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Classification of thin shell models deduced from the nonlinear three-dimensional elasticity. Part I : the shallow shells

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THE PURPOSE of this paper is to construct a classification of asymptotic shell models (inferred from the non linear three-dimensional elasticity) with respect to the applied forces and to the geometrical data. To do this, we use a constructive approach based on a dimensional analysis of the nonlinear three-dimensional equilibrium equations, which naturally gives rise to the appearance of dimensionless numbers characterizing the applied forces and the geometry of the shell. In order to limit our study to one-scale problems, these dimensionless numbers are expressed in terms of the relative thickness ε of the shell, which is considered as the perturbation parameter. This leads, on the one hand, to distinguish shallow shells from strongly curved shells which have a different asymptotic behaviour, and on the other hand, to fix the applied force level. For each of these two classes of shells, using the usual asymptotic method, we propose a complete classification of two-dimensional shell models based on decreasing force levels, from severe to low. In the first part of this paper, we present the classification for shallow shells. We obtain successively the nonlinear membrane model, another membrane model, Koiter's non linear shallow shell model, and the linear Novozhilov-Donnell one, respectively for severe, high, moderate and low forces.

1. Introduction

THE STUDY of thin shells is the subject of numerous works in mechanical structure area. The main goal of these works is to predict the behaviour of the shell, when it is subjected to a known level of applied loads. To this end, many authors have proposed two-dimensional shell models whose resolution is less difficult than the classical three-dimensional equations. These two-dimensional models may be obtained from the three-dimensional elasticity using essentially three approaches[1]. The first one is a direct approach which consists in introducing a priori assumptions in the three-dimensional Eqs. [27][17]. The second one is a

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[1] Even if other approaches exist

direct or surfacic approach where the shell is modelled as a surface embedded in
$\mathbb{R}^3$ [3]-[5][16][48][25]. Finally, the third possible approach is based on asymptotic
techniques. Contrary to the two first approaches, the asymptotic approaches,
based on mathematical techniques developed by J. L. Lions [28] for problems
containing a small parameter, lead to a rigorous justification of two-dimensional
shell models.

In linear shell theory, H. S. Rutten [40] and A. L. Goldenveizer [24] have
developed some ideas concerning the application of the asymptotic expansion
method to the shell theory. F. John [26] has also proposed another approach
which is based on the estimation of the stresses and of their derivatives in the
interior of the domain.

However, the first rigorous results have been obtained by P. Destuynder
[4][5] within the framework of linear elasticity. In these works, the author uses
an intrinsic variational approach which makes appear explicitly the curvature of
the shell middle surface. The application of the asymptotic expansion method
leads to the Novozhilov-Donnell model in the case of shallow shells and to the
linear membrane model in the case of strongly curved shells, also called general
shells by other authors.

Another approach using local coordinates has been developed in [11][21][22].
For "general shells", the asymptotic expansion of linear three-dimensional vari-
ational equations leads to the classical linear membrane or to the pure bending
model [14][21], according to whether the middle surface of the shell admits or
not inextensional displacements. The importance of such inextensional displace-
ments in shell theory, which does not modify the metric of the middle surface,
is known since V. V. Novozhilov [17] and A. L. Goldenveizer [9]. The
study of inextensional displacements in linear theory has been systematized by
P. Destuynder [14], E. Sanchez-Palencia [19][20], G. Geymona *et al.* [8]
and D. Choï [2].

These two asymptotic approaches have been extended to non-linear shell
theory. For shallow shells, the Koiter nonlinear shell model[2] and the non-linear
Marguerre-von Kármán one have been deduced from three-dimensional non lin-
ear Eqs. [7][19][10]. For general shells, the non linear membrane model has been
obtained whatever the geometric rigidity of the middle surface is [13]. The
nonlinear pure bending model has been deduced in the case of non-inhibited
shells[3] [11]. Let us cite also the works of W. Z. Chien [6] who tried to clas-
sify the geometrically two-dimensional nonlinear shell models by evaluating the
respective order of magnitude of the membrane and bending stress tensor con-
tribution in 2D general equations.

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[2] Which is also named Donnell-Mushtari-Vlasov model.

[3] In the nonlinear sense.

However, on the one hand the these approaches generally use a priori scaling assumptions on displacements which are unknowns of the problem. On the other hand, the results obtained do not enable to deduce a general classification which specifies the domain of validity of the two-dimensional shell models. In particular, the following paradox still subsists: when the nonlinear Koiter shallow shell model and the linear Novozhilov-Donnell one are deduced, respectively from the nonlinear and the linear elasticity, they are obtained for the same level of forces and the same deflections. However, it is well known that these two models reflect qualitatively different types of behaviour.

The aim of this paper is to present *a constructive method of classification* of asymptotic shell models from the *nonlinear three-dimensional elasticity*, which specifies the domain of validity of the obtained models. To do this, the asymptotic models are deduced from the level of applied forces and from the geometric properties of the middle surface of the shell.

In this approach, we use the following classical assumptions which enable to simplify considerably the calculations :

• Shallow shells are assumed to be totally clamped on the lateral surface, to avoid boundary layers. The study of these boundary layers is not the subject of this paper[4].

• The fields of applied forces and displacements are assumed not to vary rapidly, in order to be able to use only one scale to characterize the displacements, and only one scale to characterize the forces[5]. This is equivalent, as in [26][27][18], to consider that the wave length of strain is of L_0 order.

However, contrary to most of works on asymptotic justification of shell models, the scale of the displacements[6] is not a data of the problem, but *is deduced from the scale of applied forces*. It is in this sense that the expression "without any a priori assumption" can be used to characterize the approach presented in this paper. This approach has been already validated in nonlinear plate theory [34] – [35], in linear shell theory [15] and extended to nonlinear shell theory [6]. Let us notice as well that our approach can be applied to elastic-plastic plate and shells [36].

First, using geometrical and physical reference quantities, we write the three-dimensional nonlinear equilibrium equations in a dimensionless form. This naturally makes appear dimensionless numbers $\mathcal{F}$ and $\mathcal{G}$ characterizing the level of body and surface forces, and two shape factors ε and $\mathcal{C}$, which characterize the

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[4] Thus we have no number characterizing the distance from a current point to the lateral boundary, as in [26].

[5] This assumption is not necessary and can be dropped. In this case, we have multi-scale problems which are much more complicated. It is not the subject of this paper.

[6] Which characterizes the order of magnitude of the displacements and is a priori an unknown of the problem.

thickness and the curvature of the shell. Then, to obtain a one-scale problem, the dimensionless numbers $\mathcal{F}, \mathcal{G}$ and $\mathcal{C}$ are linked to ε. This leads to distinguish, as in [4][5], shallow shells (where $\mathcal{C} = \varepsilon^2$) from strongly curved shells (where $\mathcal{C} = \varepsilon$), which have different asymptotic behaviours. For a given force level[7], we make the asymptotic expansion of equations with respect to the small parameter ε.

Finally, to make out the classification of asymptotic shell models, we study decreasing force levels, from severe to low. For each force level, the order of magnitude of displacements and the corresponding two-dimensional model are deduced from asymptotic expansion of three-dimensional nonlinear equations. To each two-dimensional model that we obtain, we associate a minimization problem. When we consider lower force levels, two cases are possible. If the leading term of the expansion of the displacement is equal to zero, then we make a new dimensional analysis of the displacement. If it is different from zero, we continue the expansion of the three-dimensional equations. This constitutes the original character of this approach. Indeed, the scalings on displacements are progressively deduced from the level of applied forces.

For shallow shells, the classification leads to four kinds of models : a nonlinear membrane model, another membrane model[8], the non-linear Koiter shallow shell model and the linear Novozhilov-Donnell model, obtained respectively for severe, high, moderate and low force levels. These results can be summarized in the following table:

Shell model	In-plane surface forces	Normal surface forces
Nonlinear membrane model	ε	ε
Another membrane model	ε^2	ε^3
Koiter's non linear model	ε^3	ε^4
Linear Novozhilov-Donnell model	$\varepsilon^{n \geq 4}$	ε^{n+1}

In the case of strongly curved shells, the classification obtained is more complex. It depends not only on the force levels, but also on the inhibited or not inhibited character of the middle surface in the linear and nonlinear sense. The classification of asymptotic shell models for strongly curved shells will not be presented here. It will be the subject of the second part of this paper.

2. The three-dimensional problem

In what follows, we index by a star (*) all dimensional variables. On the other hand, within the framework of large displacements, the reference and the current

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[7] A force level corresponds to a relation between $\mathcal{F}$, $\mathcal{G}$ and ε^p, which is chosen as the reference scale.

[8] Which has to our knowledge no equivalent in the literature.

configuration cannot be confused. So the reference configuration variables will be indexed by $(_0)$.

Let ω_0^* be a connected surface embedded in $\mathbb{R}^3$, whose diameter is L_0, with a "smooth enough" boundary γ_0^*. We note N_0 the unit normal to ω_0^* and C_0^* its curvature operator. Let $i_{\omega_0^*}^*$ be the identity mapping on ω_0^*, $T\omega_0^*$ the tangent bundle of ω_0^* (the collection of all tangent spaces corresponding to all points p_0^* of ω_0^*), I_0^* the identity on $T\omega_0^*$, Π_0^* the orthogonal projection onto $T\omega_0^*$ and J_0^* the $\pi/2$ rotation around N_0.

