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Inertial effects of the gas motion upon the linear and nonlinear waves in Kelvin–Helmholtz flow

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AN INVESTIGATION is made of the inertial effects of the gas motion upon the linear and nonlinear stability characteristics of the wave motion at the interface between a gas stream and a liquid. The analysis considers a body force directed towards the liquid as well as the effects of surface tension of the liquid. The liquid is assumed to be initially quiescent. The gas flow is considered to be subsonic. The treatment of the linear problem shows that the inertial effects of the gas motion lead to overstability. For the nonlinear problem, the Poincaré–Lighthill–Kuo method is used to obtain solutions as perturbations about the neutrally-stable linear oscillation. Detailed discussion of solutions at various values of wave-numbers is given.

Przeprowadzono badania wpływu efektów dynamicznych w ruchu gazu na liniowe i nieliniowe charakterystyki stateczności ruchu falowego na powierzchni rozdzielającej gaz od cieczy. Analiza uwzględnia wpływ sił masowych skierowanych do cieczy, jak również wpływ napięcia powierzchniowego cieczy. Zakłada się, że ciecz znajduje się w chwili początkowej w spoczynku, a przepływ gazu jest poddźwiękowy. Analiza przypadku liniowego wskazuje, że efekty inercyjne w ruchu gazu prowadzą do nadstateczności. W przypadku nieliniowym zastosowano metodę Poincaré–Lighthilla–Kuo, otrzymując rozwiązania w postaci perturbacji wokół liniowych drgań ustalonych. Przeprowadzono szczegółową dyskusję rozwiązań dla różnych wartości liczb falowych.

Проведены исследования влияния динамических эффектов в движении газа на линейные и нелинейные характеристики устойчивости волнового движения на поверхности разделяющей газ от жидкости. Анализ учитывает влияние массовых сил направленных в жидкость как тоже влияние поверхностного натяжения жидкости. Предполагается, что жидкость находится в начальный момент в покое, а течение газа является дозвуковым. Анализ линейного случая указывает, что инерционные эффекты в движении газа приводят к сверхустойчивости. В нелинейном случае применен метод Пуанкаре–Лайтхилла–Куо, получая решения в виде пертурбаций вокруг линейных установившихся колебаний. Проведено детальное обсуждение решений для разных значений волновых чисел.

1. Introduction

THE STUDIES of the wave motion at the interface between a liquid layer and a gas flowing past it are of interest in the problems of transpiration cooling of the reentry vehicles and in particular, in determining the amount of the liquid entrained by the gas. CHANG and RUSSELL [2] made a study of the linear stability characteristics of the wave motion at the interface between a liquid layer and a gas stream adjacent to it and found that the nature of the waves generated at the interface depends markedly on the state of the gas. For a supersonic gas flow, the gas pressure at the interface is out of phase with the surface tension so that a purely oscillatory constant-amplitude motion of the interface is not possible. For a subsonic gas flow, however, the stabilising effect of the surface tension gives rise to cut-off frequencies. NAYFEH anf SARIC [5] later extended the study for the nonlinear case, but their treatment suffers from an error made early in the analysis.

In both of these treatments of the linear and nonlinear problems, the inertial effects of the gas motion have been ignored, and probably on this account the linear results, particularly, in the above are in disagreement with the experimental findings, e.g. those of GATER and L'ECUYER [3], where the interfacial wave motion was found to be unstable even for a subsonic gas flow. In any event, the inertial effects of the gas motion become important for waves with speeds of propagation comparable with the gas speed.

The following analysis is an investigation of the inertial effects of the gas motion on the linear and nonlinear stability characteristics of the wave motion at the interface between a liquid and a gas stream adjacent to it. The analysis considers a body force directed towards the liquid, and the effects of the surface tension of the liquid. The liquid is assumed to be initially quiescent, and the gas flow is considered to be subsonic. For the nonlinear problem, the Poincaré–Lighthill–Kuo method is used to obtain solutions as perturbations about the neutrally-stable linear oscillation.

2. Method of perturbations

Consider an initially quiescent liquid of infinite depth whose mean level of contact with a gas flowing past it is the horizontal surface $y = 0$ (see Fig. 1). Both the liquid and

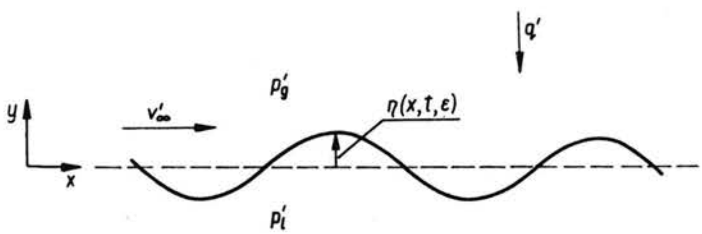

FIG. 1. The Kelvin–Helmholtz problem.

