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Deflections of elastic-plastic hyperstatic beams under cyclic loading

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EXPERIMENTS on shakedown and incremental collapse are described. The test data are presented and discussed as obtained for two-span small-scale continuous beams subjected to variable repeated loads. The loading is such that one span remains under a sustained load whereas the other is subjected to cycling. The magnitudes of elastic-plastic deflections of mid spans are compared with upper bounds to shakedown deformations. The results show clearly the importance of the spread of plastic zones on the stiffness of an elastic-plastic structure. The plastic hinges hypothesis does not appear appropriate to evaluate deflection in elastic-plastic structures.

Przedstawiono badania doświadczalne przystosowania się i zniszczenia przyrostowego sprężysto-plastycznych belek. Wyniki badań prezentowane są i dyskutowane dla dwuprzęsłowych belek obciążanych cyklicznie. Rejestrowane wielkości sprężysto-plastycznych ugięć są porównane z teoretycznie otrzymanym górnym oszacowaniem ugięć. Wyniki pokazują wpływ rozprzestrzeniania się stref plastycznych na zmianę sztywności konstrukcji. Z porównania wynika, że założenie idealnych przegubów plastycznych jest nieodpowiednie do analizy przemieszczeń sprężysto-plastycznych konstrukcji.

Представлены экспериментальные исследования приспособления и прогрессирующего разрушения упруго-пластических балок. Результаты исследований представлены и обсуждены для двухпролетных сплошных балок нагруженных циклически. Регистрированные величины упруго-пластических прогибов сравнены с теоретически полученной верхней оценкой прогибов. Результаты показывают важность влияния распространения пластических зон на изменение жесткости конструкции. Из сравнения следует, что предположение идеальных пластических шарниров является неподходящим для анализа перемещений упруго-пластических конструкций.

1. Introduction

THE PLASTIC design method concerns, as a rule, the design of structures intended to resist the maximum possible level of proportional loads. It is presumed, therefore, that all the loads on a given structure increase in a fixed ratio from zero to the value prescribed by the limit analysis theory. The theoretical framework of the theory of limit analysis can be found in [1, 10, 11].

The structures subjected both to variable (live) and sustained (dead) loading require a closer examination so that the effects of nonproportional loading on permanent deflections could be evaluated or at least estimated.

A redundant structure subjected to variable repeated loading either shakes down to an elastic state after previous excursions into the inelastic range or fails. The failure, in turn, may take place either by an incremental collapse resulting in displacements increasing beyond all bounds or by an alternating plasticity when the low cycle fatigue occurs. It is therefore apparent that a hyperstatic structure must be proportioned so that it shakes down when subjected to variable repeated loading. The Polish Code for Steel Structures [18] is specific as to this point.

The classical theory of shakedown is furnished with a set of dual theorems allowing to estimate whether a given structure will or will not shake down to loads varying arbitrarily within the given limits, MELAN [17], KOITER [12]. However, no information is available as to deflections of a structure prior to the state of shakedown. Only recently studies have appeared regarding methods of estimation of the shakedown deflections [13, 15, 23, 19, 4, 5, 6]. Applications of these methods in structural design requires experimental research allowing to estimate the practical usefulness and accuracy of the bounding principles developed.

The available experimental data on shakedown can be arranged into two groups. The first one concerns the behaviour of machine parts subjected to cyclic heating. This group is specifically focused on ratchetting and on the increase of permanent strains beyond all bounds. Ample information regarding these questions can be found in [9]. In the second group beams and frames made of commercially available steels and subjected to repeating loading were studied [2, 7, 8, 16, 22]. The state of the art report regarding shakedown is presented in [21 and 14].

The present note describes experiments on the behaviour of continuous beams beyond the elastic range. The paper [3] reports experiments on beams of rectangular cross section, subjected to loads increasing in proportion. The obtained experimental material was presented there quite extensively for further comparisons with that on the behaviour of the same beams under repeated loads.

The purpose of this study is to investigate the performance of continuous two-span beams with respect to shakedown. Special attention will be paid to elastic-plastic displacements within the shakedown range. Test results are reported and discussed for two-span beams subjected to load cycling.

A description of tested beams is given in Sect. 2 and the test stand is described in the next section. This allows to present the test program in Sect. 4. Section 5 gives the test results referring to the shakedown and incremental collapse. Comparisons of the experimental data with the predictions of earlier developed analytical methods is given in Sect. 6. The conclusions are given in the last section.