Let us consider a shell of $2h_0$ thickness, whose middle surface is ω_0^*. The shell itself occupies the domain $\overline{\Omega}_0^*$ in its reference configuration where $\Omega_0^* = \omega_0^* \times] - h_0, h_0[$ is an open set of $\mathbb{R}^3$. We denote q_0^* the generic point of $\overline{\Omega}_0^*$ and $\Gamma_0^{*\pm} = \overline{\omega}_0 * \times \{\pm h_0\}$ the upper and lower faces of the shell.To simplify the problem without loss of generality, we assume that the shell is clamped on all its lateral surface $\Gamma_0^* = \gamma_0^* \times [-h_0, h_0]$.

We assume that the shell, subjected to applied body forces $f^* : \overline{\Omega}_0^* \to \mathbb{R}^3$ and to surface forces $g^{*\pm} : \Gamma_0^{*\pm}\mathbb{R} \to \mathbb{R}^3$, occupies the set $\overline{\Omega}^*$ in its deformed configuration. In what follows, we set $f^* = f_t^* + f_n^* N_0$ (respectively $g^{*\pm} = g_t^{*\pm} + g_n^{*\pm} N_0$), decomposition of f^* (respectively of $g^{*\pm}$) onto $T\omega_0^* \oplus \mathbb{R}N_0$. Moreover we consider only thin shells (such as $h_0 << L_0$ and $h_0 \|C_0^*\|_\infty << 1$), subjected to dead loads which are independent of the configuration.

The unknown of the problem is then the displacement $U^* : \overline{\Omega}_0^* \to \mathbb{R}^3$ (or the mapping $\phi^* : \overline{\Omega}_0^* \to \mathbb{R}^3$) such that if $q_0^* \in \overline{\Omega}_0^*$ denotes the initial position of a material point, its position in the deformed configuration is $\phi^*(q_0^*) = q_0^* + U^*(q_0^*)$.
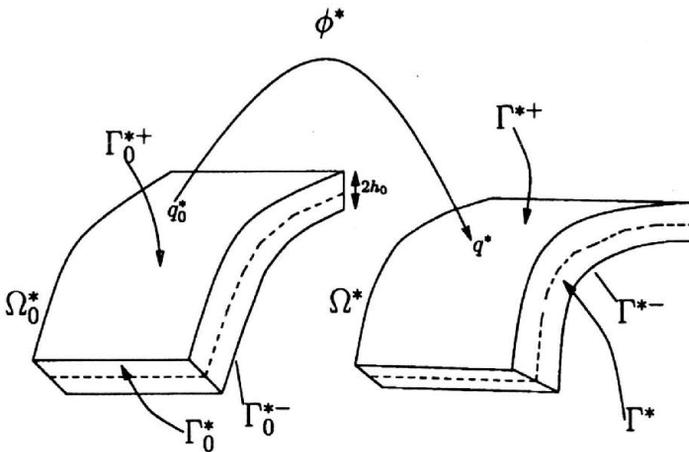

FIG. 1. Initial and final shell configuration.

In this paper we will use the following notations: $\dfrac{\partial}{\partial q_0^*}$ and Div^* denote respectively the gradient and the divergence in the three-dimensional space, $\dfrac{\partial}{\partial p_0^*}$, $\dfrac{\hat{\partial}}{\partial p_0^*}$ and div^* denote respectively the two-dimensional gradient, the covariant derivative and the two-dimensional divergence defined on ω_0^*. The overbar denotes the transposition operator with respect to the metric and $\mathrm{Tr}A$ the trace of the endomorphism A.

Within the framework of nonlinear elasticity, the displacement $U^*:\overline{\Omega_0^*}\to\mathbb{R}^3$ and the second Piola-Kirchhoff tensor Σ^* solve the nonlinear equilibrium equations

$$\mathrm{Div}^*(\Sigma^*\overline{F^*}) = -\overline{f^*} \qquad \text{in}\quad \Omega_0^*,$$

(2.1)
$$(F^*\Sigma^*)^{\pm}.N_0 = \pm g^{*\pm} \qquad \text{on}\quad \Gamma_0^{*\pm},$$

$$U^* = 0 \qquad \text{on}\quad \Gamma_0^*,$$

where $F^* = \dfrac{\partial\phi^*(q_0^*)}{\partial q_0^*} = I_3 + \dfrac{\partial U^*}{\partial q_0^*}$ denotes the linear tangent map to the mapping function $q_0^* \to \phi^*(q_0^*) = q_0^* + U^*(q_0^*)$. Limiting our study to Saint-Venant Kirchhoff materials, the constitutive relation takes the following form :

$$\Sigma^* = \lambda\,\mathrm{Tr}(E^*)I_3 + 2\mu E^*$$

where $E^* = (\overline{F}^*F^* - I_3)/2$ denotes the nonlinear Green-Lagrange strain tensor, I_3 the identity of $\mathbb{R}^3$, λ and μ the Lamé constants of the material.

These equilibrium equations can be completed with the equation of continuity $\rho^*\det F^* = \rho_0^*$ where ρ_0^* and ρ^* denote respectively the voluminal mass of the material in the reference and in the deformed configuration. In what follows, we assume ρ^* to be bounded, which can be written

(2.2)
$$\det F^* = \det\left(\frac{\partial\phi^*(q_0^*)}{\partial q_0^*}\right) \geq a > 0 \quad \text{in}\quad \Omega_0^*,$$

where $a > 0$ is a constant independent of the geometry. This condition will be used later.

In the case of thin shells, for all material point q_0^* in $\overline{\Omega_0^*}$, we have the unique decomposition $q_0^* = p_0^* + z^*N_0$ where $p_0^* \in \overline{\omega_0^*}$ and $z^* \in [-h_0, h_0]$ denote respectively the orthogonal projection of q_0^* onto $\overline{\omega_0^*}$ and onto the normal N_0. It is then possible to decompose the displacement U^* onto $T\omega_0^* \oplus \mathbb{R}N_0$ as follows:

$$U^* = V^* + u^*N_0$$

where V^* is a field of tangent vectors and u^* a scalar field on ω_0^*.
In what follows, we set

$$G^* = \frac{\partial U^*}{\partial q_0^*} \quad \text{and} \quad \mathcal{H}^* = \Sigma^* \overline{F^*}.$$

Then, using the matrix notation, G^* can be decomposed onto $T\omega_0^* \oplus \mathbb{R}N_0$ as:

(2.3)
$$G^* = \begin{bmatrix} G_t^* & G_s^* \\ \overline{G}_s^{'*} & G_n^* \end{bmatrix}$$

with

$$G_t^* = \left(\frac{\hat{\partial} V^*}{\partial p_0^*} - u^* C_0^* \right) \kappa_0^{*-1}, \qquad G_s^* = \frac{\partial V^*}{\partial z^*},$$

$$G_s^{'*} = \kappa_0^{*-1} \left(C_0^* V^* + \frac{\overline{\partial u^*}}{\partial p_0^*} \right), \qquad G_n^* = \frac{\partial u^*}{\partial z^*},$$

and $\kappa_0^* = I_0 - z^* C_0^*$. As κ_0^* is invertible[9], we have : $\kappa_0^{*-1} = I_0 + z^* C_0^* + z^{*2} C_0^{*2} + \ldots$.
In the same way, E^* can be written :

(2.4)
$$E^* = \begin{bmatrix} E_t^* & E_s^* \\ \overline{E}_s^* & E_n^* \end{bmatrix}$$

where

$$2E_t^* = \overline{G}_t^* G_t^* + G_s^{'*} \overline{G}_s^{'*} + \overline{G}_t^* + G_t^*,$$

$$2E_s^* = \overline{G}_t^* G_s^* + G_n^* G_s^{'*} + G_s^* + G_s^{'*},$$

$$2E_n^* = \overline{G}_s^* G_s^* + G_n^{*2} + 2G_n^*.$$

Therefore, the second Piola-Kirchhoff tensor Σ^* takes the following form:

(2.5)
$$\Sigma^* = \begin{bmatrix} \Sigma_t^* & \Sigma_s^* \\ \overline{\Sigma}_s^* & \Sigma_n^* \end{bmatrix}$$

with

$$\Sigma_t^* = \lambda(\text{Tr}\,(E_t^*) + E_n^*) + 2\mu E_t^*, \qquad \Sigma_s^* = 2\mu E_s^*,$$

$$\Sigma_n^* = \lambda \text{Tr}\,(E_t^*) + (\lambda + 2\mu) E_n^*.$$

Finally, the matrix form of $\mathcal{H}^* = \Sigma^* \overline{F^*}$ becomes:

(2.6)
$$\mathcal{H}_t^* = \begin{bmatrix} \mathcal{H}_s^*, & \mathcal{H}_s^* \\ \overline{\mathcal{H}}_s^{'*} & \mathcal{H}_n^* \end{bmatrix}$$

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[9] Because $h_0 \|C_0^*\|_\infty \ll 1$ for a thin shell.

with

$$\mathcal{H}_t^* = \Sigma_t^* + \Sigma_t^* \overline{G}_t^* + \Sigma_s^* \overline{G}_s^*, \qquad \mathcal{H}_s^* = \Sigma_s^* + G_n^* \Sigma_s^* + \Sigma_t^* G_s'^*,$$

$$\mathcal{H}_s'^* = \Sigma_s^* + G_t^* \Sigma_s^* + \Sigma_n^* G_s^*, \qquad \mathcal{H}_n^* = \Sigma_n^* + G_n^* \Sigma_n^* + \overline{G'}_s^* \Sigma_s^*.$$

To finish, let us decompose the three-dimensional divergence onto $T\omega_0^* \oplus \mathbb{R}N_0$. Then the equilibrium Eqs. (2.1) can be written in $T\omega_0^* \oplus \mathbb{R}N_0$ as :