the gas are assumed to be inviscid and the effects of the viscous boundary layer at the interface are ignored. If the motion of the whole system is supposed to start from rest, it may be assumed to be irrotational. The following analysis takes into account the surface tension of the liquid as well as a body force acting normal to the interface and directed towards the liquid. If a typical interfacial disturbance is characterised by a sinusoidal travelling wave with an amplitude a' and wavelength λ', then all the physical quantities in the following are nondimensionalised with respect to a reference length $(\lambda'/2\pi)$ and time $(\lambda'/2\pi g')^{1/2}$; and the inertial effects of the gas motion are characterised by a time scale U'_∞/g', where g' denotes the acceleration due to gravity, U'_∞ the ambient gas speed, and the primes here denote the dimensional quantities. The gas density ϱ'_g is small so that the corresponding body force is negligible. The potential function of the motion of the liquid and the gas are taken to be, respectively,

$$(g')^{1/2}(\lambda'/2\pi)^{3/2}\varphi(x, y, t;\ \varepsilon)$$

and

$$U'_\infty\left(\frac{\lambda'}{2\pi}\right)\left[x+\phi(x, y, t;\ \varepsilon)\right],$$

where

(2.1) $$y<\eta:\ \varphi_{xx}+\varphi_{yy}=0,$$

(2.2) $$y>\eta:\ (1-M_\infty^2)\phi_{xx}+\phi_{yy}=M_\infty^2\left\{\left[(\gamma+1)\phi_x+\frac{\gamma+1}{2}\phi_x^2+\frac{\gamma-1}{2}\phi_y^2\right.\right.$$
$$\left.+\delta(\gamma-1)\phi_t\right]\phi_{xx}+\left[(\gamma-1)\phi_x+\frac{\gamma+1}{2}\phi_y^2+\frac{\gamma-1}{2}\phi_x^2+\delta(\gamma-1)\phi_t\right]\phi_{yy}$$
$$\left.+[2\phi_y(1+\phi_x)]\phi_{xy}+2\delta[(1+\phi_x)\phi_{xt}+\phi_y\phi_{yt}]\right\}$$

and

$$\delta=\left(\frac{\lambda'g'}{2\pi}\right)^{1/2}\frac{1}{U'_\infty}\sim\frac{\text{wave speed}}{\text{gas speed}}\ll 1$$

and in the present analysis only the terms of $0(\delta)$ are retained. Here $y=\eta(x, t, \varepsilon)$ denotes the disturbed shape of the interface, and M_∞ the ambient gas Mach number. One has the following boundary conditions at the interface:

1) Kinematic condition

(2.3) $$y=\eta:\ \varphi_y=\eta_t+\varphi_x\eta_x,$$

(2.4) $$y=\eta:\ \phi_y=\delta\eta_t+(1+\phi_x)\eta_x.$$

2) Dynamic condition

(2.5) $$y=\eta:\ \varphi_t+\frac{1}{2}(\varphi_x^2+\varphi_y^2)+\eta=k^2\eta_{xx}(1+\eta_x^2)^{-3/2}-\frac{\sigma}{2}kC_p,$$

where

$$k^2=\left(\frac{2\pi}{\lambda'}\right)^2\frac{T'}{\varrho'_l g'},\qquad \sigma=\frac{\varrho'_g U'^2_\infty}{\sqrt{\varrho'_l g' T'}},$$
$$C_p=\frac{2}{\gamma M_\infty^2}\left\{\left[1-\frac{\gamma-1}{2}M_\infty^2(2\delta\phi_t+2\phi_x+\phi_x^2+\phi_y^2)\right]^{\frac{\gamma}{\gamma-1}}-1\right\}$$

or, on expansion,

$$C_p\cong -2\phi_x-2\delta\phi_t-[(1-M_\infty^2)\phi_x^2+\phi_y^2-2\delta M_\infty^2\phi_x\phi_t]$$
$$+\left[M_\infty^2\left\{1+\frac{M_\infty^2}{3}(\gamma-2)\right\}\phi_x^3+M_\infty^2\left\{[1+M_\infty^2(\gamma-2)]\delta\phi_t\phi_x^2+\delta\phi_t\phi_y^2+\phi_x\phi_y^2\right\}\right]$$

and T' denotes the surface tension.

NAYFEH and SARIC [5] have missed the term other than unity in the coefficient of ϕ_x^3 in the expression for C_p, and this error has been transmitted right through their analysis.

The infinity conditions are

$$y \to -\infty: \ \varphi_y \to 0, \tag{2.6}$$

$$y \to \infty: \ \phi_y \to 0. \tag{2.7}$$

In order to look for travelling waves, introduce

$$\xi = x - ct \tag{2.8}$$

so that Eqs. (2.1)–(2.7) become

$$y < \eta: \ \varphi_{\xi\xi} + \varphi_{yy} = 0, \tag{2.9}$$

$$\begin{aligned} y > \eta: \ (1-M_\infty^2)\phi_{\xi\xi} + \phi_{yy} = M_\infty^2 \Bigg\{ \Bigg[(\gamma-1)(1-\delta c)\phi_\xi + \frac{\gamma+1}{2}\phi_\xi^2 + \frac{\gamma-1}{2}\phi_y^2 \\ - 2\delta c \Bigg] \phi_{\xi\xi} + \Bigg[(\gamma-1)(1-\delta c)\phi_\xi + \frac{\gamma+1}{2}\phi_y^2 + \frac{\gamma-1}{2}\phi_\xi^2 \Bigg] \phi_{yy} + [2\phi_y(1+\phi_\xi-\delta c)]\phi_{\xi y} \Bigg\}, \end{aligned} \tag{2.10}$$

