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Flow and stability of second grade fluids between two parallel rotating plates

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An exact solution is given for the flow of an incompressible fluid of second grade between two infinite parallel plates rotating about a common axis. The stability of this flow subject to disturbances of finite amplitude is studied using the energy method and, further, the zone of sure stability is delineated.

Podano ścisłe rozwiązanie dla zagadnienia przepływu płynu nieściśliwego drugiego rzędu między dwiema równoległymi, nieskończonymi płytkami wirującymi wokół wspólnej osi. Stateczność tego przepływu, poddanego zakłóceniom o skończonych amplitudach, przeanalizowano za pomocą metody energetycznej, określając następnie obszar pewnej stateczności.

Приведено точное решение для задачи течения несжимаемой жидкости второго порядка между двумя параллельными, бесконечными плитами, вращающимися вокруг общей оси. Устойчивость этого течения, подвергнутого возмущениям с конечными амплитудами, проанализирована при помощи энергетического метода, определяя затем область надежной устойчивости.

1. Introduction

Abbot and Walters [1] established an exact solution for an incompressible viscous fluid between two infinite parallel plates which rotate with the same angular velocity about two noncoincident axes normal to the plates. Recently, Berker [2] exhibited the existence of an infinite set of nontrivial solutions for the flow of an incompressible viscous fluid between two parallel plates which rotate with constant angular velocity about a common axis normal to the plates. The trivial rigid body motion turns out to be a particular case of the above set. Berker [2] pointed out that his solution cannot be obtained as a limiting case of the solution due to Abbot and Walters. The study of these flow problems has relevance to the determination of the material moduli, which characterize a non-Newtonian fluid, in viscometric experiments.

In this paper we extend Berker's [2] boundary value problem to the study of a certain class of non-Newtonian fluids, namely the homogeneous incompressible Rivlin–Ericksen fluids of second grade. The equations of motion of these fluids are in general of higher order than the Navier–Stokes equations. Thus, if one does not use a perturbation approach, the solution of these equations will demand boundary conditions in addition to the usual no slip conditions. Besides the nonlinearities which occur in the Navier–Stokes equations, the equations of motion of an incompressible fluid of second grade contain higher-order nonlinearities which severely restrict the class of flows for which exact solutions can be found. For plane flows, Tanner [3] showed that the Stokes solution for the velocity field corresponding to an incompressible viscous fluid is also a solution to the equations of motion of a second grade fluid if the inertial effects can be neglected, the pressure field being different. In general, there are very few nonslow flows where exact solutions have been established for the equations of motion of a second grade fluid(1). Of course, exact solutions for the velocity field for unidirectional steady flows of the Navier–Stokes equations are also exact solutions of the equations of motion of a fluid of second grade, but these solutions for the velocity field do not depend on either of the normal stress moduli(2). In our problem we show that the velocity field is similar to that established by BERKER [2]. However, unlike the above mentioned "universal" flows, the normal stress modulus α_1 influences the velocity field. We also find that the structure of the pressure field is significantly different from that obtained by BERKER [2].

We have also studied the stability of the above flow using the energy method (cf. SERRIN [7]). As is to be expected, the stability of the flow in our problem depends on the viscosity μ and the normal stress modulus α_1 and also on the nature of the base flow. Sufficient conditions for the asymptotic stability in the mean of the base flow to arbitrary disturbances, in bounded domains, were established by DUNN and FOSDICK [8]. They showed that if the viscosity μ is sufficiently large and the base flow is slow enough in the sense that the eigenvalues of the first Rivlin–Ericksen tensor and the Laplacian of the first Rivlin–Ericksen tensor associated with the base flow are small enough, then the disturbances decay asymptotically. Since the flow domian in our problem is unbounded, we extend the analysis of Dunn and Fosdick to infinite domains. We then study the stability of the base flow in detail in terms of two nondimensional numbers R ($\Omega h^2/\nu$ — the Reynolds number based on the common angular velocity Ω and the distance between the plates) and a viscoelastic parameter Γ ($\alpha_1\Omega/\varrho\nu$, which is the ratio of elastic forces to the viscous forces). It is found that as the number Γ increases, the domain of sure stability decreases(3).

In the case of the trivial rigid body motion which is a member of the class of solutions studied by BERKER [2], it was shown by ELCRAT [10] that the solution is stable with respect to disturbances which go to zero sufficiently rapidly at infinity in the case of a classical viscous fluid. He also showed that when the angular velocities of the plates are different but sufficiently close, the flow is stable with respect to perturbations whose deformation energy is sufficiently confined to a core region. In addition to studying the stability of flow of the second grade fluid model which includes the classical viscous model as a special case, we study the stability of a wider class of flows which also includes Elcrat's analysis.