(2.7)
$$\text{div}^* \left(\kappa_0^{*-1} \mathcal{H}_t^*\right) - \text{div}^* \left(\kappa_0^{*-1}\right) \mathcal{H}_t^* - \overline{\mathcal{H}_s^*} \kappa_0^{*-1} C_0^* - \text{Tr}(\kappa_0^{*-1} C_0^*)\overline{\mathcal{H}}_s'^*$$
$$+ \frac{\overline{\partial \mathcal{H}_s'^*}}{\partial z*} = -\overline{f}_t^*,$$

$$\text{div}^* \left(\kappa_0^{*-1} \mathcal{H}_s^*\right) - \text{div}^* \left(\kappa_0^{*-1}\right) \mathcal{H}_s^* + \text{Tr}(\mathcal{H}_t^* \kappa_0^{*-1} C_0^*) - \text{Tr}(\kappa_0^{*-1} C_0^*)\overline{\mathcal{H}}_n^*$$
$$+ \frac{\partial \mathcal{H}_n^*}{\partial z*} = -f_n^*,$$

where we recall that div^* denotes the two-dimensional divergence on ω_0^*. The boundary conditions on the upper and lower faces $\Gamma_0^{*\pm}$ become:

(2.8)
$$\mathcal{H}_s'^{*\pm} = \pm g_t^{*\pm} \qquad \text{and} \qquad \mathcal{H}_n^{*\pm} = \pm g_n^{*\pm}.$$

The boundary conditions on the lateral surface Γ_0^* are given by:

(2.9)
$$V^* = 0 \qquad \text{and} \qquad u^* = 0.$$

3. Dimensional analysis of equations

3.1. The dimensionless numbers governing shell problems

Let us define the following dimensionless physical data and dimensionless unknowns of the problem :

(3.1)
$$p_0 = \frac{p_0^*}{L_0}, \qquad z = \frac{z^*}{h_0}, \qquad C_0 = \frac{C_0^*}{C_r}, \qquad V = \frac{V^*}{V_r}, \qquad u = \frac{u^*}{u_r},$$

$$f_n = \frac{f_t^*}{f_{tr}}, \qquad g_n = \frac{f_n^*}{f_{nr}}, \qquad g_n = \frac{g_t^*}{g_{tr}}, \qquad g_t = \frac{g_n^*}{g_{nr}},$$

where the variables indexed by r are the reference ones. The new variables which appear without a star are dimensionless. In particular, $C_r = \|C_0^*\|_\infty$ denotes the maximum curvature of the shell in its reference configuration.

To avoid any assumption on the order of magnitude of the normal and the tangential displacement components, the reference scales V_r and u_r are first assumed to be equal to L_0. Thus we allow a priori large displacements. If necessary, it will be always possible to define new reference scales for the displacements.

In what follows, to simplify the calculations, we set:

$$(3.2) \qquad G = \varepsilon G^*, \quad E = \varepsilon^2 E^*, \quad \Sigma = \frac{\varepsilon^2}{\mu}\Sigma^* \quad \text{and} \quad \mathcal{H} = \frac{\varepsilon^3}{\mu}\mathcal{H}^*.$$

The so defined dimensionless variables naturally depend on ε. However, it is important to notice that this definition does not constitute any assumption on the order of magnitude of G^*, E^*, Σ^* or $\mathcal{H}^*$. It only enables to use dimensionless quantities which will be more practical for the asymptotic expansion of equations.

According to the previous notation, the dimensionless components of G are given by:

$$(3.3) \qquad G_t = \left(\varepsilon\frac{\partial\hat{V}}{\partial_{p0}} - u\mathcal{C}C_0\right)\kappa_0^{-1}, \quad G_s = \frac{\partial V}{\partial z},$$

$$G_s' = \kappa_0^{-1}\left(\mathcal{C}C_0 V + \varepsilon\frac{\overline{\partial u}}{\partial_{p0}}\right), \quad G_n = \frac{\partial u}{\partial z},$$

where

$$\kappa_0 = I_0 - \mathcal{C}zC_0 \qquad \text{and} \qquad \kappa_0^{-1} = I_0 + z\mathcal{C}C_0 + z^2\mathcal{C}^2 C_0^3 + \dots$$

This dimensional analysis naturally makes appear the dimensionless numbers $\varepsilon = h_0/L_0$ and $\mathcal{C} = h_0 C_r$ which characterize the geometry of the shell:

i) The first one, $\varepsilon = h_0/L_0$, ratio of the initial half-thickness of the shell to the diameter of the middle surface ω_0^* is a known parameter of the problem.

ii) The second one, $\mathcal{C} = h_0 C_r$, product of the half-thickness by the reference curvature of the shell, is as well a known parameter of the problem. For thin shells, ε and $\mathcal{C}$ are small parameters.

On the other hand, the dimensionless components of E are given by:

$$(3.4) \qquad \begin{aligned} 2E_t &= \overline{G_t}G_t + G_s'\overline{G_s'} + \varepsilon(\overline{G_t} + G_t), \\ 2E_s &= \overline{G_t}G_s + G_n G_s' + \varepsilon(G_s' + G_s), \\ 2E_n &= \overline{G_s}G_s + G_n^2 + 2\varepsilon G_n, \end{aligned}$$

and the dimensionless components of Σ become:

$$(3.5) \qquad \Sigma_t = \beta(Tr(E_t) + E_n)I_0 + 2E_t, \qquad \Sigma_s = 2E_s,$$

$$\Sigma_n = \beta Tr(E_t) + (\beta + 2)E_n,$$

where $\beta = \dfrac{\lambda}{\mu}$. Finally, the dimensionless components of $\mathcal{H}$ are given by:

(3.6)
$$\mathcal{H}_t = \varepsilon\Sigma_t + \Sigma_t\overline{G_t} + \Sigma_s\overline{G_s}, \quad \mathcal{H}_s = \varepsilon\Sigma_s + G_n\Sigma_s + \Sigma_t G'_s,$$

$$\mathcal{H}'_s = \varepsilon\Sigma_s + G_t\Sigma_s + \Sigma_n G_s, \quad \mathcal{H}_n = \varepsilon\Sigma_n + G_n\Sigma_n + \overline{G'_s}\Sigma_s.$$

Accordingly, the equilibrium Eq. (2.7) can be written in $\Omega_0 = \omega_0 \times]-1, +1[$ in the dimensionless form:

(3.7)
$$\varepsilon\left(\mathrm{div}\left(\kappa_0^{-1}\mathcal{H}_t\right) - \mathrm{div}\left(\kappa_0^{-1}\right)\mathcal{H}_t\right) - \mathcal{C}\left(\overline{\mathcal{H}}_s\kappa_0^{-1}C_0 + \mathrm{Tr}(\kappa_0^{-1}C_0)\overline{\mathcal{H}'}_s\right)$$

$$+\frac{\overline{\mathcal{H}'_s}}{\partial z} = -\varepsilon^3\mathcal{F}\overline{f}_t,$$

$$\varepsilon\left(\mathrm{div}\left(\kappa_0^{-1}\mathcal{H}_s\right) - \mathrm{div}\left(\kappa_0^{-1}\right)\mathcal{H}_s\right) + \mathcal{C}\left(\mathrm{Tr}(\mathcal{H}_t\kappa_0^{-1}C_0) - \mathrm{Tr}(\kappa_0^{-1}C_0)\mathcal{H}_n\right)$$

$$+\frac{\partial\mathcal{H}_n}{\partial z} = -\varepsilon^3\mathcal{F}f_n.$$

The dimensionless boundary conditions (2.8) on the upper and lower faces $\Gamma_0^{\pm} = \overline{\omega}_0 \times \{\pm 1\}$ are given by:

(3.8)
$$\mathcal{H}'^{\pm}_s = \pm\varepsilon^3\mathcal{G}g_t^{\pm} \quad \text{and} \quad \mathcal{H}^{\pm}_n = \pm\varepsilon^3\mathcal{G}g_n^{\pm}$$

and the boundary conditions (2.9) on the lateral surface $\Gamma_0 = \gamma_0 \times [-1, 1]$ lead to:

(3.9)
$$V = 0 \quad \text{and} \quad u = 0.$$

Thus the dimensional analysis of equations makes appear the other dimensionless numbers, already obtained in [33]-[35], which characterize the applied forces:

(3.10)
$$\mathcal{F}_t = \frac{h_0 f_{tr}}{\mu}, \quad \mathcal{F}_n = \frac{h_0 f_{nr}}{\mu}, \quad \mathcal{G}_t = \frac{g_{tr}}{\mu} \quad \text{and} \quad \mathcal{G}_n = \frac{g_{nr}}{\mu}.$$

Indeed, the numbers $\mathcal{F}_t$ and $\mathcal{F}_n$ (respectively $\mathcal{G}_t$ and $\mathcal{G}_n$) represent the ratio of the resultant on the thickness of the body forces (respectively the ratio of the surface forces) to μ considered as a reference stress. These numbers only depend on known physical quantities and must be considered as known data of the problem.

3.2. Reduction to a one-scale problem

To obtain a one-scale problem, ε is chosen as the reference parameter. Therefore the other dimensionless numbers must be linked to ε.