$$y = \eta: \ \varphi_y = (\varphi_\xi - c)\eta_\xi, \tag{2.11}$$

$$y = \eta: \ \phi_y = (1+\phi_\xi - \delta c)\eta_\xi, \tag{2.12}$$

$$\begin{aligned} y = \eta: \ \eta = \left[c\varphi_\xi - \frac{1}{2}(\varphi_\xi^2 + \varphi_y^2) \right] + k\sigma \Bigg[(1-\delta c)\phi_\xi + \frac{1}{2}(\gamma^2\phi_\xi^2 + \phi_y^2) \\ - \frac{1}{2}M_\infty^2 \left\{ (1-\delta c) + M_\infty^2(\gamma-2)\left(\frac{1}{3} - \delta c\right) \right\} \phi_\xi^3 - \frac{1}{2}M_\infty^2(1-\delta c)\phi_\xi\phi_y^2 \Bigg] \\ + k^2\eta_{\xi\xi}(1+\eta_\xi^2)^{-3/2}, \end{aligned} \tag{2.13}$$

$$y \to -\infty: \ \varphi_y \to 0,$$

$$y \to \infty: \ \phi_y \to 0,$$

where

$$\gamma^2 = 1 - M_\infty^2 + 2\delta c M_\infty^2.$$

Seek solutions of the form, with $\varepsilon = a' 2\pi/\lambda' \ll 1$,

$$\varphi(\xi, y, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \varphi_n(\xi, y), \tag{2.14}$$

$$\phi(\xi, y, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \phi_n(\xi, y), \tag{2.15}$$

$$\eta(\xi, \varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n \eta_n(\xi), \tag{2.16}$$

$$c(k, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n c_n(k), \tag{2.17}$$

$$k(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n k_n. \tag{2.18}$$

One then obtains from Eqs. (2.9)–(2.13), (2.6) and (2.7), upon substituting Eqs. (2.14)–(2.18) the set of equations (A.1)–(A.21) as given in the Appendix.

3. Linear problem

Let

$$\eta_1(\xi) = A\cos\xi, \tag{3.1}$$

then, from Eqs. (A.1)–(A.4), (A.6) and (A.7) one obtains

$$\varphi_1(\xi, y) = Ac_0 e^y \sin\xi, \tag{3.2}$$

$$\phi_1(\xi, y) = \frac{A}{\gamma_0}(1-\delta c_0)e^{-\gamma_0 y}\sin\xi. \tag{3.3}$$

Using Eqs. (3.1)–(3.3) in Eq. (A.5), one obtains

$$c_0^2 + \frac{k_0\sigma(1-2\delta c)}{\gamma_0} - (1+k_0^2) = 0 \tag{3.4}$$

from which one finds

$$c_0 = k_0\sigma\delta\left(\frac{1}{m}+\frac{M_\infty^2}{2m^3}\right) \pm \Delta, \tag{3.5}$$

where

$$\Delta = \sqrt{k_0^2 - \frac{k_0\sigma}{m}+1}, \quad m = \sqrt{1-M_\infty^2}.$$

It is obvious that the inertial effects of the gas motion lead to overstability (see CHANDRASEKHAR, [1] Chapt. I).

The cut-off wavenumbers correspond to

$$k_{0n_{1,2}} = \frac{\sigma}{2m} \mp \sqrt{\frac{\sigma^2}{4m^2}-1}. \tag{3.6}$$

Thus there are two cut-off wavenumbers, and all disturbances with wavenumbers above or below these values propagate without growth or decay.

For the cases $M_\infty \ll 1$, note from Eq. (3.5),

$$-c_0 + \frac{k_0\sigma\delta}{m} = -\Delta. \tag{3.7}$$

4. Nonlinear analysis for wavenumbers away from the linear cut-off and the second-harmonic resonant values

Let

$$\eta_2 = B\cos 2\xi, \tag{4.1}$$

so that one obtains from Eqs. (A.8)–(A.11), (A.13) and (A.14)

$$\varphi_2(\xi, y) = c_0\left(B-\frac{A^2}{2}\right)e^{2y}\sin 2\xi + c_1 Ae^y\sin\xi, \tag{4.2}$$

$$(4.3)\qquad \phi_2(\xi, y) = \Bigg[-\frac{M_\infty^2 A^2 D(1-\delta c_0)}{4\gamma_0}\left(y+\frac{1}{2\gamma_0}\right) + \frac{\left\{2B+\frac{A^2}{2}\left(\gamma_0+\frac{1}{\gamma_0}\right)\right\}(1-\delta c_0)}{2\gamma_0}\Bigg] e^{-2\gamma_0 y}\sin 2\xi - 2c_1\delta\left[\frac{M_\infty^2}{\gamma_0^2}\left(y+\frac{1}{\gamma_0}\right)+\frac{1}{\gamma_0}\right]Ae^{-\gamma_0 y}\sin\xi,$$