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(1) There exists a general class of plane flows where exact solutions can be established for the equations of motion of an incompressible fluid of second grade by virtue of the higher order nonlinearities being self-cancelling though individually nonvanishing (cf. [4]). Also an exact solution has been established in the case of flow between infinite parallel plates rotating about non-coincident axes [5].

(2) It is well known (cf. FOSDICK and TRUESDELL [6]) that all "universal" solutions to the Navier–Stokes equations also satisfy the equations of motion of a fluid of second grade and the velocity field does not depend on either of the normal stress moduli. We use the terminology *second grade* for fluids whose material model α_1 and α_2 satisfy $\alpha_1 \geqslant 0$ and $\alpha_1+\alpha_2 = 0$.

(3) It was shown in [9] that the uniqueness of the solutions for the equations of motion for a fluid of second grade for plane flows depends on Γ.

After a few preliminary remarks and the statement of the problem in Sect. 2, we prove the main theorem regarding the existence of an infinity of velocity-pressure pairs which satisfy the equations of motion of a fluid of second grade in Sect. 1. Finally, in Sect. 4 we study the stability of these solutions.

2. Preliminaries

The Cauchy stress $\mathbf{T}$ in a homogeneous incompressible Rivlin–Ericksen fluid of second grade is related to the fluid motion in the following manner (cf. [11], [12])

$$\mathbf{T} = -p\mathbf{1}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_1\mathbf{A}_1^2, \tag{2.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli, $-p\mathbf{1}$ denotes the indeterminate pressure and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematical Rivlin–Ericksen tensors defined through

$$\mathbf{A}_1 = \operatorname{grad}\mathbf{v}+(\operatorname{grad}\mathbf{v})^T, \tag*{(2.2)$_1$}$$

and

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1+\mathbf{A}_1(\operatorname{grad}\mathbf{v})+(\operatorname{grad}\mathbf{v})^T\mathbf{A}_1, \tag*{(2.2)$_2$}$$

where the dot denotes material time differentiation and $\mathbf{v}$ denotes the velocity field.

The constitutive model (2.1) can be considered as a second-order approximation to the response functional of a simple fluid in the sense of retardation (cf. COLEMAN and NOLL [12]). However, since the model is properly frame-invariant, it can be also considered as an exact model for some fluid as is done for example when $\alpha_1 = \alpha_2 = 0$, i.e. the case of the classical Navier–Stokes theory. When the model (2.1) is required to be compatible with thermodynamics in the sense that all motions of the fluid meet the Clausius–Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid be a minimum when the fluid is locally at rest under isothermal conditions, it follows that the material moduli have to meet the following restrictions (cf. DUNN and FOSDICK [8]):

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0 \quad \text{and} \quad \alpha_1+\alpha_2 = 0. \tag{2.3}$$

The results expressed in Eq. (2.3) are the subject of much controversy and involves the works of Coleman, Dunn, Fosdick, Mizel, Noll, Rajagopal, Ting, Truesdell and others. We refer the reader to [8] for a discussion of the same. Henceforth we shall assume (2.3) holds.

We now develop the main field equations for the velocity $\mathbf{v}$. When the constitutive expression (2.1) is substituted into the balance of linear momentum

$$\operatorname{div}\mathbf{T}+\varrho\mathbf{b} = \varrho\dot{\mathbf{v}}, \tag{2.4}$$

where $\mathbf{b}$ denotes the body force field, one obtains that

$$\mu\Delta\mathbf{v}+\alpha_1(\Delta\boldsymbol{\omega}\times\mathbf{v})+\alpha_1\Delta\mathbf{v}_t-\varrho\mathbf{v}_t-\varrho(\boldsymbol{\omega}\times\mathbf{v}) = \operatorname{grad}\hat{P}, \tag{2.5}$$

where

$$\hat{P} \equiv p-\alpha_1[(\mathbf{v}\cdot\Delta\mathbf{v})+\frac{1}{4}|\mathbf{A}_1|^2]+\varrho\phi+\frac{1}{2}\varrho|\mathbf{v}|^2 \tag{2.6}$$

and

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}.$$

Here we have assumed that $\mathbf{b}$ is conservative, so that $\mathbf{b} = -\operatorname{grad}\phi$. Also the suffix t denotes the partial derivative with respect to time and $|\mathbf{A}_1|^2$ denotes the usual trace norm for the tensor $\mathbf{A}_1$. Since the fluid is incompressible, it can undergo only isochoric motions and hence

$$\operatorname{div}\mathbf{v} = 0. \tag{2.7}$$

While determining the velocity field, we shall find it convenient to eliminate the gradient of pressure by operating on Eq. (2.5) by the curl operator:

$$\mu\Delta\boldsymbol{\omega} + \alpha_1 \operatorname{curl}(\Delta\boldsymbol{\omega}\times\mathbf{v}) + \alpha_1\Delta\boldsymbol{\omega}_t - \varrho\boldsymbol{\omega}_t - \varrho\operatorname{curl}(\boldsymbol{\omega}\times\mathbf{v}) = 0. \tag{2.8}$$