On one hand, $\mathcal{C}$ must be linked to ε. This leads to distinguish shallow shells where $\mathcal{C} = \varepsilon^2$ (whose middle surface is close to a plate) from strongly curved shells where $\mathcal{C} = \varepsilon$. These two shell families have different asymptotic behaviours which have been already studied in the linear case [4][5]. In the first part of this paper, we will limit our study to shallow shells. On the other hand, the force ratios $\mathcal{F}_t$, $\mathcal{F}_n$, $\mathcal{G}_t$ and $\mathcal{G}_n$ must be linked to the powers of ε as well. This determines the level of applied forces. To make out a general classification for all the force levels, we should consider all the combinations $\mathcal{F}_t = \varepsilon^m$, $\mathcal{F}_n = \varepsilon^l$, $\mathcal{G}_t = \varepsilon^p$ and $\mathcal{G}_n = \varepsilon^q$, for m, l, p and q strictly positive integers. However, such a tiresome work is not necessary because the different two-dimensional shell models obtained are essentially determined by the first member of equilibrium equations. Hence, the study of all the combinations of force levels can be reduced to some particular ones. Let us define

$$(3.11) \qquad \tau = \mathrm{Max}(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n)$$

which will determine the corresponding two-dimensional model. If $\tau = \varepsilon^p$, we will say that the force level is of ε^p order. The classification will be deduced with respect to decreasing values of τ, from $\tau = \varepsilon$ (severe force level) to $\tau = \varepsilon^n$, $n \geq 4$ (low force level).

- For severe applied forces we will consider the same level of normal and tangential forces $\mathcal{F}_t = \mathcal{G}_t = \mathcal{F}_n = \mathcal{G}_n = \varepsilon$.
- For high to low applied forces, we will consider a level of tangential forces more important than the level of normal forces : $\mathcal{F}_t = \mathcal{G}_t = \varepsilon^m$ and $\mathcal{F}_n = \mathcal{G}_n = \varepsilon^{m+1}$ for $m \geq 2$. This gap which naturally appears is due to the fact that the tangential and the normal direction do not play a symmetrical role for shallow shells. Once reduced to a one-scale problem, we make the asymptotic expansion of Eqs. (3.7)–(3.9) for decreasing force levels.

4. The nonlinear membrane model

4.1. Asymptotic expansion of 3D equilibrium equations

Let us consider in this section a shallow shell where $\mathcal{C} = \varepsilon^2$, subjected to a severe force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon$. Thus problem (3.7)–(3.9) is now reduced to a one-scale problem with ε as the small parameter. The standard asymptotic expansion method leads to write the dimensionless solution (V, u) as a formal expansion with respect to ε:

$$(4.1) \quad V = V^0 + \varepsilon^1 V^1 + \varepsilon^2 V^2 + \ldots \quad \text{and} \quad u = u^0 + \varepsilon^1 u^1 + \varepsilon^2 u^2 + \ldots$$

The expansion (4.1) of (V, u) implies an expansion of G, E, Σ and $\mathcal{H}$:

(4.2)

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \ldots \qquad E = E^0 + \varepsilon E^1 + \varepsilon^2 E^2 + \ldots$$

$$\Sigma = \Sigma^0 + \varepsilon \Sigma^1 + \varepsilon^2 \Sigma^2 + \ldots \qquad \mathcal{H}^0 = \mathcal{H}^0 + \varepsilon \mathcal{H}^1 + \varepsilon^2 \mathcal{H}^2 + \ldots$$

To simplify the presentation, the expressions of G^0, G^1 ..., E^0, E^1 ..., Σ^0, Σ^1 ... and $\mathcal{H}^0$, $\mathcal{H}^1$ ... will be detailed later when necessary. We only recall that, for shallow shells where $\mathcal{C} = \varepsilon^2$, the dimensional components (3.3) of G become :

(4.3)

$$G_t = \varepsilon \left(\frac{\hat{\partial} V}{\partial p_0} - \varepsilon u C_0 \right) \kappa_0^{-1}, \qquad G_s = \frac{\partial v}{\partial z},$$

$$G_s' = \varepsilon \kappa_0^{-1} \left(\varepsilon C_0 V + \overline{\frac{\partial u}{\partial p_0}} \right), \qquad G_n = \frac{\partial u}{\partial z},$$

where $\kappa_0 = I_0 - \varepsilon^2 z C_0$ and $\kappa_0^{-1} = I_0 + \varepsilon^2 z C_0 + \varepsilon^4 (z C_0)^2 + \ldots$. Then we have the following result:

RESULT 1.

For a severe force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon$, the leading term (V^0, u^0) of the asymptotic expansion of (V, u) only depends on p_0 and solves the following non-linear membrane problem:

$$\mathrm{div} \left(n_t^0 \left(I_0 + \overline{\frac{\hat{\partial} V^0}{\partial p_0}} \right) \right) = -\overline{p_t} \qquad \text{in} \quad \omega_0,$$

$$\mathrm{div} \left(n_t^0 \overline{\frac{\partial u^0}{\partial p_0}} \right) = -p_n \qquad \text{in} \quad \omega_0,$$

$$(V^0, u^0) = (0, 0), \quad \text{on} \quad \gamma_0$$

where $\quad n_t^0 = \dfrac{4\beta}{2+\beta} \, \mathrm{Tr}(\Delta_t^0) \, I_0 + 4\Delta_t^0,$

$$2\Delta_t^0 = \overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\hat{\partial} V^0}{\partial p_0}} \frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\partial u^0}{\partial p_0}} \frac{\partial u^0}{\partial p_0},$$

$$p_t = g_t^+ + g_t^- + \int_{-1}^{+1} f_t dz \quad \text{and} \quad p_n = g_n^+ + g_n^- + \int_{-1}^{+1} f_n dz.$$

P r o o f. The proof of this result is decomposed into 3 steps from i) to iii)

i) (V^0, u^0) depends only on p_0

We replace $\mathcal{H}$ by its expansion in the dimensionless equilibrium Eqs. (3.7) – (3.9) where the dimensionless numbers have been replaced by $\mathcal{C} = \varepsilon^2$, $\mathcal{G}_t = \mathcal{G}_n = \mathcal{F}_t = \mathcal{F}_n = \varepsilon$. We then obtain a chain of coupled problems $\mathcal{P}^0$, $\mathcal{P}^1$, $\mathcal{P}^2$ ..., corresponding respectively to the cancellation of the factor of ε^0, ε^1, ε^2 ...

Using (3.4) – (3.6) and (4.3), let us write explicitly the expressions of G^0, E^0, Σ^0 and $\mathcal{H}^0$. We have:

(4.4)

$$G_t^0 = 0, \quad G_s^0 = \frac{\partial V^0}{\partial z}, \quad G'^0_s = 0, \quad G_n^0 = \frac{\partial u^0}{\partial z},$$

$$2E_t^0 = 0, \quad 2E_s^0 = 0, \quad 2E_n^0 = \overline{G_s^0}\, G_s^0 + G_n^{0\,2},$$

$$\Sigma_t^0 = \beta E_n^0\, I_0, \quad \Sigma_s^0 = 0, \quad \Sigma_n^0 = (\beta + 2)E_n^0,$$

$$\mathcal{H}_t^0 = 0, \quad \mathcal{H}_s^0 = 0, \quad \mathcal{H}'^0_s = G_s^0\Sigma_n^0, \quad \mathcal{H}_n^0 = G_n^0\,\Sigma_n^0.$$

The cancellation of the factor of ε^0 in the expansion of dimensionless equations (3.7) – (3.9) leads to problem $\mathcal{P}^0$:

$$\frac{\partial \mathcal{H}'^0_s}{\partial z} = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}_n^0}{\partial z} = 0 \quad \text{in} \quad \Omega_0,$$

$$\mathcal{H}'^{0\pm}_s = 0 \quad \text{and} \quad \mathcal{H}_n^{0\pm} = 0 \quad \text{on} \quad \Gamma_0^{\pm}.$$

So we have

$$\mathcal{H}'^0_s = 0 \quad \text{and} \quad \mathcal{H}_n^0 = 0 \quad \text{in} \quad \Omega_0.$$

Using the expression (4.4) of $\mathcal{H}'^0_s$ and $\mathcal{H}_n^0$, we obtain:

$$\left(\left\|\frac{\partial V^0}{\partial z}\right\|^2 + \left(\frac{\partial u^0}{\partial z}\right)^2\right)\frac{\partial V^0}{\partial z} = 0 \quad \text{and} \quad \left(\left\|\frac{\partial V^0}{\partial z}\right\|^2 + \left(\frac{\partial u^0}{\partial z}\right)^2\right)\frac{\partial V^0}{\partial z} = 0$$

$$\text{in} \quad \Omega_0$$

which leads to

$$\frac{\partial V^0}{\partial z} = 0 \quad \text{and} \quad \frac{\partial u^0}{\partial z} = 0 \quad \text{in} \quad \Omega_0$$

and implies that

(4.5)
$$V^0 = V^0(p_0) \quad \text{and} \quad u^0 = u^0(p_0).$$

Hence we get $\mathcal{H}^0 = \mathcal{H}^1 = \mathcal{H}^2 = 0$, and problems $\mathcal{P}^1$ and $\mathcal{P}^2$ are trivially satisfied.

ii) Expression of (V^1, u^1)