where

$$D = \frac{1}{2}(1-\delta c_0)(\gamma+1)\left(1-\frac{1}{\gamma_0^2}\right).$$

Using Eqs. (4.1)–(4.3) in Eq. (A.12), one finds that the removal of the secular terms requires (or, equivalently, one is free to choose)

$$(4.4)\qquad k_1 = 0,$$

$$(4.5)_1\qquad c_1\left[c_0-\frac{k_0\sigma\delta}{m}\right] = 0$$

or

$$(4.5)_2\qquad 2c_1\Delta = 0,$$

so that

$$(4.6)\qquad c_1 = 0, \quad \textit{if} \quad \Delta \neq 0, \quad \text{i.e., if} \quad k \neq k_{0n_{1,2}}.$$

One then finds from Eq. (A.12)

$$(4.7)\qquad B = \mu A^2,$$

where

$$\mu = \frac{\frac{c_0^2}{2}-\frac{\sigma k_0(1-2\delta c_0)}{4\gamma_0^2}(2-M_\infty^2 D)}{1-2k_0^2}.$$

The case

$$k_{0s} = \pm\sqrt{\frac{1}{2}},$$

where μ becomes unbounded, corresponds to the well-known second-harmonic resonance which we shall treat in a later section.

Using Eqs. (3.1)–(3.3), (4.1)–(4.3) and (4.5)–(4.7), one finds from Eqs. (A.15)–(A.18), (A.20) and (A.21)

$$(4.8)\qquad \varphi_3(\xi, y) = A\left[-c_0 A^2\left(3\frac{\mu}{2}+\frac{1}{8}\right)+c_2\right]e^y\sin\xi + \text{higher harmonics},$$

$$(4.9)\qquad \phi_3(\xi, y) = [A^2(\mathcal{J}y+\mathcal{K})(1-\delta c_0)]Ae^{-3\gamma_0 y}\sin\xi + \left[-2c_2\delta\frac{M_\infty^2}{\gamma_0^2}\left(y+\frac{1}{\gamma_0}\right)-\frac{2c_2\delta}{\gamma_0}-A^2(1-\delta c_0)\mathcal{L}+\frac{A^2}{\gamma_0}(\mathcal{J}-3\gamma_0\mathcal{K})(1-\delta c_0)\right] \times Ac^{-}\quad \sin\xi + \text{higher harmonics},$$

where

$$\mathscr{J} = \frac{(1-2\delta c_0)M_\infty^2 D}{32\gamma_0^2}(\gamma+1)\left(1-\frac{1}{\gamma_0^2}\right),$$

$$\mathscr{K} = \frac{1}{8\gamma_0^2}\Bigg\{[1-2\delta c_0]\Bigg[M_\infty^2 D\left(-\frac{\gamma+1}{8\gamma_0^3}+\frac{\gamma-1}{8\gamma_0}\right) + \left\{2\mu+\frac{1}{2}\left(\gamma_0+\frac{1}{\gamma_0}\right)\right\}\left(\frac{\gamma+1}{2\gamma_0^2}+\frac{\gamma-3}{2}\right)+(\gamma+1)\left(-\frac{1}{8\gamma_0^3}+\frac{3\gamma_0}{8}\right) - \frac{\gamma-1}{4\gamma_0}+\frac{1}{2\gamma_0}+\frac{3M_\infty^2 D}{16\gamma_0}(\gamma+1)\left(1-\frac{1}{\gamma_0^2}\right)\Bigg]\Bigg\},$$

$$\mathscr{L} = \frac{1}{\gamma_0}\Bigg[M_\infty^2 D\left(\frac{1}{4}-\frac{1}{8\gamma_0^2}\right)+\left\{2\mu+\frac{1}{2}\left(\gamma_0+\frac{1}{\gamma_0}\right)\right\} \times\left(\frac{1}{2\gamma_0}-\gamma_0\right)+\mu\left(\frac{\gamma_0}{2}-\frac{1}{\gamma_0}\right)+\frac{\gamma_0^2}{4}+4\Bigg].$$

Using Eqs. (3.1)–(3.3), (4.1)–(4.3) and (4.5)–(4.9), one finds that the removal of the secular terms in Eq. (A.19) requires

$$c_2\left[2c_0-\frac{2k_0\sigma\delta}{m}\right]-A^2\Bigg\{c_0^2\left(\mu+\frac{1}{2}\right) +\sigma k_0(1-2\delta c_0)\Bigg[2\mathscr{K}-\frac{\mathscr{J}}{\gamma}+\mathscr{L}+\frac{5\mu}{2}+\frac{3\gamma}{4}+\frac{1}{2\gamma}+\frac{M_\infty^2 D}{8\gamma} +\frac{3}{8\gamma^3}M_\infty^2\left\{(1-2\delta c_0)+\frac{M_\infty^2(\gamma-2)}{3}(1-4\delta c_0)\right\}-\frac{1}{8\gamma}M_\infty^2(1-2\delta c_0)\Bigg]-\frac{3}{8}k_0^2\Bigg\}=0$$