2. Tested beams

All the beams of cross section 20×10 mm were machined from a metal bar of mild steel St 35 in the rolling direction. The surfaces were polished. The cross section boundaries deviated at the most ± 0.05 mm from the required dimensions [3]. The specimens were annealed during one hour at the temperature 650°C and cooled to the ambient temperature during 30 hours.

The material properties tests provided the following characteristics: the Young modulus $E = 210\,680$ MPa, the yield point $\sigma_0 = 315.29$ MPa, the elastic deformation at the yield point $\varepsilon_E = 1498 \cdot 10^{-6}$, the plastic platform $\varepsilon_p = 2.92\varepsilon_E$.

The span to depth ratio of 20 was selected for the tested two-span beams in order to arrive at failure only by excessive deflections. A tested beam under loading arrangement is schematically shown in Fig. 1. The full plastic moment of a beam is $M_0 = 15\,764$ Ncm and the maximum elastic moment $M_E = 2/3\ M_0$.

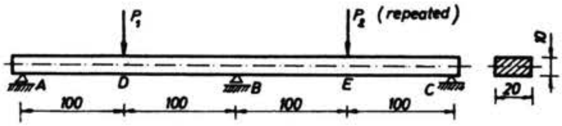

FIG. 1 Tested beam.

3. Test stand

The test stand was essentially the same as for the static bending tests, for which the details are summarized in [3]. A modification was introduced allowing for a cyclic variation of the loading, Fig. 2. The stand consists of a stiff supporting frame (*1*), two hinged

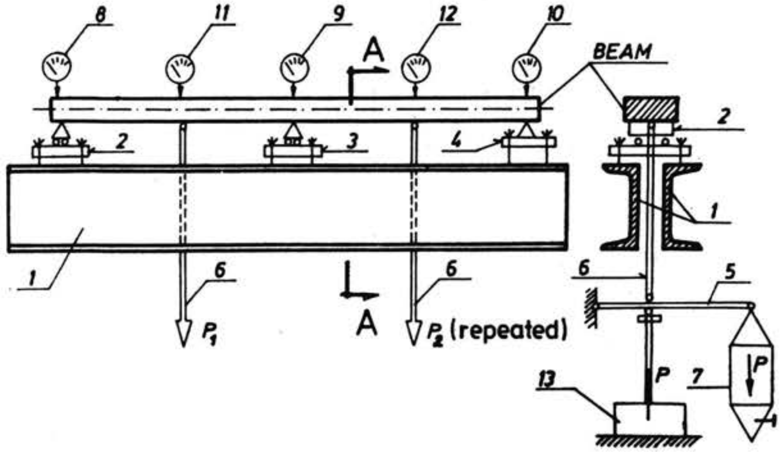

FIG. 2. Test stand.

supports (*2*) and (*3*) and an immovable hinged support (*4*). The loading device is made of two levers (*5*) and a tendon (*6*). The loads P_1 and P_2 are increased when tokens are put in the containers (*7*). The repeated load P_2 is realized by a hydraulic hand pump (*13*). Deflection measurements are made by means of dial gauges at each point of loading (*11*), (*12*) as well as at the supports (*8*), (*9*) and (*10*). This type of loading arragement allows to have a prescribed static load. All dynamics effects are attenuated before the deflection reading is taken. The time required for such an attenuation increases with the load, and towards the end of the test this delay amounted to about 15 minutes.

4. Test program

In advance of the shakedown tests, static bending tests on single span and two-span beams under the loads continuously increasing were carried out. The purpose of such tests and the results are discussed in [3]. The standard tests were as well intended to furnish data for comparisons of the behaviour under proportionally increased and cyclic loading for the considered type of structures.

The loading program for each of the beams was as given in Fig. 3a. First, the beams were loaded at one span $P_1 = P$ (stage I). Consecutively, the load $P_2 = P$ was applied to the other span keeping the P_1 acting constant (stage II). In the next step the load P_2