The cancellation of the factor of ε^3 in Eqs. (3.7) – (3.9) leads to the problem $\mathcal{P}^3$:

$$\frac{\partial \mathcal{H}_s'^3}{\partial z} = \frac{\partial \mathcal{H}_n^3}{\partial z} = 0 \quad \text{in} \quad \Omega_0,$$

$$\mathcal{H}_s'^{3\pm} = \mathcal{H}_n^{3\pm} = 0 \quad \text{on} \quad \Gamma_0^\pm,$$

where the expression of $\mathcal{H}^3$ is obtained from (3.4) – (3.6) and (4.3) as follows:

$$\mathcal{H}_t^3 = \Sigma_t^2(\overline{G_t^1} + I_0) + \Sigma_s^2\overline{G_s^1}, \qquad \mathcal{H}_s^3 = \Sigma_s^2(G_n^1 + 1) + \Sigma_t^2 G_s'^1,$$

$$\mathcal{H}_s'^3 = (G_t^1 + I_0)\Sigma_s^2 + G_s^1\Sigma_t^2, \qquad \mathcal{H}_n^3 = (G_n^1 + 1)\Sigma_n^2 + \overline{G_s'^1}\Sigma_s^2,$$

$$\Sigma_t^2 = \beta(\text{Tr}(E_t^2) + E_n^2)I_0 + 2E_t^2,$$

$$\Sigma_s^2 = 2E_s^2,$$

(4.6)

$$\Sigma_n^2 = \beta\text{Tr}(E_t^2) + (\beta + 2)E_n^2,$$

$$2E_t^2 = \overline{G_t^1}G_t^1 + \overline{G_t^1} + G_t^1 + G_s'^1\overline{G_s'^1},$$

$$2E_s^2 = \overline{G_t^1}G_s^1 + G_n^1G_s'^1 + G_s'^1 + G_s^1,$$

$$2E_n^2 = \overline{G_s^1}G_s^1 + (G_n^1)^2 + 2G_n^1,$$

$$G_t^1 = \frac{\hat{\partial}V^0}{\partial p_0}, \quad G_s^1 = \frac{\partial V^1}{\partial z}, \quad G_s'^1 = \frac{\overline{\partial u^0}}{\partial p_0}, \quad G_n^1 = \frac{\partial u^1}{\partial z}.$$

Problem $\mathcal{P}^3$ then gives us :

$$\mathcal{H}_s'^3 = \mathcal{H}_n^3 = 0 \quad \text{in} \quad \Omega_0$$

Replacing $\mathcal{H}_s'^3$ et $\mathcal{H}_n^3$ by their expressions (4.6), we get:

(4.7) $(G_t^1 + I_0)\Sigma_s^2 + G_s^1\Sigma_n^2 = 0$ and $\overline{G_s'^1}\Sigma_s^2 + (G_n^1 + 1)\Sigma_n^2 = 0$ in Ω_0

or equivalently, using matrix notations :

(4.8) $$(I_3 + G^1) \begin{bmatrix} \Sigma_s^2 \\ \Sigma_n^2 \end{bmatrix} = 0 \quad \text{in} \quad \Omega_0.$$

Now, let us use of the equation of continuity (2.2)

$$\det F^* \geq a > 0$$

where a is independent of the geometry, hence independent of $\varepsilon = h_0/L_0$. As $G^0 = 0$, the expansion of G becomes

$$G = \varepsilon G^1 + \varepsilon^2 G^2 + \ldots$$

According to (3.2), the expansion of $F^* = I_3 + G^*$ reduces to

$$F^* = (I_3 + G^1) + \varepsilon G^2 + \cdots$$

and the equation of continuity leads to:

$$\det (I_3 + G^1) > 0$$

So $I_3 + G^1$ is invertible and Eq. (4.8) then implies that:

$$\Sigma_s^2 = 0 \quad \text{and} \quad \Sigma_n^2 = 0.$$

Hence, using the expressions (4.6) of Σ_s^2 and Σ_n^2, we get:

$$E_s^2 = 0 \qquad \text{and} \qquad E_n^2 = -\frac{\beta}{\beta+2}\text{Tr}(E_t^2)$$

So we have :

$$(4.9) \quad E_t^2 = \Delta_t^0, \quad \Sigma_t^2 = \frac{2\beta}{\beta+2}\text{Tr}(\Delta_t^0)I_0 + 2\Delta_t^0, \quad \mathcal{H}_t^3 = \Sigma_t^2\left(I_0 + \frac{\overline{\partial \hat{V}^0}}{\partial p_0}\right),$$

$$\mathcal{H}_s^3 = \Sigma_t^2\frac{\overline{\partial u^0}}{\partial p_0}, \quad \mathcal{H}_s'^3 = \frac{\partial V^1}{\partial z}\Sigma_t^2 \quad \text{and} \quad \mathcal{H}_n^3 = 0,$$

where $\quad 2\Delta_t^0 = \dfrac{\overline{\partial \hat{V}^0}}{\partial p_0} + \dfrac{\overline{\partial V^0}}{\partial p_0} + \dfrac{\overline{\partial \hat{V}^0}}{\partial p_0}\dfrac{\partial V^0}{\partial p_0} + \dfrac{\overline{\partial u^0}}{\partial p_0}\dfrac{\partial u^0}{\partial p_0}.$

Now, let us define the mapping $\phi^1 = V^1 + (u^1 + z)N_0$ to simplify the expressions. Then Eq. (4.7) can be written with matricial notations :

$$(4.10) \quad 2E_s^2 = \left[I_0 + \frac{\overline{\partial \hat{V}^0}}{\partial p_0} \quad \frac{\overline{\partial u^0}}{\partial p_0}\right]\left[\begin{array}{c} \dfrac{\partial V^1}{\partial z} \\ \dfrac{\partial u^1}{\partial z} + 1 \end{array}\right] = \Pi_0\overline{(I_3 + G^1)}\frac{\partial \phi^1}{\partial z} = 0,$$

$$2E_n^2 = \left\|\frac{\partial \phi^1}{\partial z}\right\|^2 - 1 = -\frac{2\beta}{\beta+2}\text{Tr}(E_t^2),$$

where Π_0 denotes the orthogonal projection onto $T\omega_0$.

In the first above equation $I_3 + G^1$ is invertible in $\mathbb{R}^3$, so $\Pi_0\overline{(I_3 + G^1)}$ is an operator of rank 2. On the other hand, we have :

$$\Pi_0\overline{(I_3 + G^1)} = \left[I_0 + \frac{\overline{\partial V^0}}{\partial p_0} \quad \frac{\overline{\partial u^0}}{\partial p_0} \right].$$

So we deduce that $\Pi_0\overline{(I_3 + G^1)}$ is independent of z. Accordingly, the image of $\mathbb{R}^3$ by this operator is a plane independent of z. Then the relation $\Pi_0\overline{(I_3 + G^1)}\dfrac{\partial \phi^1}{\partial z} = 0$ implies that $\dfrac{\partial \phi^1}{\partial z} = \theta_0 N$, $\theta_0 \in \mathbb{R}$, where N denotes the normal to the plane image of $\mathbb{R}^3$ by the application $(I_3 + G^1)\Pi_0$.

As $2E_t^2 = \dfrac{\overline{\partial V^0}}{\partial p_0} + \dfrac{\partial V^0}{\partial p_0} + \dfrac{\overline{\partial V^0}}{\partial p_0}\dfrac{\partial V^0}{\partial p_0} + \dfrac{\overline{\partial u^0}}{\partial p_0}\dfrac{\partial u^0}{\partial p_0}$ is independent of z, the

second Eq. (4.10) then implies that $\left\| \dfrac{\partial \phi^1}{\partial z} \right\| = \sqrt{1 - \dfrac{2\beta}{\beta+2}\mathrm{Tr}\,(E_t^2)}$ is independent of z as well. So we have :

(4.11) $$\phi^1 = U^1(p_0) + z\theta_0 N$$

with $\theta_0 = \sqrt{1 - \dfrac{2\beta}{\beta+2}\,\mathrm{Tr}\,(E_t^2)}$,

where N is an unit vector such as $\Pi_0\overline{(I_3 + G^1)}N = 0$, and where U^1 depends only on p_0

iii) Nonlinear membrane equations

The cancellation of the factor of ε^4 in the expansion of Eqs. $(3.7)-(3.9)$ leads to problem $\mathcal{P}^4$:

$$\mathrm{div}\left(\mathcal{H}_t^3\right) + \frac{\overline{\partial \mathcal{H}_s'^4}}{\partial z} = -\overline{f_t} \quad \text{in} \quad \Omega_0 \qquad \mathcal{H}_s'^{4\pm} = \pm g_t^\pm \quad \text{on} \quad \Gamma_0^\pm$$

$$\mathrm{div}\left(\mathcal{H}_s^3\right) + \frac{\partial \mathcal{H}_n^4}{\partial z} = -\overline{f_n} \quad \text{in} \quad \Omega_0 \qquad \mathcal{H}_n^{4\pm} = \pm g_t^\pm \quad \text{on} \quad \Gamma_0^\pm$$

Using (4.9), an integration from -1 to 1 with respect to z of the above equations leads to the equilibrium equations of Result 1.

The boundary conditions on γ_0 are obtained by assuming that the leading term (V^0, u^0) of the expansion of (V, u) satisfies the clamped condition on γ_0.

$\square$

4.2. Comparison with existing models

In the literature, the two-dimensional shell models are generally obtained in a weak formulation. So the nonlinear membrane model obtained in Result 1 in a local formulation must be written in a weak formulation to be compared to other existing ones. To do this, let us define the space of admissible displacements :

$$V(\omega_0) = \left\{ U = (V, u) \ : \ \omega_0 \longrightarrow \mathbb{R}^3, \text{ "smooth"}, \ (V, u) = (0, 0) \ \text{on} \ \gamma_0 \right\}$$

where V and u denote the tangential and normal components of the displacement U. Then, the two-dimensional membrane equations of Result 1 can be written in the following weak formulation:

RESULT 2.