or

$$c_2 = \Bigg\{c_0^2\left(\mu+\frac{1}{2}\right)+\sigma k_0(1-2\delta c_0)\Bigg[2\mathscr{K}-\frac{\mathscr{J}}{\gamma}+\mathscr{L}+\frac{5\mu}{2} +\frac{3\gamma}{4}+\frac{1}{2\gamma}+\frac{M_\infty^2 D}{8\gamma}+\frac{3}{8\gamma^3}M_\infty^2\left\{(1-2\delta c_0)+\frac{M_\infty^2(\gamma-2)}{3}(1-4\delta c_0)\right\} -\frac{1}{8\gamma}M_\infty^2(1-2\delta c_0)\Bigg]-\frac{3}{8}k_0^2\Bigg\}A^2\Big/{}^2\Delta. \tag{4.10}$$

A comparison of this result in the limit $\delta \Rightarrow 0$ with the corresponding one given by Nayfeh and Saric is not possible for the reasons already mentioned. It is clear that this result is not valid for wavenumbers near the linear cut-off values $k_{0n_{1,2}}$ where $\Delta = 0$ and the second-harmonic resonant values k_{0s}, where μ becomes unbounded. In the limit $\sigma \Rightarrow 0$, Eq. (4.10) agrees with the one deduced by NAYFEH [4].

5. Nonlinear analysis for wavenumbers near the linear cut-off values

In order to treat the case with wavenumbers near the linear cut-off values $k_{0n_{1,2}}$, one goes back to Eq. $(4.5)_2$, and notices that for $k \approx k_{0n_{1,2}}$, $\Delta \approx 0$, so that c_1 is arbitrary in the $0(\varepsilon^2)$ problem.

Using Eqs. (3.1)–(3.3), (4.1)–(4.3), (4.4) and (4.7), one finds from Eqs. (A.15)–(A.18), (A.20) and (A.21)

$$\varphi_3(\xi, y) = A\left[-c_0 A^2\left(\frac{\mu}{2}+\frac{3}{8}\right)\right] e^y \sin\xi + \text{higher harmonics}, \tag{5.1}$$

$$\begin{aligned}\phi_3(\xi, y) = {} & [A^2(\mathcal{J}y+\mathcal{K})(1-\delta c_0)]Ae^{-3\gamma y}\sin\xi + \\ & + \left[\frac{A^2}{\gamma^2}(\mathcal{J}-3\gamma\mathcal{K})(1-\delta c_0) - A^2(1-\delta c_0)\mathcal{L}\right] Ae^{-\gamma y}\sin\xi + \text{higher harmonics}.\end{aligned} \tag{5.2}$$

Using Eqs. (3.1)–(3.3), (4.1)–(4.3), (4.5), (4.7), (5.1) and (5.2), one finds that the removal of the secular terms in Eq. (A.19) requires that

$$\begin{aligned} c_1^2 + k_2\left[-2k_0 + \frac{\sigma(1-2\delta c_0)}{\gamma}\right] - A^2\Bigg\{c_0^2\left(\mu+\frac{1}{2}\right) \\ + \sigma k_0(1-2\delta c_0)\left[2\mathcal{K} - \frac{\mathcal{J}}{\gamma} + \mathcal{L} + \frac{5\mu}{2} + \frac{3\gamma}{8} + \frac{1}{2\gamma} + \frac{M_\infty^2 D}{8\gamma} + \frac{3}{8\gamma^3} M_\infty^2\Bigg\{(1-2\delta c_0)\right. \\ \left. + \frac{M_\infty^2(\gamma-2)}{3}(1-4\delta c_0)\Bigg\} - \frac{1}{8\gamma} M_\infty^2(1-2\delta c_0)\right] - \frac{3}{8}k_0^2\Bigg\} = 0 \end{aligned}$$

from which

$$c_1 = \sqrt{\beta\left(k_2 - \frac{\alpha}{\beta}A^2\right)}, \tag{5.3}$$

where

$$\beta = 2k_0 - \frac{\sigma}{\gamma}(1-2\delta c_0),$$

$$\begin{aligned}\alpha = \frac{3}{8}k_0^2 - c_0^2\left(\mu+\frac{1}{2}\right) - \sigma k_0(1-2\delta c_0)\Bigg[2\mathcal{K} - \frac{\mathcal{J}}{\gamma} + \mathcal{L} + \frac{5\mu}{2} \\ + \frac{3\gamma}{4} + \frac{1}{2\gamma} + \frac{M_\infty^2 D}{8\gamma} + \frac{3}{8\gamma^3} M_\infty^2\Bigg\{(1-2\delta c_0) \\ + \frac{M_\infty^2(\gamma-2)}{3}(1-4\delta c_0)\Bigg\} - \frac{1}{8\gamma} M_\infty^2(1-2\delta c_0).\end{aligned}$$

Again, a comparison of this result in the limit $\delta \Rightarrow 0$ with the corresponding one deduced by Nayfeh and Saric is not possible.