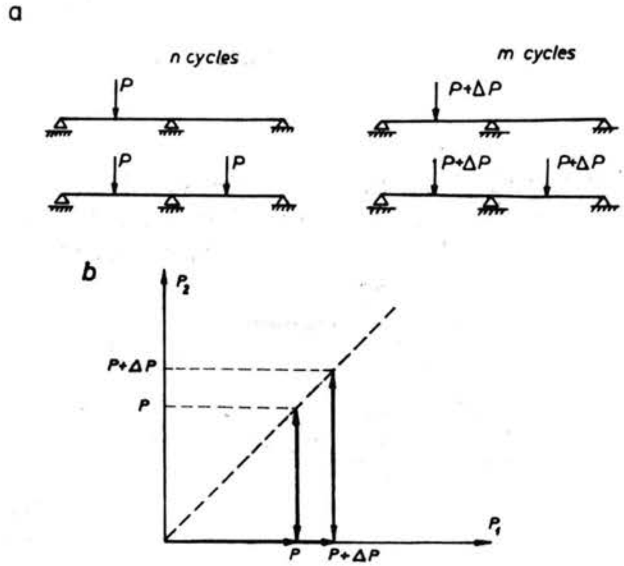

FIG. 3. Loading program.

was brought down to $P_2 = 0$ by using the hydraulic hand pump. In this manner the loads were cycled a prescribed number of times. In each stage of the cycle the vertical displacements were measured. In the next step the loads were incremented, $P+\Delta P$, and the beam was cycled again, Fig. 3b.

The program of cyclic loading was broken into the following series:

Series D, increments ΔP amounting to about 294 N $= 0.115\ P/P_E$ of the amplitude of cyclic loading. Five specimens were tested in the same conditions.

Series E, with increments ΔP about 88 N $= 0.034\ P/P_E$ of the amplitude. Five specimens were tested in the same conditions.

Series F, each of five beams was subjected to a different program of cyclic loading.

The objective of the program of testing was to provide a direct comparison of the obtained response depending on the different increments of cyclic loading amplitude.

5. Test results

Characteristic results recorded for the series D and E are given in Tables 1 and 2. It can be concluded from the Tables that the shakedown was characterized by a gradual reduction in the deflection increments. The shakedown situation normally requires a few cycles to occur.

Table 1. Deflections under cyclic loading, Series *D*.

Load [N]	$\frac{P}{P_S}$	cycle number	Deflection u_{DI} mean [mm]	deviation [mm]	$\frac{u_{DI}}{u_e}$
2069	0.52	1	0.70	0.007	0.99
2599	0.65	2	0.90	0.019	1.01
		3	0.88	0.016	0.99
2835	0.71	4	0.95	0.016	0.98
		6	0.96	0.016	1.00
3364	0.84	7	1.21	0.022	1.05
		9	1.22	0.021	1.06
3894	0.98	10	2.11	0.113	1.59
		19	3.79	0.581	2.85
4012	1.01	20	4.09	0.500	3.35
		27	4.96	0.467	3.62
4306	1.08	28	5.71	0.515	3.89
		39	7.82	0.405	5.32
4600	1.16	40	8.52	0.442	5.43
		51	10.60	0.429	6.15
4895	1.23	52	11.25	0.500	6.73
		63	13.31	0.649	7.97
5189	1.30	64	13.92	0.654	7.86
		75	15.94	0.658	9.00
5483	1.38	76	16.43	0.697	8.78
		87	18.19	0.468	9.72
5778	1.45	88	18.63	0.665	9.45
		100	20.67	0.543	10.48

For series *D* and *E* the number of load cycles on each level of loading was assumed before the test. This number of cycles was 5 at the beginning of the test, whereas towards the end it increased to about 40 cycles. The test was terminated arbitrarily when the maximum deflection exceeded the elastic limit deflection by more than 5 times. The maximum amplitude of the cyclic load exceeded the theoretical shakedown load by about 45 per cent for beams series *D*, by 8 per cent for series *E* and 18 per cent for series *F*.

The theoretical value P_E of the perfectly elastic behaviour, the shakedown value P_S, and the collapse load P_L are, respectively:

$$P_E = \frac{128}{69}\frac{M_0}{L} = 2580 \text{ N},$$

$$P_S = \frac{96}{19}\frac{M_0}{L} = 3983 \text{ N},$$

$$P_L = 6\frac{M_0}{L} = 4728 \text{ N}.$$

Table 2. Deflections under cyclic loading. Series E.