We conclude this section with a formal statement of the boundary value problem. We wish to determine the velocity-pressure pair which satisfies the equations of motion (2.5) for an incompressible second grade fluid for the problem of the flow between two parallel infinite plates rotating with constant angular velocity Ω about a common fixed axis normal to the plates. A Cartesian coordinate system $Oxyz$ with the z-axis in the direction of the axis of rotation is located so that the equations of the top and bottom plates correspond to $z = h$ and $z = -h$, respectively. We are interested in motions wherein streamlines in any z = constant plane are concentric circles. The locus of the centers of these circles as the z = constant plane shifts from $z = -h$ to $z = h$ is in general a curve in space. From a physical point of view this curve represents the axis of a curvilinear vortex. Following Berker [2], we shall seek steady solutions for the velocity field of the form(4)

$$u = -\Omega\big(y-g(z)\big), \tag*{(2.9)$_1$}$$

$$v = \Omega\big(x-f(z)\big), \tag*{(2.9)$_2$}$$

and

$$w = 0, \tag*{(2.9)$_3$}$$

where u, v, and w are the components of velocity $\mathbf{v}$ in the x, y, and z coordinate directions, respectively. Here $x = f(z)$ and $y = g(z)$ are the equations which define the locus of the centers. Since the locus passes through $(0, 0, -h)$ and $(0, 0, h)$, it follows from the no-slip conditions at the two plates and Eqs. $(2.9)_{1,2,3}$ that

$$f(h) = f(-h) = 0, \tag*{(2.10)$_1$}$$

and

$$g(h) = g(-h) = 0. \tag*{(2.10)$_2$}$$

If the locus of the centers intersects the $z = 0$ plane at the point Q with the coordinates $(l, 0, 0)$ (one can always choose the x and y axes in such a manner that this is so), where $l \geqslant 0$, it follows that

$$f(0) = l, \quad g(0) = 0, \tag*{(2.11)$_{1,2}$}$$

since the velocity of the fluid at Q is zero. Equations $(2.10)_{1,2}$ and $(2.11)_{1,2}$ provide the boundary conditions which are necessary to obtain the velocity field from Eqs. (2.7) and (2.8).

(4) This class of motions belong to the family of pseudo-plane motions considered by Berker [13].

3. Exact solutions

In this section we obtain an infinite set of exact solutions to Eq. (2.4) and (2.8). Since we seek solutions of the form $(2.9)_{1,2,3}$, on substituting the same into Eq. (2.8) we find that

$$\mu\Omega f''' + \alpha_1\Omega^2 g''' + \varrho\Omega^2 g' = 0, \tag{3.1$_1$}$$

and

$$\mu\Omega g''' - \alpha_1\Omega^2 f''' - \varrho\Omega^2 f' = 0, \tag{3.1$_2$}$$

where the prime denotes differentiation with respect to z. The velocity field given by Eqs. $(2.9)_{1,2,3}$ is clearly compatible with Eq. (2.7). Note that the order of the differential equation which is obtained when $\alpha_1 \neq 0$ is the same as that when $\alpha_1 = 0$. Thus the usual no-slip boundary conditions will suffice to determine the velocity field completely.

On defining the function F through $F = f + ig$, where $i = \sqrt{-1}$, it follows from Eqs. $(3.1)_1$ and $(3.1)_2$ that

$$F''' - (m+in)^2 F' = 0, \tag{3.2}$$

where m and n are defined through

$$m^2 = \frac{\varrho\{[(\mu/\Omega)^2 + \alpha_1^2]^{1/2} - \alpha_1\}}{2[(\mu/\Omega)^2 + \alpha_1^2]}, \tag{3.3$_1$}$$

and

$$n^2 = \frac{\varrho\{[(\mu/\Omega)^2 + \alpha_1^2]^{1/2} + \alpha_1\}}{2[(\mu/\Omega)^2 + \alpha_1^2]}. \tag{3.3$_2$}$$