One has for $k \approx k_{0n_2}$ (the larger cut-off value)

$$k_2 < \frac{\alpha A^2}{\beta}: \quad \text{instability},$$

$$k_2 = \frac{\alpha A^2}{\beta}: \quad \text{neutral stability},$$

$$k_2 > \frac{\alpha A^2}{\beta}: \quad \text{stability}$$

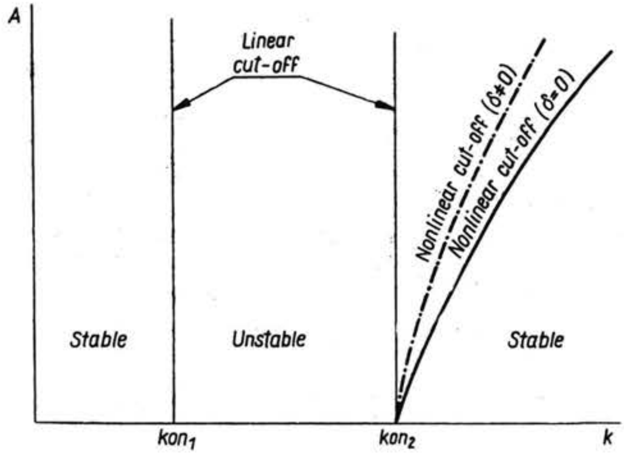

FIG. 2.

and corresponding to neutral stability, one has

$$k = k_{0n_2} + \varepsilon^2 \frac{\alpha}{\beta} A^2, \tag{5.4}$$

which is graphically represented in Fig. 2. Thus equilibriun solutions can exist for wavenumbers sufficiently close to the linear cut-off values, and these solutions have a definite minimum amplitude.

It is obvious that

1) the interfacial waves grow even at $k = k_{0n_2}$, despite the cut-off predicted by the linear theory, however, such unstable waves do not grow indefinitely in time but reach a steady-state amplitude,

2) since for $k \approx k_{0n_2}$, $c_0 = 0(\delta)$, for cases $\varrho'_g/\varrho'_l \ll 1$, one may approximate

$$\alpha \approx \frac{3}{8} k_0^2 - \frac{7}{4} c_0,$$
$$\beta \approx 2k_0$$

so that corresponding to the case $\delta \neq 0$, the unstable region shrinks further and therefore the inertial effects of the gas motion tend to be stabilising upon the interfacial wave-motion in the nonlinear case.

6. Nonlinear analysis for wavenumbers near the second-harmonic resonant values

It can be readily verified that corresponding to the second-harmonic resonant case,

$$k_{0s} = \pm \sqrt{\frac{1}{2}}, \tag{6.1}$$

the fundamental component

$$\eta_1^{(1)} = A \cos\xi, \quad \phi_1^{(1)} = \frac{A}{\gamma}(1-\delta c_0) e^{-\gamma y} \sin\xi,$$
$$\varphi_1^{(1)} = A c_0 e \sin\xi$$

and its second harmonic

$$\eta_1^{(1)} = \hat{B}\cos 2\xi, \quad \phi_1^{(2)} = \frac{\hat{B}}{\gamma}(1-\delta c_0)e^{-2\gamma y}\sin 2\xi,$$
$$\varphi_1^{(2)} = \hat{B}c_0 e^{2y}\sin 2\xi,$$

have the same linear wave velocity c_0.

In order to treat this case of nonlinear resonant interaction, put

$$\eta_1 = A\cos\xi + \hat{B}\cos 2\xi, \tag{6.2}$$

$$\varphi_1 = c_0[Ae^{y}\sin\xi + \hat{B}e^{2y}\sin 2\xi], \tag{6.3}$$

$$\phi_1 = \frac{(1-\delta c_0)}{\gamma}[Ae^{-\gamma y}\sin\xi + \hat{B}e^{-2\gamma y}\sin 2\xi]. \tag{6.4}$$

Using Eqs. (6.2)–(6.4), one finds from Eqs. (A.8)–(A.11), (A.13) and (A.14)

$$\varphi_2(\xi, y) = \left(-\frac{3A\hat{B}}{2}c_0 + Ac_1\right)e^{y}\sin\xi + (-A^2c_0 + 2c_1\hat{B})\frac{1}{2}e^{2y}\sin 2\xi + \text{higher harmonics}, \tag{6.5}$$

$$\begin{aligned}\phi_2(\xi, y) = {} & \Bigg[-\frac{M_\infty^2}{\gamma^3}2\delta Ac_1\left(y+\frac{1}{\gamma}\right)e^{-\gamma y} - \frac{M_\infty^2(1-3\delta c_0)}{8\gamma^4}\{(\gamma+1)-\gamma^2(\gamma-1)\} \\ & \times A\hat{B}(e^{-3\gamma y}-e^{-\gamma y}) - \frac{1}{\gamma}\left\{-\left(\frac{1-\delta c_0}{2}\right)3A\hat{B}\gamma + \delta c_1 A\right\}e^{-\gamma y}\Bigg]\sin\xi \\ & + \Bigg[-M_\infty^2(1-3\delta c_0)\frac{A^2}{4\gamma^2}\left\{-\frac{1}{2}(\gamma+1)+\frac{\gamma^2}{2}(\gamma+1)\right\}\left(y+\frac{1}{2\gamma}\right)e^{-2\gamma y} \\ & - \frac{1}{2\gamma}\{-(1-\delta c_0)\gamma A^2 + 2\delta_1 c\hat{B}\}e^{-2\gamma y} - \frac{M_\infty^2}{\gamma^2}2\delta\hat{B}c_1\left(y+\frac{1}{2\gamma}\right)e^{-2\gamma y}\Bigg]\sin 2\xi \\ & + \text{higher harmonics}.\end{aligned} \tag{6.6}$$