Load [N]	$\frac{P}{P_{sh}}$	cycle number	Deflection u_{DI} mean mm	deviation mm	$\frac{u_{DI}}{u_e}$
2364	0.59	1	0.77	0.067	0.96
		3	0.77	0.069	0.96
2600	0.65	4	0.87	0.064	0.98
		6	0.87	0.061	0.98
2776	0.70	7	0.93	0.062	0.98
		9	0.93	0.057	0.98
2953	0.74	10	1.00	0.069	0.99
		12	1.00	0.069	0.99
3130	0.79	13	1.07	0.072	1.00
		15	1.07	0.072	1.00
3247	0.82	16	1.11	0.074	1.00
		20	1.12	0.076	1.01
3335	0.84	21	1.15	0.087	1.01
		25	1.16	0.086	1.02
3423	0.86	26	1.20	0.098	1.02
		30	1.21	0.104	1.03
3541	0.89	31	1.34	0.136	1.08
		35	1.33	0.156	1.10
3630	0.91	36	1.37	0.161	1.10
		40	1.41	0.203	1.13
3718	0.93	41	1.54	0.298	1.21
		48	1.69	0.487	1.33
3835	0.96	49	1.85	0.578	1.42
		56	2.29	1.024	1.75
3924	0.99	57	2.49	1.170	1.86
		66	2.87	1.380	2.14
4012	1.01	67	3.03	1.397	2.21
		80	3.73	1.600	2.72
4130	1.04	81	4.05	1.502	2.87
		99	5.38	1.169	3.82
4218	1.06	100	5.55	1.163	3.85
		119	6.46	0.956	4.49
4306	1.08	120	6.61	1.009	4.50
		140	7.35	1.005	5.01

The averages from the results of the shakedown tests of beams of series D and E are given respectively in Figs. 4 and 5, whereas Fig. 6 gives the results of the series F. The results present the numbers of cycles versus the midspan deflections u_A, u_B, with both cycle and deflection being cumulative. Each figure gives the entire load history for the

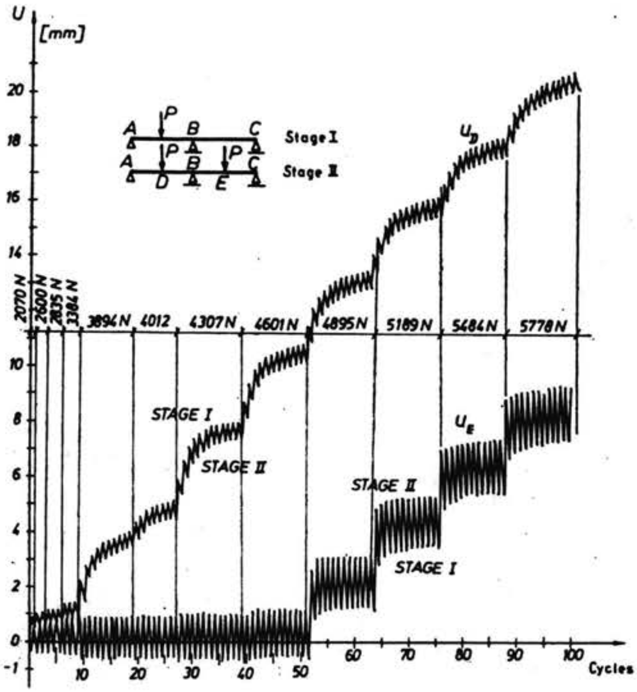

FIG. 4. Deflexion increase at cycling $\Delta P = 0.115\ P/P_E$.

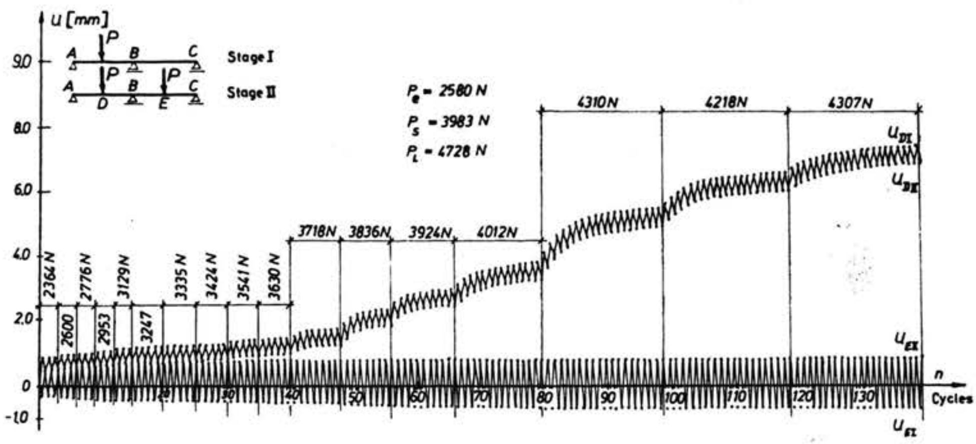

FIG. 5. Deflexion increase at cycling $\Delta P = 0.034\ P/P_E$.

respective beams and graphically illustrates the shakedown or inadaptation at each load level. The curves of Figs. 4 and 5 and certain curves of Fig. 6 clearly show the shakedown state for amplitudes of loads lower than or equal to the theoretical shakedown load $P \leqslant \leqslant 3983$ N, and a marked increase in deflections under any amplitude of loading beyond the shakedown limit. The slope of the deflection does not tend then to a horizontal line.