On integrating (3.2) one obtains that

$$F(z) = C_1 e^{(m+in)z} + C_2 e^{-(m+in)z} + C_3. \tag{3.4}$$

We obtain the boundary conditions which are required to determine the constants C_1, C_2 and C_3 from Eqs. $(2.10)_{1,2}$ and $(2.11)_{1,2}$ which imply that

$$F(0) = l, \tag{3.5}$$

and

$$F(h) = F(-h) = 0. \tag{3.6}$$

It follows from Eqs. (3.4), (3.5) and (3.6) that

$$C_1 = C_2 = \frac{l}{2[1-\cosh(m+in)h]} \tag{3.7$_1$}$$

and

$$C_3 = \frac{l\cosh(m+in)h}{[\cosh(m+in)h-1]}. \tag{3.7$_2$}$$

On substituting Eqs. $(3.7)_1$ and $(3.7)_2$ into Eq. (3.4) we find that

$$F(z) = \frac{l\{\cosh(m+in)h - \cosh(m+in)z\}}{[\cosh(m+in)h-1]}. \tag{3.8}$$

A straightforward computation from Eq. (3.8) yields

$$(3.9)_1 \quad \frac{f(z)}{l} = \frac{1}{(\cosh mh \cos nh - 1)^2 + (\sinh mh \sin nh)^2} \{(\cosh mh \cos nh - \cosh mz \times \cos nz)(\cosh mh \cos nh - 1) + (\sinh mh \sin nh - \sinh mz \sin nz)(\sinh mh \sin nh)\}$$

and

$$(3.9)_2 \quad \frac{g(z)}{l} = \frac{1}{(\cosh mh \cos nh - 1)^2 + (\sinh mh \sin nh)^2} (\sinh mh \sin nh - \sinh mz \times \sin nz)(\cosh mh \cos nh - 1) - (\cosh mh \cos nh - \cosh nz \cos nz)(\sinh mh \sin nh)\},$$

where

$$(3.10)_1 \qquad mh = \frac{R}{2(1+\Gamma^2)}[(1+\Gamma^2)^{1/2} - \Gamma],$$

and

$$(3.10)_2 \qquad nh = \frac{R}{2(1+\Gamma^2)}[(1+\Gamma^2)^{1/2} + \Gamma].$$

These expressions for mh and nh follow from Eqs. $(3.3)_1$ and $(3.3)_2$. In the above equations, $R(=\Omega h^2/\nu)$ is a Reynolds number based on the angular velocity and the distance between the plates, and $\Gamma(=\alpha_1\Omega/\varrho\nu)$ is a vicsoelastic parameter characterizing the ratio of the elastic forces to the viscous forces.

We next derive an expression for the pressure field. It follows from Eq. (2.5) that

$$(3.11)_1 \qquad \frac{1}{\varrho}\frac{\partial P}{\partial x} = \Omega^2 x + \Omega(\nu g'' - \Omega f),$$

$$(3.11)_2 \qquad \frac{1}{\varrho}\frac{\partial P}{\partial y} = \Omega^2 y - \Omega(\nu f'' + \Omega g),$$

and

$$(3.11)_3 \qquad \frac{1}{\varrho}\frac{\partial P}{\partial z} = \frac{\alpha_1}{\varrho}\Omega^2[f'''(x-f) + g'''(y-g)],$$

where

$$(3.11)_4 \qquad P \equiv p - \alpha_1[(\mathbf{v}\cdot\Delta\mathbf{v}) + \frac{1}{4}|\mathbf{A}_1|^2] + \varrho\phi.$$

Note that $\hat{P} = P + \frac{1}{2}\varrho|\mathbf{v}|^2$. On integrating Eqs. $(3.11)_{1,2,3}$ it follows that

$$(3.12) \quad \frac{1}{\varrho}P = \frac{\Omega^2}{2}[(x-h_1(z))^2 + (y-h_2(z))^2] - \frac{\Omega^2}{2}[h_1^2(z) + h_2^2(z)] + \frac{\alpha_1\Omega^2}{\varrho}\{xf'' + yg''\} + \frac{\alpha_1\Omega^2}{\varrho}\{gg'' - \frac{1}{2}(g')^2 - ff'' + \frac{1}{2}(f')^2\} + C,$$

where $h_1(z)$ and $h_2(z)$ are defined through

$$(3.13)_1 \qquad h_1(z) = -\nu g''(z) + \Omega f(z),$$

$$(3.13)_2 \qquad h_2(z) = \nu f''(z) + \Omega g(z),$$

and C is an arbitrary constant.