For applied forces such as $\mathcal{G}_t = \mathcal{G}_n = \mathcal{F}_t = \mathcal{F}_n = \varepsilon$, the leading term (V^0, u^0) of the expansion of the displacement is solution of the weak problem :

Find $(V^0, u^0) \in V(\omega_0)$ such that:

$$(4.12) \qquad \int_{\omega_0} \text{Tr} \, (n_t^0 \delta \Delta_t^0) d\omega_0 = \int_{\omega_0} (\overline{p_t} \delta V^0 + p_n \delta u^0) d\omega_0 \qquad \forall (\delta V^0, \delta u^0) \in V(\omega_0)$$

with :

$$n_t^0 = \frac{4\beta}{2+\beta} \, \text{Tr}(\Delta_t^0) I_0 + 4\Delta_t^0 \quad \text{and} \quad 2\Delta_t^0 = \overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\hat{\partial} V^0}{\partial p_0}} \frac{\hat{\partial} V^0}{\partial p_0} + \overline{\frac{\partial u^0}{\partial p_0}} \frac{\partial u_0}{\partial p_0}$$

where $\delta \Delta_t^0$ denotes the variation of Δ_t^0 due to the virtual displacements $(\delta V^0, \delta u^0)$ associated to (V^0, u^0).

The proof of this result is classical and uses the Stokes theorem. It will not be discussed here.

Thus, for a severe force level, we obtain a nonlinear membrane model whose weak formulation of Result 2 is different from the one obtained in [13] for "general shells"[10]. However, it seems to be a generalization to shallow shells of the nonlinear membrane model obtained for plates in [21][34].

4.3. Back to physical variables

The return to physical variables in equations of Result 1 leads to define :

$$V^{*0} = V_r V^0 = L_0 V^0 \qquad \text{and} \qquad u^{*0} = u_r^0 u^0 = L_0 u^0$$

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[10] In the papers using a description of the shell with local coordinates, the "general shells" corresponds for us to strongly curved shells. In the second part of this paper, where the strongly curved shells are studied, we will see that we obtain the same nonlinear membrane model as in [13] for a severe force level.

We then have the following result:

Result 3.

For applied forces f^* and g^* such as $\mathcal{G}_t = \mathcal{G}_n = \mathcal{F}_t = \mathcal{F}_n = \varepsilon$, the displacement (V^{*0}, u^{*0}) depends only on p_0^* and verifies the following nonlinear membrane problem :

$$h_0 \operatorname{div}\left(n_t^{*0}\left(I_0 + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} \right) \right) = -\overline{p_t^*} \quad \text{in} \quad \omega_0^*,$$

$$h_0 \operatorname{div}^*\left(n_t^{*0} \overline{\frac{\partial u^{*0}}{\partial p_0^*}} \right) = -p_n^* \quad \text{in} \quad \omega_0^*,$$

$$V^{*0} = 0 \quad \text{and} \quad u^{*0} = 0 \quad \text{on,} \quad \gamma_0^*$$

where:

$$n_t^{*0} = \frac{4\lambda\mu}{\lambda + 2\mu} \operatorname{Tr}(\Delta_t^{*0})\, I_0 + 4\mu\Delta_t^{*0},$$

$$2\Delta_t^{*0} = \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} + \frac{\hat{\partial} V^{*0}}{\partial p_0^*} + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} \frac{\hat{\partial} V^{*0}}{\partial p_0^*} + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} \frac{\hat{\partial} V^{*0}}{\partial p_0^*},$$

$$p_t^* = g^{*+}_t + g^{*-}_t + \int\limits_{-h_0}^{h_0} f_t^* dz^* \quad \text{and} \quad p_n^* = g^{*+}_n + g^{*-}_n + \int\limits_{-h_0}^{h_0} f_n^* dz^*.$$

P r o o f. Let us define $n_t^{*0} = \mu n_t^0$, $p_t^* = g^{*+}_t + g^{*-}_t + \dfrac{\mathcal{G}_t}{\mathcal{F}_t} \displaystyle\int\limits_{-h_0}^{h_0} f_t^* dz^*$ and

$p_n^* = g^{*+}_n + g^{*-}_n + \dfrac{\mathcal{G}_n}{\mathcal{F}n} \displaystyle\int\limits_{-h_0}^{h_0} f_n^* dz^*$. Going back to the physical variables in Result 1, we get :

$$L_0 \operatorname{div}\left(n_t^{*0}\left(I_0 + \overline{\frac{\hat{\partial} V^{*0}}{\partial p_0^*}} \right) \right) = -\frac{1}{\mathcal{G}_t}\overline{p_t^*} \quad \text{in} \quad \omega_0^*,$$

$$L_0 \operatorname{div}^*\left(n_t^{*0} \overline{\frac{\partial u^{*0}}{\partial p_0^*}} \right) = -\frac{1}{\mathcal{G}_n}p_n^*, \quad \text{in} \quad \omega_0^*,$$

$$V^{*0} = 0 \quad \text{and} \quad u^{*0} = 0 \quad \text{on,} \quad \gamma_0^*.$$

$\square$

According to the force level considered here, we obtain the equations of Result 3. In what follows, to save space, we won't give the dimensional equations associated

to the other dimensionless models. However, it would be possible to obtain them in the same way.

5. Another membrane model

In this section, we assume that the shell is subjected to a high force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^2$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^3$. First we show that the displacement (V^0, u^0) which is a solution of the Result 1 is equal to zero. This proves that the reference scales of the displacement $V_r = u_r = L_0$ have been not properly chosen and we will make a new dimensional analysis of equilibrium equations with $V_r = u_r = h_0$. We then show that the new asymptotic expansion of equations leads to another membrane model.

5.1. Determination of the reference scales of the displacement

For a force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^2$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^3$, following the proof of Results 1 and 2, we obtain the same weak formulation without a right side, whose associated minimization problem is the following one :

Find $(V^0, u^0) \in V(\omega_0)$ *which minimizes on* $V(\omega_0)$ *the functional* $\mathcal{J} = \displaystyle\int_{\omega_0} \alpha d\omega_0$

where $\quad \alpha = \dfrac{2\beta}{\beta + 2} [\mathrm{Tr}(\Delta_t)]^2 + 2\mathrm{Tr}(\Delta_t{}^2),$

and $\quad 2\Delta_t(V, u) = \dfrac{\overline{\partial V}}{\partial p_0} + \dfrac{\hat{\partial V}}{\partial p_0} + \dfrac{\overline{\partial V}}{\partial p_0}\dfrac{\hat{\partial V}}{\partial p_0} + \dfrac{\overline{\partial u}}{\partial p_0}\dfrac{\partial u}{\partial p_0}.$

It is easy to show that the displacement (V^0, u^0) which minimizes the functional $\mathcal{J}$ defined above is solution of Result 2.

As the density of energy α is positive and is equal to zero if and only if $\Delta_t = 0$, this minimization problem implies that $\Delta_t(V^0, u^0) = 0$. We now have to use the following lemma to prove that $(V^0, u^0) = (0, 0)$:

LEMMA 1. In $V(\omega_0)$, the solution of the equation

$$\frac{\overline{\partial V}}{\partial p_0} + \frac{\hat{\partial V}}{\partial p_0} + \frac{\overline{\partial V}}{\partial p_0}\frac{\hat{\partial V}}{\partial p_0} + \frac{\overline{\partial u}}{\partial p_0}\frac{\partial u}{\partial p_0} = 0$$

is $(V^0, u^0) = (0, 0)$ in ω_0

P r o o f. Let us explain $\displaystyle\int_{\omega_0} \mathrm{Tr}(\Delta_t) d\omega_0$. We have :

$$\mathrm{Tr}(\Delta_t) = \mathrm{Tr}\left(\frac{\hat{\partial V}}{\partial p_0}\right) + \frac{1}{2}\mathrm{Tr}\left(\frac{\overline{\partial V}}{\partial p_0}\frac{\hat{\partial V}}{\partial p_0}\right) + \frac{1}{2}\left\|\frac{\overline{\partial u}}{\partial p_0}\right\|^2$$

As $V = 0$ on γ_0, using the Stokes formula, we get:

$$\int_{\omega_0} \mathrm{Tr}\left(\frac{\hat{\partial} V}{\partial p_0}\right) d\omega_0 = 0.$$

So we have:

$$\int_{\omega_0} \mathrm{Tr}\left(\Delta_t\right) d\omega_0 = \frac{1}{2} \int_{\omega_0} \left(\mathrm{Tr}\left(\frac{\overline{\hat{\partial} V}}{\partial p_0} \frac{\hat{\partial} V}{\partial p_0}\right) + \left\|\frac{\overline{\partial u}}{\partial p_0}\right\|^2 \right) d\omega_0.$$

Thus the equation $\Delta_t(V, u) = 0$ implies that $\displaystyle\int_{\omega_0} \mathrm{Tr}(\Delta_t) d\omega_0 = 0$ and we have:

$$\int_{\omega_0} \left(\mathrm{Tr}\left(\frac{\overline{\hat{\partial} V}}{\partial p_0} \frac{\hat{\partial} V}{\partial p_0}\right) + \left\|\frac{\overline{\partial u}}{\partial p_0}\right\|^2 \right) d\omega_0 = 0.$$

As $\mathrm{Tr}\left(\dfrac{\overline{\hat{\partial} V}}{\partial p_0} \dfrac{\hat{\partial} V}{\partial p_0}\right) + \left\|\dfrac{\overline{\partial u}}{\partial p_0}\right\|^2$ is positive, we obtain:

$$\mathrm{Tr}\left(\frac{\overline{\hat{\partial} V}}{\partial p_0} \frac{\hat{\partial} V}{\partial p_0}\right) = 0 \quad \text{and} \quad \left\|\frac{\overline{\partial u}}{\partial p_0}\right\|^2 = 0 \quad \text{in} \quad \omega_0$$

which leads to :

$$\frac{\hat{\partial} V}{\partial p_0} = 0 \quad \text{and} \quad \frac{\overline{\partial u}}{\partial p_0} = 0 \quad \text{in} \quad \omega_0.$$

Finally we have:

$$\frac{\partial \|V\|^2}{\partial p_0} = 2\overline{V}\frac{\hat{\partial} V}{\partial p_0} = 0 \quad \text{and} \quad \frac{\overline{\partial u}}{\partial p_0} = 0 \quad \text{in} \quad \omega_0$$

which proves that $\|V\|$ and u are constant in ω_0. As V and u are zero on the boundary γ_0, they are equal to zero in all ω_0. This ends the proof of Lemma 1.