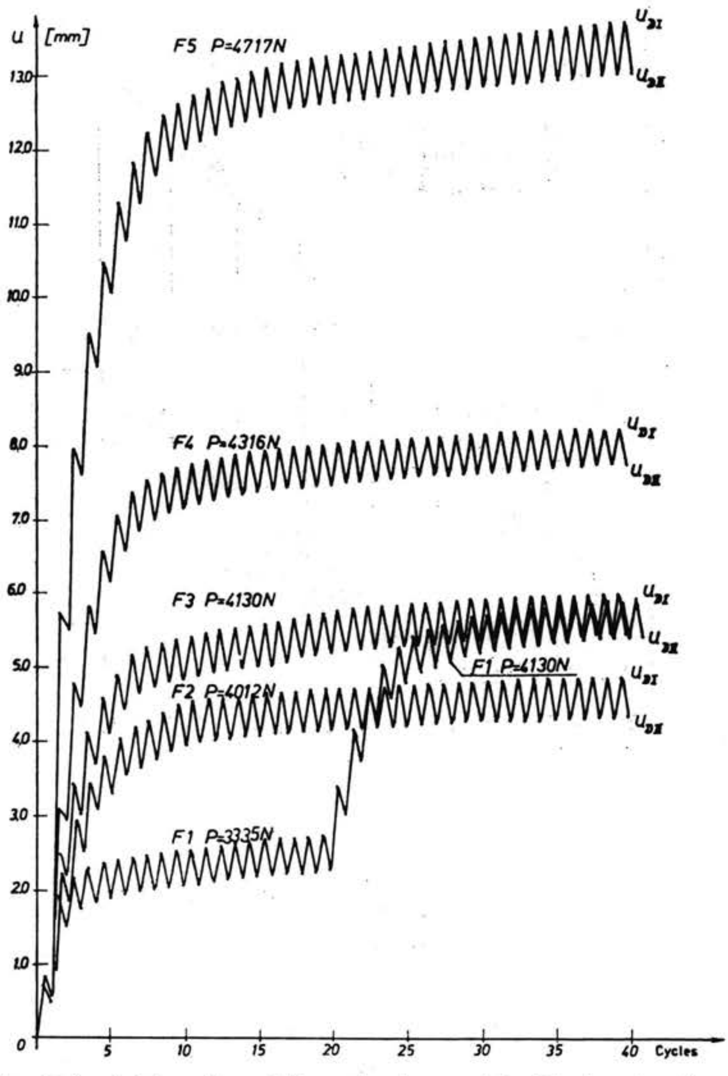

FIG. 6. Load deformation relations at various sustained loads and cycling.

Comparison of the effect of different load levels and different increments of amplitude of cyclic loading is given in Fig. 7. The lines corresponding to deflections of specific points of the beam in the stabilized range under different increments of load are similar in the majority of cases.

In Fig. 8 the tested beams are shown as compared to those subjected to load increasing in proportion and represented on the left side of the figure (cf. [3]).

In Figs. 9 and 10 as well as in Tables 1 and 2 the deflections at stage I at the loading point D are referred to the elastic deflections u_e calculated under the assumption of ideal elastic response of the beams and traced versus the dimensionless load P/P_S. The vertical lines represent dimensionless increments of the deflections during a fixed number of cycles