Hence, we have proved the following:

THEOREM 1. *Let an incompressible fluid of second grade occupy the region between two infinite parallel plates, distant 2h apart, rotating about a fixed normal axis with constant angular velocity* Ω. *Let the coordinate axes Oxyz be so chosen that the z-axis coincides with the axis of rotation and the equations of the two plates are* $z = h$ *and* $z = -h$. *If Q is an arbitrary point with the coordinates* $(l, 0, 0)$, $l \geq 0$, *then the velocity-pressure pair for the flow is defined through* Eqs. $(2.9)_{1,2,3}$ *and* (3.12) *where the functions* f, g, h_1 *and* h_2 *are defined through* Eqs. $(3.9)_{1,2}$, (3.10), $(3.13)_1$ and $(3.13)_2$, *respectively.*

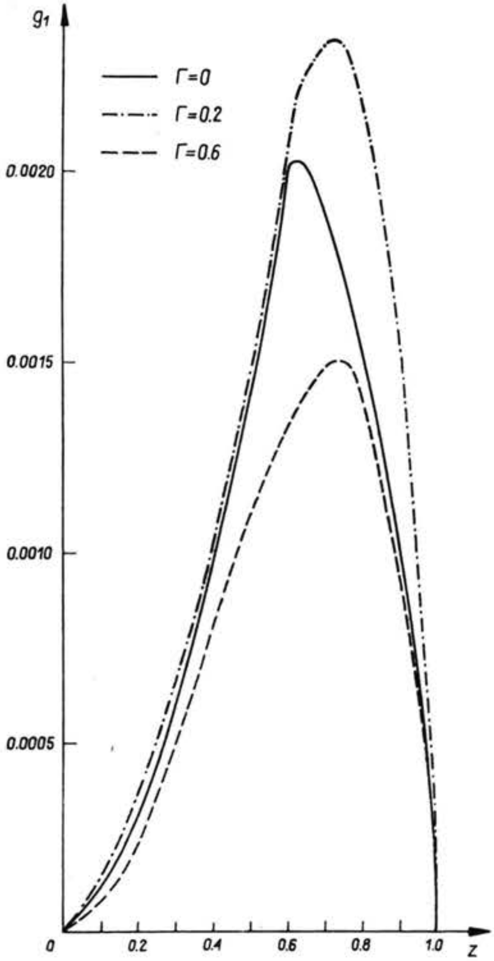

FIG. 1.

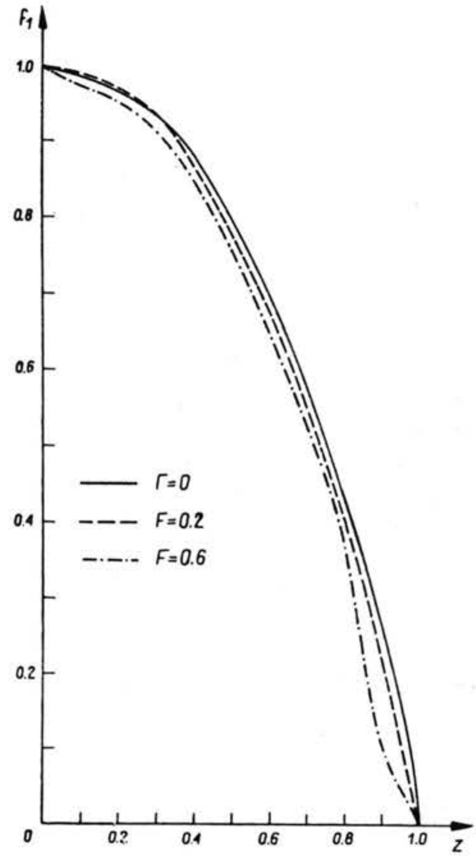

FIG. 2.

It may be noticed that the velocity field **v** defined through Eqs. $(2.9)_{1,2,3}$ depends on the normal stress modulus α_1. These expressions agree with BERKER [2] when $\alpha_1 = 0$. This result is in sharp contrast with the velocity field in steady unidirectional flows as observed earlier. Figures 1 and 2 display the variation of the functions f/l and g/l versus z for various values of the viscoelastic nondimensional parameter Γ for a fixed value of the Reynolds number R. As the functions f and g are even, f and g are symmetric with respect to the $z = 0$ plane. It is found that the magnitude of g/l is very small everywhere. Figure 1 shows that over a major portion of the flow domain f/l decreases, i.e. the y-component of the velocity increases with increasing Γ. It can be seen from Fig. 1 that the curve for $\Gamma = 0.6$ has an inflection point near the upper plate. Figure 1 shows that f decreases with increasing Γ over a major portion of the flow.