Using Eqs. (6.2)–(6.6), one finds that the removal of the secular terms in Eq. (A.12) requires that

$$k_0^2 p\hat{B} + 2k_0c_1Q + k_1\left[\frac{2\sigma(1-2\delta c_0)}{\gamma}\right] - 2k_0k_1 = 0, \tag{6.7}$$

$$\frac{k_0^2Ap}{2} + 4k_0\hat{B}c_1Q + 2\hat{B}k_1\left[\frac{2\sigma(1-2\delta c_0)}{\gamma}\right] - 8k_0k_1\hat{B} = 0, \tag{6.8}$$

where

$$p = \frac{1}{k_0^2}\left[-c_0^2 + \frac{k_0\sigma}{4\gamma^4}M_\infty^2(1-4\delta c_0)\{(\gamma+1)-\gamma^2(\gamma-1)\}\right],$$

$$Q = \frac{1}{2k_0}\left[2c_0 - \frac{2k_0\sigma\delta}{m}\left(1-\frac{M_\infty^2}{m^2}\right)\right] = \frac{\Delta}{k_0}.$$

From Eqs. (6.7) and (6.8), one obtains

$$\hat{B} = -\frac{k_1}{pk_0} \pm \sqrt{\frac{k_1^2}{p^2k_0} + \frac{A^2}{4}}, \tag{6.9}$$

$$c_1 = \frac{k_1(3k_0-p)}{2\Delta} \mp \frac{k_0^2 p}{2\Delta}\sqrt{\frac{k_1^2}{p^2k_0} + \frac{A^2}{4}}, \tag{6.10}$$

which for $k_1 = 0$, agree with those deduced by WILTON [6] long ago for the case of capillary waves on water. Thus purely phase-modulated waves are possible for wavenumbers near the second-harmonic resonant values. Note that these results are not valid if $k_{0n_{1,2}} = k_{0s}$.

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Appendix

$0(\varepsilon)$:

(A.1) $y < 0$: $\varphi_{1\xi\xi} + \varphi_{1yy} = 0,$

(A.2) $y > 0$: $\gamma_0^2\phi_{1\xi\xi} + \phi_{1yy} = 0,$

(A.3) $y = 0$: $\varphi_{1y} = -c_0\eta_{1\xi},$

(A.4) $y = 0$: $\phi_{1y} = (1-\delta c_0)\eta_{1\xi},$

(A.5) $y = 0$: $\eta_1 = c_0\varphi_{1\xi} + k_0\sigma(1-\delta c_0)\phi_{1\xi} + k_0^2\eta_{1\xi\xi},$

(A.6) $y \to -\infty$: $\varphi_{1y} \to 0,$

(A.7) $y \to \infty$: $\phi_{1y} \to 0.$

(cont. on p. 795)

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Internal effects of the gas motion upon the linear and nonlinear waves in Kelvin-Helmholtz flow

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Appendix (*)

[cont.]

$0(\varepsilon^2)$:

(A.8) $$y<0: \quad \varphi_{2\xi\xi}+\varphi_{2yy}=0,$$

(A.9) $$y>0: \quad \gamma_0^2\phi_{2\xi\xi}+\phi_{2yy}=M_\infty^2\{[(\gamma+1)(1-\delta c_0)\phi_{1\xi}-2\delta c_1]\phi_{1\xi\xi} + [(\gamma-1)(1-\delta c_0)\phi_{1\xi}]\phi_{1yy}+[2\phi_{1y}(1-\delta c_0)]\phi_{1\xi y}\},$$

(A.10) $$y=0: \quad \varphi_{2y}+\varphi_{1yy}\eta_1=-c_0\eta_{2\xi}+(\varphi_{1\xi}-c_1)\eta_{1\xi},$$

(A.11) $$y=0: \quad \phi_{2y}+\phi_{1yy}\eta_1=(1-\delta c_0)\eta_{2\xi}+(\phi_{1\xi}-\delta c_1)\eta_{1\xi},$$

(A.12) $$y=0: \quad \eta_2=c_0(\varphi_{2\xi}+\varphi_{1y\xi}\eta_1)-\frac{1}{2}(\varphi_{1\xi}^2+\varphi_{1y}^2)+c_1\varphi_{1\xi} + \sigma k_0[(1-\delta c_0)(\phi_{2\xi}+\phi_{1\xi y}\eta_1)+\frac{1}{2}(\gamma_0^2\phi_{1\xi}^2+\phi_{1y}^2)-\delta c_1\phi_{1\xi}] + k_0^2\eta_{2\xi\xi}+2k_0k_1\eta_{1\xi\xi}+\sigma k_1(1-\delta c_0)\phi_{1\xi},$$