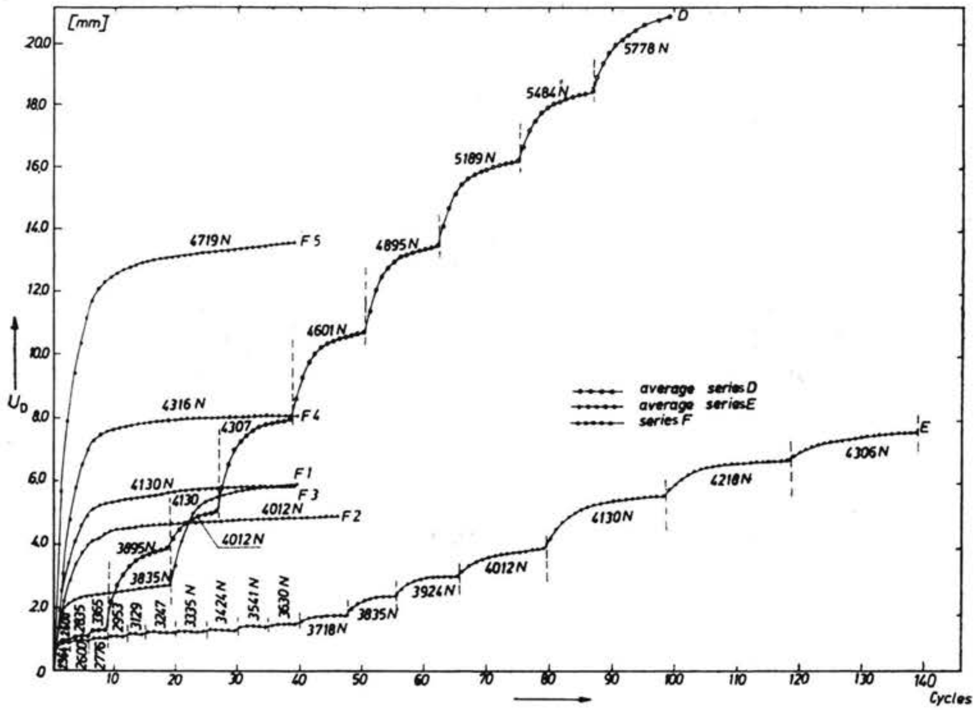

FIG. 7. Shakedown or incremental collapse under cycling.

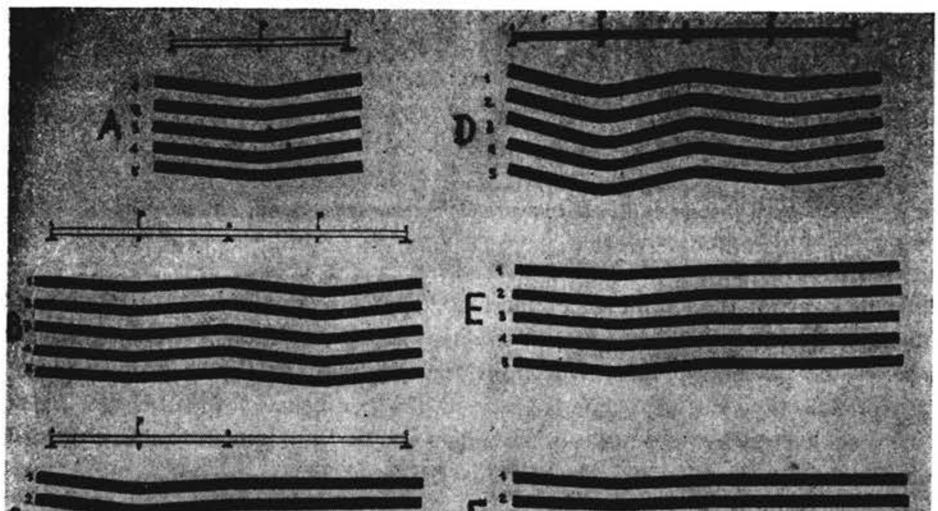

FIG. 8. Specimens after testing, a) uniformly increasing loads, *A*, *B*, *C*, b) cyclic under one load sustained, *D*, *E*, *F*.