Equation (4.12) shows that in any $z =$ constant plane the curves $P =$ constant are circles. Note that this result is different from Berker's in that the modified pressure P defined in Eq. (3.11) involves the normal stress modulus α_1. More importantly, it follows from Eq. (3.12) that $\partial P/\partial z$ is not zero unlike the result obtained by Berker. In fact it is a non-constant function of z. This implies that the contribution due to the pressure to the normal forces exerted on the top and bottom plates are different. Hence, in order to keep the plates at a fixed distance $2h$ apart, different forces are to be exerted at the top and bottom plates. From the velocity field we can determine the stress at any point on the plate (and hence the torque and the normal force exerted by the fluid on the plate). On the other hand, it is possible to measure experimentally the stress on a rotating plate by using the orthogonal rheometer of MAXWELL and CHARTOFF [14]. This will enable us to determine the normal stress moduli. Extensive work has been done in this area, the details of which can be found in HUILGOL [15].

4. Stability analysis

In this concluding section we study the stability of base flows to arbitrary disturbances. Let $\mathbf{v}$ denote the velocity of the base flow and p the associated pressure field which satisfy Eqs. (2.5) and (2.7). Let $\mathbf{v}'$ and p' denote another velocity-pressure pair obeying Eqs. (2.5). and (2.7) and the same boundary conditions as those satisfied by $(\mathbf{v}, p)$ but possibly different initial conditions. We shall denote the difference fields $(\mathbf{u}, \hat{p})$ through

$$\mathbf{u} \equiv \mathbf{v}' - \mathbf{v}, \quad \hat{p} = p' - p, \tag{4.1}$$

DUNN and FOSDICK [8] established sufficient conditions for the asymptotic stability of the base flow to arbitrary disturbances. They showed that in bounded domains if (cf. Theorem [8]) the material moduli are such that

$$\frac{2\alpha_1}{\varrho}(M+\overline{M})+\varkappa\left[\frac{\alpha_1}{\varrho}N+\overline{M}\right]-\frac{2\mu}{\varrho} < 0, \tag{4.2}$$

where M and $-\overline{M}$ denote the maximum and minimum of the eigenvalues of the Rivlin–Erickson tensor $\mathbf{A}_1$ associated with the base flow, N denotes the maximum of the eigenvalue of $\Delta\mathbf{A}_1$ and $\varkappa$ the Poincaré coefficient for the domain, then the base flow is stable in the mean in the sense that

$$\left[\|\mathbf{u}(t)\|^2+\frac{\alpha_1}{\varrho}\|\operatorname{grad}\mathbf{u}(t)\|^2\right] \leqslant \left[\|\mathbf{u}(0)\|^2+\frac{\alpha_1}{\varrho}\|\operatorname{grad}\mathbf{u}(\mathrm{o})\|^2\right]e^{\gamma t}, \tag{4.3}$$

where the norm $\|\cdot\|$ is defined as

$$\|\mathbf{f}(t)\|^2 = \int_B |\mathbf{f}(\mathbf{x}, t)|^2 dv.$$

The positive definite quantity $\|\mathbf{u}(t)\|^2+\frac{\alpha_1}{\varrho}\|\operatorname{grad}\mathbf{u}(t)\|^2$ is a measure of the kinetic energy and the energy due to stretching in the fluid. If Eq. (4.2) holds. then the exponent γ which appears as the exponent in Eq. (4.3) is negative and hence both $||\mathbf{u}(t)||^2$ and $||\operatorname{grad}\mathbf{u}(t)||^2$ are bounded by an exponentially decaying function.

We make the following assumptions regarding the asymptotic behavior of $\mathbf{u}$ and p to extend the result of Dunn and Fosdick to unbounded domains, namely

$$\mathbf{u} = 0(r^{-k}), \quad \operatorname{grad}\mathbf{u} = 0(r^{-k-1}) \quad \text{and} \quad \hat{P} = P'-P = 0(r^{-k+1}), \tag{4.4}$$

where $k > 1$. These conditions will ensure that appropriate surface integrals will vanish in the limit $r \to \infty$. It was shown by SERRIN [16] that for the flow between two infinite parallel plates, the Poincaré constant $\varkappa$ for the domain is given through $\varkappa = 4h^2/\pi^2$, i.e.,

$$\int_{\Omega} |\mathbf{v}(\mathbf{x}, t)|^2 dv \leqslant \frac{4h^2}{\pi^2} \int_{\Omega} |\operatorname{grad}\mathbf{v}(\mathbf{x}, t)|^2 dv. \tag{4.5}$$

It can be easily shown that Eq. (4.2) follows from Eq. (4.4) and (4.5) on writing the equations of motion (2.5) for $(\mathbf{v}, p)$ and $(\mathbf{v}', p')$, subtracting the former from the latter and forming the scalar product of the same with $\mathbf{u}$ and integrating over the domain.