(A.13) $$y\to-\infty: \quad \varphi_{2y}\to 0,$$

(A.14) $$y\to\infty: \quad \phi_{2y}\to 0,$$

$0(\varepsilon^3)$:

(A.15) $$y<0: \quad \varphi_{3\xi\xi}+\varphi_{3yy}=0,$$

(A.16) $$y>0: \quad \gamma_0^2\phi_{3\xi\xi}+\phi_{3yy}=M_\infty^2\Bigg\{\Bigg[(\gamma+1)(1-\delta c_0)\phi_{2\xi} - \delta c_1(\gamma+1)\phi_{1\xi}+\frac{\gamma+1}{2}\phi_{1\xi}^2+\frac{\gamma-1}{2}\phi_{1y}^2-2\delta c_2\Bigg]\phi_{1\xi\xi} + \Bigg[(\gamma-1)(1-\delta c_0)\phi_{2\xi}-\delta c_1(\gamma-1)\phi_{1\xi}+\frac{\gamma+1}{2}\phi_{1y}^2+\frac{\gamma-1}{2}\phi_{1\xi}^2\Bigg]\phi_{1yy} + [2\phi_{2y}(1-\delta c_0)+2\phi_{1y}(\phi_{1\xi}-\delta c_1)]\phi_{1\xi y}\Bigg\},$$

(*) This part of the Appendix has been sent by the author too late (July 1982) to be included in the original text (Ed. Com.).

(A.17) $y = 0$: $$\phi_{3y} + (\varphi_{2yy}\eta_1 + \varphi_{1yy}\eta_2) + \frac{1}{2}\varphi_{1yyy}\eta_1^2 = -c_0\eta_{3\xi} + (\varphi_{1\xi} - c_1)\eta_{2\xi} + (\varphi_{2\xi} + \varphi_{1\xi y}\eta_1 - c_2)\eta_{1\xi},$$

(A.18) $y = 0$: $$\phi_{3y} + (\phi_{2yy}\eta_1 + \phi_{1yy}\eta_2) + \frac{1}{2}\phi_{1yyy}\eta_1^2 = (1 - \delta c_0)\eta_{3\xi} + (\phi_{1\xi} - \delta c_1)\eta_{2\xi} + (\phi_{2\xi} + \phi_{1\xi y}\eta_1 - \delta c_2)\eta_{1\xi},$$

(A.19) $y = 0$: $$\eta_3 = c_0\left(\varphi_{3\xi} + \varphi_{2\xi y}\eta_1 + \varphi_{1\xi y}\eta_2 + \frac{1}{2}\varphi_{1\xi yy}\eta_1^2\right) + c_1(\varphi_{2\xi} + \varphi_{1\xi y}\eta_1) + c_2\varphi_{1\xi} - \varphi_{1\xi}(\varphi_{2\xi} + \varphi_{1\xi y}\eta_1) - \varphi_{1y}(\varphi_{2y} + \varphi_{1yy}\eta_1)$$
$$+ \sigma k_0\left[(1 - \delta c_0)\left(\phi_{3\xi} + \phi_{2\xi y}\eta_1 + \phi_{1\xi y}\eta_2 + \frac{1}{2}\phi_{1\xi yy}\eta_1^2\right) - \delta c_1(\phi_{2\xi} + \phi_{1\xi y}\eta_1 - M_{00}^2\phi_{1\xi}) - \delta c_2\phi_{1\xi} + \gamma_0^2\phi_{1\xi}(\phi_{2\xi} + \phi_{1\xi y}\eta_1) + \phi_{1y}(\phi_{2y} + \phi_{1yy}\eta_1) - \frac{1}{2}M_\infty^2\left\{(1 - \delta c_0) + M_\infty^2(\gamma - 2)\left(\frac{1}{3} - \delta c_0\right)\right\}\phi_{1\xi}^3 - \frac{1}{2}M_\infty^2(1 - \delta c_0)\phi_{1\xi}\phi_{1y}^2\right] + \sigma k_1\left[(1 - \delta c_0)(\phi_{2\xi} + \phi_{1\xi y}\eta_1) + \frac{1}{2}(\gamma^2\phi_{1\xi}^2 + \phi_{1y}^2) - \delta c_1\phi_{1\xi}\right] + \sigma k_2(1 - \delta c_0)\phi_{1\xi} + k_0^2\eta_{3\xi\xi}$$
$$+ 2k_0k_1\eta_{2\xi\xi} + 2k_0k_2\eta_{1\xi\xi} - \frac{3}{2}k_0^2\eta_{1\xi\xi}\eta_{1\xi}^2,$$

(A.20) $y \to -\infty$: $\varphi_{3y} \to 0$,

(A.21) $y \to \infty$: $\phi_{3y} \to 0$,

where
$$\gamma_0^2 = m^2 + 2\delta c_0 M_\infty^2.$$