\omega t)}+\mathrm{const}, \\ W &= 0. \end{aligned} \tag{3.8}$$

Substituting the expressions (3.8) into Eq. (3.7), performing the differentiation and dividing by V_0, we are led to the following characteristic equation:

$$\omega^2 = k^2\left(A^2-\frac{1}{\hat{\varrho}}\hat{P}_{33}\right)+Rk^4, \tag{3.9}$$

Here $A^2 \equiv 4\dfrac{\partial^2 w}{\partial g_{23}\partial g_{23}}\lambda_3^2\lambda_2^2|_{\mathbf{g}=\hat{\mathbf{g}}}$, $\hat{P}_{33} = -\hat{T}_{33}$ ($\hat{T}_{33}$ is the physical component of stress in the direction of fibers), and $R \equiv 2\left(\dfrac{\partial w}{\partial J_6}\dfrac{1}{\lambda_2}+\dfrac{\partial w}{\partial J_7}\dfrac{1}{\lambda_2^3}\right)$.

Confine our considerations to the solutions periodic in z; this means that $k \in \mathrm{Re}$. It follows directly from Eq. (3.9) that ω may assume either purely imaginary or real values, and if ω is a root of Eq. (3.9), then $(-\omega)$ is also a root of that equation; in the case of real-valued ω Eq. (3.8) describes a dispersive wave, and for $\omega = 0$ the material is statically "wrinkled". In contrast, imaginary values of ω always lead to exponential instability in time. It is noted that A^2 may be interpreted as the shear modulus measured in the direction perpendicular to the fibers and so, in accordance with Eq. (3.9), the problem of stability loss is physically sensible only in the case of reinforcement characterized by the sufficiently high Young's modulus and sufficiently low shear modulus of the matrix.

Let us now consider the limiting case: compression of the material along inextensible fibers; it follows that $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\hat{\varrho} = \varrho_0$,

$$A^2 = c_{00}^2 = \lim_{k\to 0} \frac{\omega^2}{k^2}\bigg|_{\hat{P}_{33}=0}. \tag{3.10}$$

This means that c_{00} is the limiting phase velocity of the wave at $P_{33} = 0$, $k \to 0$.

For the group velocity $c_g \equiv \dfrac{\partial\omega}{\partial k}$, we obtain

$$c_g = \frac{k\left(A^2 - \frac{1}{\varrho_0}\hat{P}_{33}\right) + 2Rk^3}{\sqrt{k^2\left(A^2 - \frac{1}{\varrho_0}\hat{P}_{33}\right) + Rk^4}} = \frac{1}{k}\,\frac{\omega^2 + Rk^4}{\omega} = \frac{\omega}{k} + Rk^2\frac{k}{\omega} = c + \frac{Rk^2}{c}, \tag{3.11}$$

where $c = c(k, \hat{P}_{33})$ — phase velocity. Consequently,

$$R = \frac{c(c_g - c)}{k^2} = \frac{c^3(c_g - c)}{\omega^2}. \tag{3.12}$$

Expression (3.12) may serve for the effective determination of R and as an important criterion of correctness of the theory; if all the preceding considerations were true, the right-hand side of Eq. (3.12) should not depend on P_{33} and k (or ω). Denoting by k_{cr} and P_{33cr} the respective values of k and p_{33} corresponding to the point $\omega = 0$ of the stability loss, we obtain

$$\hat{P}_{33cr} = \varrho_0(c_{00}^2 + Rk_{cr}). \tag{3.13}$$

It means that in an infinite domain $\hat{P}_{33cr}$ is independent of R since, once the value $\varrho_0 c_{00}^2$ is exceeded, one can always find such value of k at which the stability will be lost.

If now the compression of a layer between two rough plates is considered, it is seen that, depending on the assumption whether rotations at the points of contact with the plates are allowed or not (what corresponds to the manner of fixing the ends of fibers in the plates), two cases of minimal $\hat{P}_{33cr}$ are possible:

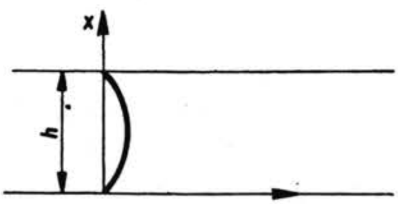

FIG. 1.

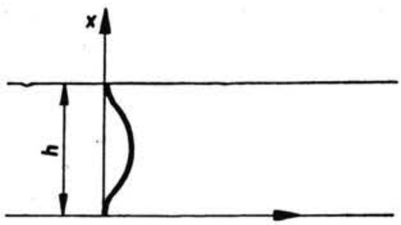

FIG. 2.

(a) $V = V_0 \sin\dfrac{\pi x}{h}$, $k = \dfrac{\pi}{h}$ i.e. $\hat{P}_{33kr} = \varrho_0\left(c_{00}^2 + \dfrac{\pi^2}{h^2}R\right)$ (Fig. 1),

(b) $V = V_0\left(\cos\dfrac{2\pi x}{h} - 1\right)$; $k = \dfrac{2\pi}{h}$ i.e. $\hat{P}_{33kr} = \varrho_0\left(c_{00}^2 + \dfrac{4\pi^2}{h^2}R\right)$ (Fig. 2).

Final remarks

The fragmentary and incomplete study in the second part of this paper, illustrated by only two examples, is a mere presentation of certain possibilities of utilization of the model introduced in the first part. In order to establish whether the model might be used in solving certain practical engineering problems, one should be able to evaluate the necessary material functions. Due to the rather complex character of interaction between the reinforcement and the matrix, this is not necessarily a trivial task. A study of the wave propagation and stability problems under slightly more general assumptions as to deformation geometry and material properties would also be of interest. These problems will be dealt with in the author's next paper.

References

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INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received January 29, 1981.