It is our aim to study the implications of Eq. (4.2) in detail. A straightforward computation from Eqs. $(2.9)_{1,2,3}$ yields

$$M = M \geqslant \Omega\{(g')^2+(f')^2\}^{1/2}, \quad -h \leqslant z \leqslant h, \tag{4.6$_1$}$$

and

$$N \geqslant \Omega\{(g''')^2+(f''')^2\}^{1/2}, \quad -h \leqslant z \leqslant h. \tag{4.6$_2$}$$

On substituting Eqs. $(3.9)_1$ and $(3.9)_2$ for f and g into Eqs. $(4.6)_1$ and $(4.6)_2$, it follows that

$$M = \overline{M} \geqslant 2\Omega[(mK-nS)^2+(nK+mS)^2]^{1/2}\{\sinh^2 mz\cos^2 nz+\cosh^2 mz\sin^2 nz\}, \quad -h \leqslant z \leqslant h, \tag{4.7$_1$}$$

and

$$N \geqslant \Omega\{[Km^3-3Kmn^2-3Sm^2n+Sn^3]^2+[Kn^3-3Km^2n+3Smn^2-Sm^3]^2\}^{1/2} \times \{\sinh^2 mz\cos^2 nz+\cosh^2 mz\sin^2 nz\}, -h \leqslant z \leqslant h, \tag{4.7$_2$}$$

where

$$K = \frac{l[1-\cosh mh\cos nh]}{(1-\cosh mh\cos nh)^2+(\sinh mh\sin nh)^2}, \tag{4.7$_3$}$$

and

$$S = \frac{l\sinh mh\sin nh}{(1-\cosh mh\cos nh)^2+(\sinh mh\sin nh)^2}. \tag{4.7$_4$}$$

We are now in a position to prove the following:

THEOREM 2. *Let $(\mathbf{v}, p)$ be a velocity-pressure pair which is a solution to the boundary value problem being considered. Suppose $(\mathbf{u}, p)$ is a disturbance pair which obeys the asymptotic structure* (4.3). *Then the basic flow $(\mathbf{v}, p)$ is asymptotically stable in the mean with respect to arbitrary disturbances $(\mathbf{u}, \hat{p})$ if*

$$(4.8)\quad \left[\left(\frac{2\alpha_1\Omega}{\varrho\nu}+\frac{2h^2\Omega}{\pi^2\nu}\right)\{mK-nS)^2+(nK+mS)^2\}^{1/2}+\right.$$
$$+\frac{2h^2\alpha_1\Omega}{\pi^2\varrho\nu}\{(Km^3-3Kmn^2-3Sm^2n+Sn^3)^2+$$
$$\left.+(Kn^3-3Km^2n+3Smn^2-Sm^3)^2\}^{1/2}\right](\cosh^2 nh-\cos^2 nh)^{1/2}<1,$$

where m, n, K and S are defined by Eqs. $(3.3)_1$, $(3.3)_2$, $(4.7)_3$ *and* $(4.7)_4$, respectively.

P r o o f. On substituting for M, $\overline{M}$ and N from Eq. $(4.7)_{1,2}$ into Eq. (4.2), it follows that

$$(4.9)\quad \left[\left(\frac{2\alpha_1\Omega}{\varrho\nu}+\frac{2h^2\Omega}{\pi^2\nu}\right)\{(mK-nS)^2+(nK+mS)^2\}^{1/2}\right.$$
$$+\frac{2h^2\alpha_1\Omega}{\pi^2\varrho\nu}\{(Km^3-3Kmn^2-3Smn^2-3Sm^2n+Sn^3)^2$$
$$\left.+(Kn^3-3Km^2n+3Smn^2-Sm^3)^2\}^{1/2}\right](\cosh^2 mz-\cos^2 nz)^{1/2}<1.$$

Next, observe that for any $n>0$, $-h\leqslant z\leqslant h$,

$$(4.10)\qquad \cosh^2 nz-\cos^2 nz\leqslant\cosh^2 nh-\cos^2 nh.$$

Since $\alpha_1\geqslant 0$, by $(3.3)_1$ and $(3.3)_2$,

$$\cosh^2 mz\leqslant\cosh^2 nz,$$

and thus

$$\cosh^2 mz-\cos^2 nz\leqslant\cosh^2 nh-\cos^2 nh.$$

Hence the theorem.

REMARK. For large values of α_1, a sharper bound for $\cosh^2 mz-\cos^2 nz$ can be obtained as $\cosh^2 mh$. However, if α_1 is small in comparison with μ/Ω, the bound provided in the theorem would be sharper.