$\square$

Remarking that since $(V^0, u^0) = (0, 0)$, the expressions (4.11) of θ_0 and N reduce to $\theta_0 = 1$ and $N = N_0$. So we get:

$$(5.1) \qquad\qquad V^1 = V^1(p_0) \quad \text{and} \quad u^1 = u^1(p_0).$$

As we have proved that $(V^0, u^0) = (0, 0)$, we get

$$V = \frac{V^*}{V_r} = \frac{V^*}{L_0} = \varepsilon V^1 + \varepsilon^2 V^2 + \cdots$$

$$u = \frac{u^*}{u_r} = \frac{u^*}{L_0} = \varepsilon u^1 + \varepsilon^2 u^2 + \cdots$$

which is equivalent to :

$$\widetilde{V} = \frac{V^*}{\varepsilon V_r} = \frac{V^*}{h_0} = V^1 + \varepsilon V^2 + \cdots = \widetilde{V^0} + \varepsilon \widetilde{V^1} + \varepsilon^2 \widetilde{V^2} + \cdots$$

$$\widetilde{u} = \frac{u^*}{\varepsilon u_r} = \frac{u^*}{h_0} = u^1 + \varepsilon u^2 + \cdots = \widetilde{u^0} + \varepsilon \widetilde{u^1} + \varepsilon^2 \widetilde{u^2} + \cdots$$

Accordingly we have proved that for the level forces considered here, the reference scales of the displacement $V_r = u_r = L_0$ have not been properly chosen. For the leading term of the expansion of the displacement to be of the order of one unit, the reference scales of the displacement must verify $(V_r, u_r) = (h_0, h_0)$. Therefore the dimensionless equilibrium equations must be written again with $V_r = h_0$ and $u_r = h_0$ as the new reference scales. The new dimensionless displacements will still be denoted $V = V^*/V_r$ and $u = u^*/u_r$.

5.2. The associated asymptotic model

With these new reference scales of the displacement $(V_r, u_r) = (h_0, h_0)$, which are directly deduced from the force level considered, we make a new dimensional analysis of equilibrium Eqs. (2.7) – (2.9). We obtain the same dimensionless Eqs. (3.7) – (3.9) where V and u must be changed into εV and εu in the expression (4.3) of G. Thus we have with these new reference scales of the displacement :

(5.2)
$$G_t = \varepsilon^2 \left(\frac{\hat{\partial} V}{\partial p_0} - u \varepsilon C_0 \right) \kappa^{-1}, \qquad G_s = \varepsilon \frac{\partial V}{\partial z},$$

$$G'_s = \varepsilon^2 \kappa^{-1} \left(\varepsilon C_0 V + \frac{\overline{\partial u}}{\partial p_0} \right), \qquad G_n = \varepsilon \frac{\partial u}{\partial z}.$$

The new expressions of E, Σ and H can be obtained from (3.4) – (3.6) and (5.2).

On the other hand, the asymptotic expansion of equations enables to write again the new dimensionless solution (V, u) of the new dimensionless problem as a formal expansion with respect to ε. This is equivalent to changing (V^i, u^i) into (V^{i-1}, u^{i-1}) for $i \geq 1$ in the previous results. In particular, relation (5.1) becomes:

$$V^0 = V^0(p_0) \quad \text{and} \quad u^0 = u^0(p_0)$$

Then, we have the following result:

RESULT 4.

For a force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^2$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^3$, the leading term (V^0, u^0) of the expansion of (V, u) depends only on p_0 and satisfies the membrane equations

$$\text{div}\left(n_t^0\right) = -\overline{p_t} \qquad\qquad in \quad \omega_0,$$

(5.3)
$$\text{div}\left(n_t^0 \frac{\overline{\partial u^0}}{\partial p_0}\right) + \text{Tr}\left(n_t^0 C_0\right) = -p_n - \text{div}(M_t) \quad in \quad \omega_0,$$

$$V^0 = 0 \quad\text{ and }\quad u^0 = 0 \qquad\qquad in \quad \gamma_0,$$

where
$$n_t^0 = \frac{4\beta}{2+\beta}\text{Tr}\left(\Delta_t^0\right) I_0 + 4\Delta_t^0, \quad 2\Delta_t^0 = \frac{\overline{\hat{\partial}V^0}}{\partial p_0} + \frac{\hat{\partial}V^0}{\partial p_0}$$

$$p_t = g_t^+ + g_t^- + \int\limits_{-1}^{1} f_t\,dz, \quad M_t = g_t^+ - g_t^- + \int\limits_{-1}^{1} z\,f_t\,dz \quad\text{and}\quad p_n = g_n^+ + g_n^- + \int\limits_{-1}^{1} f_n\,dz.$$

P r o o f.

i) Expression   of (V^1, u^1)

According to the new dimensional analysis (5.2), we have:

$$G_t^2 = \frac{\hat{\partial}V^0}{\partial p_0}, \quad G_s^2 = \frac{\partial V^1}{\partial_z}, \quad G_s'^2 = \frac{\overline{\partial u^0}}{\partial p_0}, \quad G_n^2 = \frac{\partial u^1}{\partial z},$$

(5.4)
$$2E_t^3 = \frac{\overline{\hat{\partial}V^0}}{\partial p_0} + \frac{\hat{\partial}V^0}{\partial p_0}, \quad 2E_s^3 = \frac{\partial V^1}{\partial z} + \frac{\overline{\partial u^0}}{\partial p_0}, \quad 2E_n^3 = 2\frac{\partial u^1}{\partial z},$$

$$\Sigma_t^3 = \beta(\text{Tr}(E_t^3) + E_n^3)I_0 + 2E_t^3, \quad \Sigma_s^3 = 2E_s^3,$$

$$\Sigma_n^3 = \beta\text{Tr}(E_t^3) + (\beta + 2)E_n^3,$$

$$\mathcal{H}_t^4 = \Sigma_t^3, \quad \mathcal{H}_s^4 = \Sigma_s^3 \quad \mathcal{H'}_s^4 = \Sigma_s^3 \quad \mathcal{H}_n^4 = \Sigma_n^3$$

The cancellation of the factor of ε^4 in the new expansion of Eqs. (3.7)–(3.9) leads to problem $\mathcal{P}^4$:

$$\frac{\partial\mathcal{H'}_s^4}{\partial z} = 0 \quad\text{and}\quad \frac{\partial\mathcal{H}_n^4}{\partial z} = 0 \quad in \quad \Omega_0, \quad \mathcal{H'}_s^{4\pm} = 0 \quad\text{and}\quad \mathcal{H}_n^{4\pm} = 0 \quad on \quad \Gamma_0^\pm$$

which implies that

$$\mathcal{H}_s'^4 = 0 \quad \text{and} \quad \mathcal{H}_n^4 = 0 \quad \text{in} \quad \Omega_0$$

Using (5.5), we get:

(5.5) $$E_s^3 = 0 \quad \text{and} \quad 2E_n^3 = -\frac{\beta}{\beta+2}\text{Tr}(E_t^3)$$

which can be written in terms of displacements :

$$\frac{\partial V^1}{\partial z} = -\frac{\overline{\partial u^0}}{\partial p_0} \quad \text{and} \quad \frac{\partial u^1}{\partial z} = -\frac{\beta}{\beta+2}\text{Tr}(\Delta_t^0) \quad \text{in} \quad \Omega_0$$

and leads to

(5.6)
$$\begin{cases} u^1 = \underline{u}^1 - z\dfrac{\beta}{\beta+2}\text{Tr}\left(\Delta_t^0\right) \\[4mm] V^1 = \underline{V}^1 - z\dfrac{\overline{\partial u^0}}{\partial p_0} \end{cases}$$

where the fields of tangent vectors $\underline{V}^1$ and of scalars $\underline{u}^1$ depend only on p_0.

On the other hand, we have according to (5.4):

(5.7) $$E_t^3 = \Delta_t^0, \quad \Sigma_t^3 = \mathcal{H}_t^4 = \frac{1}{2}n_t^0 \quad \text{and} \quad \mathcal{H}_s^4 = \mathcal{H}_s'^4 = 0$$

where the expressions of n_t^0 and Δ_t^0 are those of Result 4.

ii) First membrane equation

The first equation of problem $\mathcal{P}^5$ then reduces to :

$$\text{div}\left(\mathcal{H}_t^4\right) + \frac{\overline{\mathcal{H}_s'^5}}{\partial z} = -\overline{f_t} \quad \text{in} \quad \Omega_0 \qquad \mathcal{H}_s'^{5\pm} = \pm g_t^\pm \quad \text{on} \quad \Gamma_0^\pm,$$

(5.8)

$$\frac{\mathcal{H}_n^5}{\partial z} = 0 \quad \text{in} \quad \Omega_0 \qquad \mathcal{H}_n^{5\pm} = 0 \qquad \text{on} \quad \Gamma_0^\pm.$$