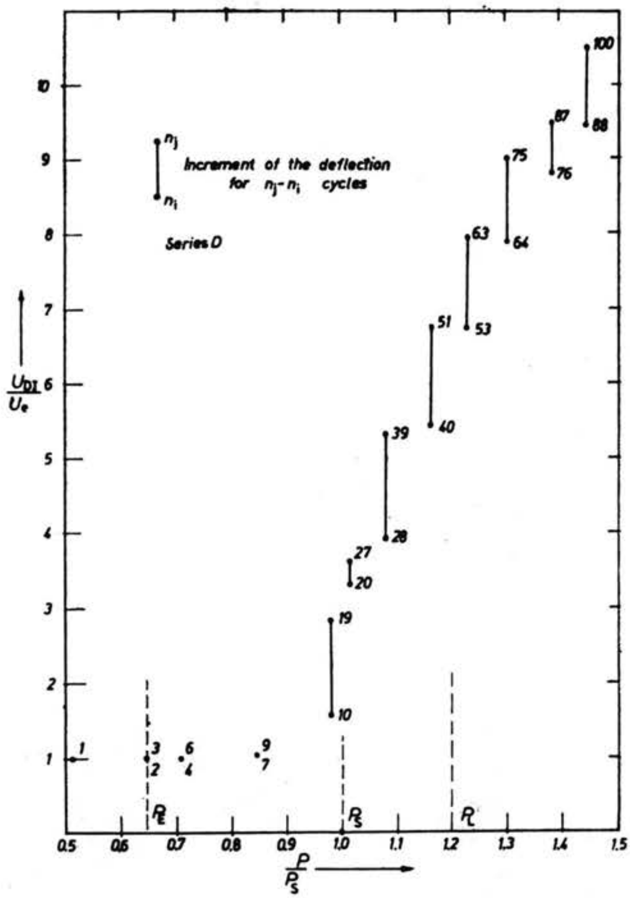

FIG. 9. Ranges of shakedown and incremental collapse, series *D*.

at different levels of the amplitude of the load. The cycle numbers are marked for the beginning and termination of cycling at specific loading conditions. It can be remarked that below the shakedown limit deflections are stabilized. At the shakedown point an increase of deflections due to cycling is observed. Further increase of load results in deflection increase. Eventually, as a result of hardening and perhaps geometric effects the deflection increase is showed down at large loads $P/P_S \approx 1.5$.

6. Comparison with theoretical predictions

The fundamental theorems of MELAN [17] and KOITER [12] allow to determine the safe (shakedown) range of the load variations. However, they do not provide any information regarding deformations prior to the shakedown state. In general, the residual deformations depend on the history of loading which is, as a rule, unknown except for the load variation intervals. From the point of view of design it is more important to estimate an upper bound to the elastic-plastic deflections prior to shakedown than to evaluate the deflections precisely for a given loading history. Some general methods of this bounding have been developed during the last few years [5, 13, 15, 19, 23]. However, the applications of those methods were limited to structures with elements of perfect I cross sections

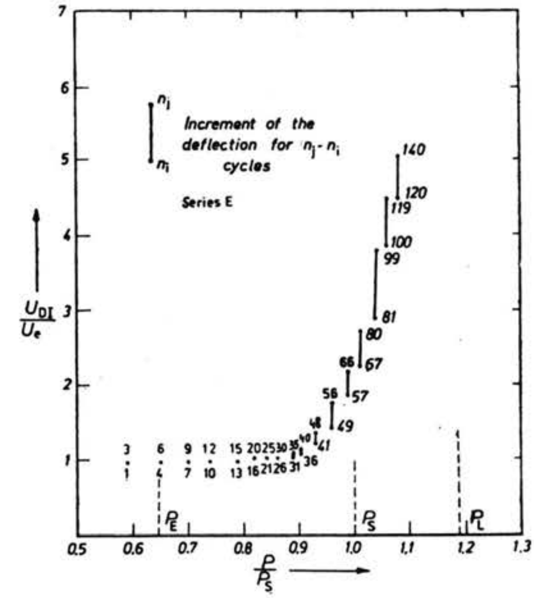

FIG. 10. Ranges of shakedown and incremental collapse, series *E*.

as they relied on the hypothesis of plastic hinges. Recently, in [6] the influence of the finite spread of plastic zones has been accounted for. Figure 11 gives the comparison of the experimental data obtained in the course of tests with predictions of the above-mentioned bounding method. It is seen that the deflection evaluated as for an ideal I-beam underestimates the real behaviour. Deflections at the shakedown load are markedly different from the elastic values.

The cumulative midspan deflections are traced against the amplitude of cyclic load. The solid line corresponds to the upper bound to the maximum shakedown deflection in the case when the influence of plastic zones is taken into account [6]. The dashed line deals with the same upper bound but is calculated under the assumption of ideal plastic hinges. This figure shows that the theory referred to gives a reasonable description of the beams response under cyclic loading. For the sake of comparison the theoretical elastic limit load, P_E shakedown load P_S and the yield point load P_L are indicated. A significant increase of deflections is observed when the load increases beyond the shakedown load. The increase is larger as the total load increases.