We now analyse in detail the implications of the above theorem. On substituting for K and S from Eqs. $(4.7)_3$ and $(4.7)_4$ into Eq. (4.8) and simplifying, we find that

$$\frac{l}{h}\left\{\left(2\Gamma+\frac{2R}{\pi^2}\right)\{[mh(1-\cosh mh\cos nh)-nh\sinh mh\sin nh]^2\right.$$
$$+[nh(1-\cosh mh\cos nh)+mh\sinh mh\sin nh]^2\}^{1/2}$$
$$+\frac{2\Gamma}{\pi^2}\{[(mh)^3(1-\cosh mh\cos nh)-3mh(nh)^2(1-\cosh mh\cos nh)$$
$$-3(mh)^2(nh)\sinh mh\sin nh+(nh)^3\sinh mh\sin nh]^2$$
$$+[(nh)^3(1-\cosh mh\cos nh)-3(mh)^3nh(1-\cosh mh\cos nh)$$
$$+3(mh)(nh)^2\sinh mh\sin nh-(mh)^3\sinh mh\sin nh]^2\}^{1/2}$$
$$<[\cosh^2 nh-\cos^2 nh]^{-1/2}[(1-\cosh mh\cos nh)^2+\sinh^2 mh\sin^2 nh],$$

where mh and nh are defined through Eqs. $(3.10)_1$ and $(3.10)_2$, respectively.

Figure 3 displays the variation of l/h vs R for several values of Γ. It can be seen that the domain of sure stability decreases with increasing Γ. It is interesting to note that the region of sure stability is significantly reduced when Γ changes from zero to a small value. However, further increase in Γ results in marginal decrease in the region of sure stability. This suggests that the viscoelasticity could possibly exert a destabilizing influence. However, this remark is of a conjectural nature because the above estimate is based on a suf-

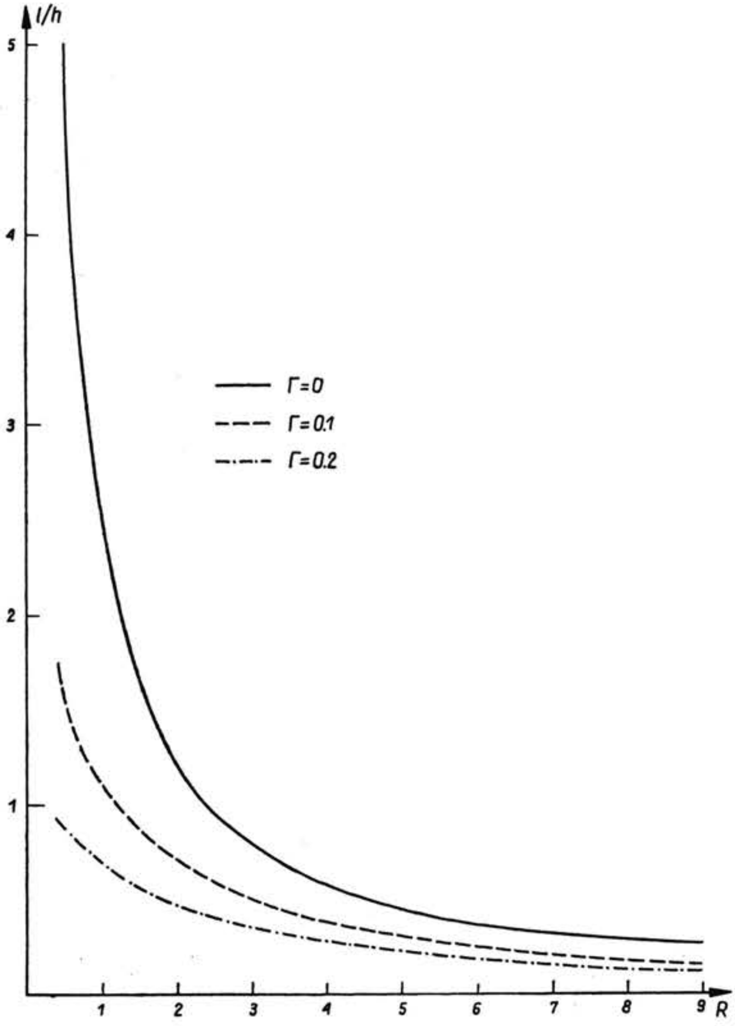

FIG. 3.

ficient condition for stability. In the event that there be no subcritical instability (i.e. the stability boundaries obtained from the finite amplitude and the linearised analysis coincide) the above result would indicate the destabilising influence of viscoelasticity. It may also be seen from Fig. 3 that the trivial rigid body motion corresponding to $l = 0$ is likely to be most stable since the Reynolds number associated with it can be the largest.

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