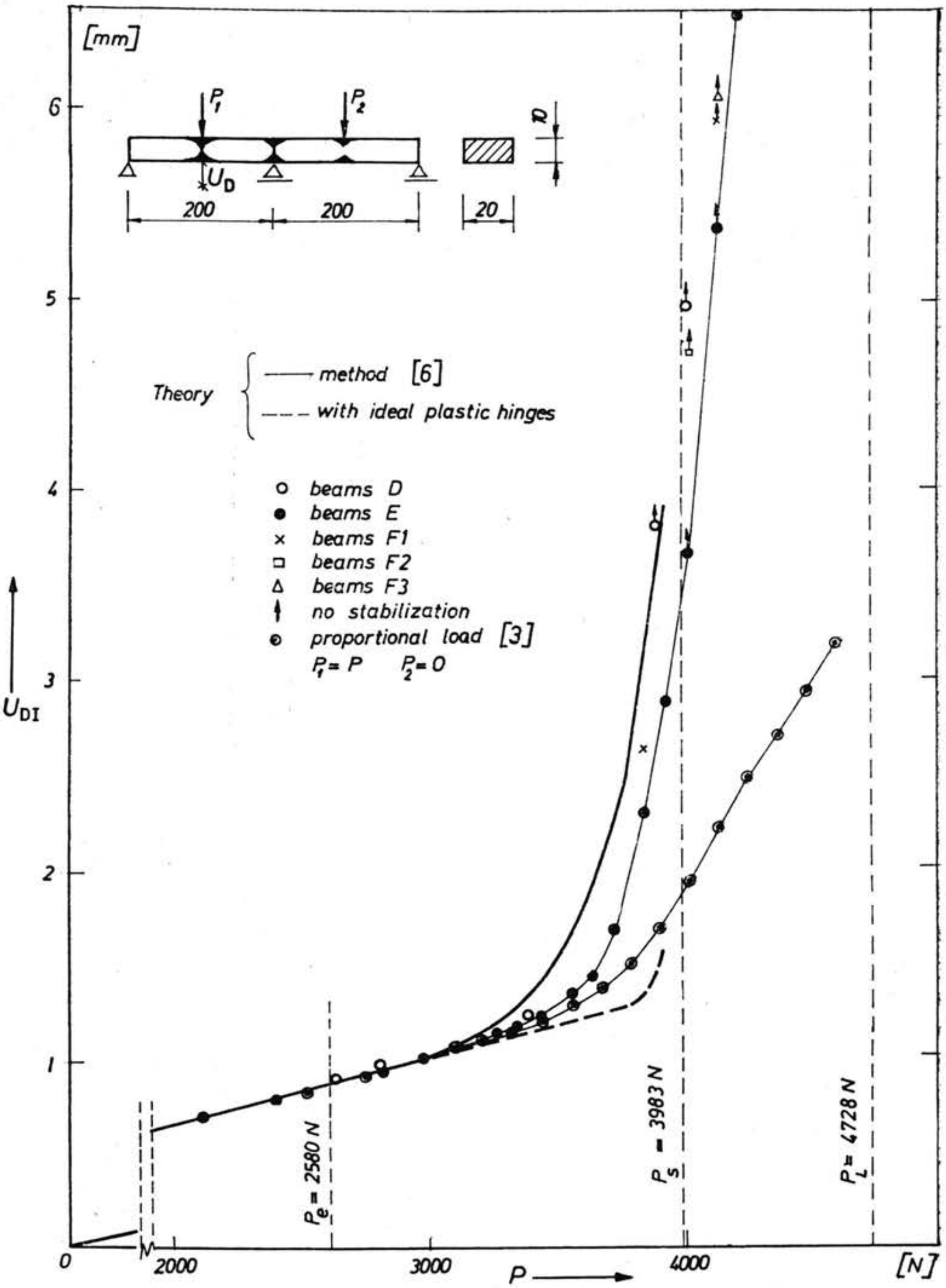

FIG. 11. Deflexion under proportional or cyclic loading.

7. Conclusions

Analysing the experimental data one can arrive at some conclusions.

In the first place it can be stated that the hypothesis of plastic hinges leads to an underestimation of deflections at the shakedown loads. Therefore, the finite spread of plastic zones should be accounted for when calculating the shakedown deflections.

Shakedown occurred in all experiments above the theoretical values. The adaptation may appear at loads beyond the theoretical shakedown load but at the expense of large deflections, about five times larger than the maximum elastic deflection.

The material hardening plays no essential role in the stabilization of deflections as far as the loads are below the shakedown limit. On the contrary, within the range of incremental collapse the strain hardening seems to influence the deflected shape.

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