Using (5.7), an integration upon the thickness (from -1 to 1 with respect to z) of the first above equation leads to the first equation of Result 4:

$$\text{div}(n_t^0) = -\overline{p_t}$$

where $p_t = g_t^+ + g_t^- + \int\limits_{-1}^{1} f_t\, dz.$

On the other hand, an integration of Eq. (5.8) with respect to z enables to calculate $\mathcal{H}_s'^5$ and $\mathcal{H}_n^5$. We obtain :

$$\mathcal{H}_s'^5 = \frac{1}{2}\left(zp_t + g_t^+ - g_t^- + \int_z^1 f_t \, dz - \int_{-1}^z f_t \, dz \right),$$

$$\mathcal{H}_n^5 = 0.$$

Moreover Eqs. (3.4) – (3.6) and (5.2), lead to :

$$\mathcal{H}_s^5 = \Sigma_s^4 + \Sigma_t^3 \overline{\frac{\partial u^0}{\partial p_0}} \qquad \text{and} \qquad \mathcal{H}_s'^5 = \Sigma_s^4$$

which implies that:

$$(5.9) \qquad \mathcal{H}_s^5 = \frac{1}{2}\left(zp_t + g_t^+ - g_t^- + \int_z^1 f_t \, dz - \int_{-1}^z f_t \, dz \right) + \Sigma_t^3 \overline{\frac{\partial u^0}{\partial p_0}}.$$

iii) Second membrane equation

The second equation of problem $\mathcal{P}^6$ reduces to :

$$\text{div}(\mathcal{H}_s^5) + \text{Tr}\left(\mathcal{H}_t^4 C_0\right) + \frac{\partial \mathcal{H}_n^6}{\partial z} = -f_n \quad \text{in} \quad \Omega_0,$$

$$\mathcal{H}_n^{6\pm} = \pm g_n^\pm \quad \text{on} \quad \Gamma_0^\pm.$$

Using (5.7) and (5.9), an integration upon the thickness leads to the second equation of Result 4:

$$\text{div}\left(n_t^0 \overline{\frac{\partial u^0}{\partial p_0}} \right) + \text{Tr}(n_t^0 C_0) = -p_n - \text{div}(M_t) \qquad \text{in} \quad \omega_0$$

with $p_n = g_n^+ + g_n^+ + \int_{-1}^1 f_n dz$ and $M_t = g_t^+ - g_t^+ + \int_{-1}^1 z\, f_t \, dz$. Let us notice that we have used the relation :

$$\int_{-1}^1 \left(\int_z^1 f_t \, dz - \int_{-1}^z f_t \, dz \right) dz = 2 \int_{-1}^1 z\, f_t \, dz.$$

The expansion of the clamped condition $U = 0$ on Γ_0 leads at the first order to :

$$(V^0, u^0) = (0,0) \qquad \text{on} \quad \gamma_0$$

which concludes the proof of Result 4.

$\square$

5.3. A few comments

The membrane model obtained in Result 4 for a high force level has to our knowledge no equivalent in the literature. If we set $C_0 = 0$, it is different from the FÖPPL plate model [20] given also in [46]. There is no nonlinear term coupling the deflection u^0 to the tangential displacement in the expression of the membrane strain, as in the von Kármán model.

On the contrary, the model obtained here is linear in the sense explained later. It can be split into two linear problems verified by V^0 and u^0, as the simplified version of the FÖPPL model given by H. M. BERGER[11] [1].

Indeed the first equation of Result 4:

$$\begin{cases} \operatorname{div}\left(n_t^0\right) = -\overline{p_t} & \text{in} \quad \omega_0, \\ V^0 = 0 & \text{on} \quad \gamma_0, \end{cases}$$

where $n_t^0 = \dfrac{4\beta}{2+\beta}\operatorname{Tr}(\Delta_t^0)\, I_0 + 4\Delta_t^0$ and $2\Delta_t^0 = \dfrac{\hat{\partial}V^0}{\partial p_0} + \dfrac{\hat{\partial}V^0}{\partial p_0}$, is a linear equation which depends only on the tangential displacement V^0. To prove that this problem has unique solution V^0 in $[H_0^1(\omega_0)]^2$, let us write the above equations in the following weak formulation:

Find $V^0 \in [H_0^1(\omega_0)]^2$ *such as*

(5.10)
$$\int_{\omega_0} \operatorname{Tr}\left(n_t^0 \delta\Delta_t^0\right) d\omega_0 = \int_{\omega_0} \overline{p_t}\delta V^0 d\omega_0 \qquad \forall \delta V^0 \in [H_0^1(\omega_0)]^2$$

where the expressions of n_t^0 *and* Δ_t^0 *are those of Result 4.*

It is possible to prove (see [4][5]) that the mapping

$$V \in [H_0^1(\omega_0)]^2 \rightarrow \left\{\int_{\omega_0} \operatorname{Tr}\left((\Delta_t^0)^2\right)\right\}^{1/2}$$

---

[11] H. M. BERGER [1] used the Föppl-von Kármán theory to formulate the strain energy density of a deformed plate. He then made the simplifying (but irrational) assumption of ignoring the term containing the second invariant of strain (relative to the first invariant of strain). This leads to a set of two uncoupled problems. For further comments see also [25].

is a norm on $[H_0^1(\omega_0)]^2$ equivalent to the usual one, provided that the Christoffel symbols of the middle surface are small enough. Therefore the first term of the weak formulation is elliptic in $[H_0^1(\omega_0)]^2$. Then if the forces are smooth enough $(L^2(\omega_0))$, this problem has an unique solution in $[H_0^1(\omega_0)]^2$.

Once V^0 and n_t^0 determined, the second equation of Result 4 becomes a linear second order equation with respect to the normal displacement u^0.

Therefore, in the case of shallow shells and for a high level of surface forces, this asymptotic approach enables to construct a membrane model which is in fact linear, but which cannot be deduced from the linear three-dimensional elasticity.

6. The Koiter's nonlinear shallow shell model

6.1. New reference scales of the displacements

In this section, we consider a shallow shell subjected to a moderate force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^3$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^4$. Then, for this force level, following the proof of Result 4, it is possible to prove that the tangential displacement V^0 is solution of $(5.3)_1$ without a right side. As this equation has an unique solution provided that V^0 is smooth enough, we have $V^0 = 0$. Let us notice that the second equation of Result 4 is then trivially satisfied and the normal displacement u^0 is undetermined.

Therefore, as $V^0 = 0$ for the force level considered here, the reference scale of the tangential displacement $V_r = h_0$ is still not properly chosen. We have to consider $V_r = \varepsilon h_0$ for V^0 to be different from zero. Thus the dimensionless equilibrium equations must be written again with $V_r = \varepsilon h_0$ and $u_r = h_0$ as reference scales. The dimensionless components of the displacements will still be denoted V and u. As previously, the new dimensionless equations so obtained from (2.7)-(2.9) are the same as (3.7)-(3.9) where V must be changed into εV in the expression (5.2) of G. Thus we have :

(6.1)
$$G_t = \varepsilon^3 \left(\frac{\hat{\partial} V}{\partial p_0} - u C_0 \right) \kappa^{-1}, \qquad G_s = \varepsilon^2 \frac{\partial V}{\partial z},$$

$$G_s' = \varepsilon^2 \kappa^{-1} \left(\varepsilon^2 C_0 V + \frac{\overline{\partial u}}{\partial p_0} \right), \qquad G_n = \varepsilon \frac{\partial u}{\partial z}.$$

The new expressions of E, Σ and H are then deduced from $(3.4)-(3.6)$ and (6.1).

6.2. The asymptotic model

We write again the new tangential displacement V of the new dimensionless problem obtained with $V_r = \varepsilon h_0$ as a formal expansion with respect to ε. This is

equivalent to change V^i into V^{i-1} for $i \geq 1$ in the results of the previous section. Thus relation (5.6) becomes :

$$(6.2) \qquad u^0 = \zeta_n^0(p_0), \quad V^0 = \zeta_t^0(p_0) - z \frac{\overline{\partial \zeta_n^0}}{\partial p_0} \quad \text{and} \quad u^1 = u^1(p_0)$$

and implies that (V^0, u^0) is a Kirchhoff-Love displacement. Then, the new asymptotic expansion of equilibrium equations leads to the following result:

RESULT 5.

For given applied forces such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^3$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^4$, the leading term (V^0, u^0) of the expansion of (V, u) is a Kirchhoff-Love displacement which satisfies:

i) $u^0 = \zeta_n^0(p_0)$ and $V^0 = \zeta_t^0(p_0) - z \dfrac{\overline{\partial \zeta_n^0}}{\partial p_0}.$

ii) $\zeta^0 = (\zeta_t^0, \zeta_n^0)$ is a solution of the following equations :

$$\operatorname{div}\left(n_t^0\right) = -\overline{p_t} \qquad\qquad\qquad \text{in} \quad \omega_0,$$

$$\operatorname{div}(\overline{\operatorname{div} m_t^0}) + \operatorname{div}\left(n_t^0 \frac{\overline{\partial \zeta_n^0}}{\partial p_0}\right) + \operatorname{Tr}(n_t^0 C_0) = -p_n - \operatorname{div}(M_t) \quad \text{in} \quad \omega_0,$$

$$\zeta_n^0 = \frac{\partial \zeta_n^0}{\partial \nu_0} = 0 \qquad \text{and} \qquad \zeta_t^0 = 0 \qquad\qquad \text{on} \quad \gamma_0,$$

where ν_0 denotes the unit external normal to γ_0 and where

$$n_t^0 = \frac{4\beta}{2 + \beta} \operatorname{Tr}(\Delta_t^0) I_0 + 4\Delta_t^0, \qquad 2\Delta_t^0 = \frac{\overline{\partial \zeta_t^0}}{\partial p_0} + \frac{\partial \zeta_t^0}{\partial_0} + \frac{\overline{\partial \zeta_n^0}}{\partial p_0} \frac{\partial \zeta_n^0}{\partial p_0} - 2\zeta_n^0 C_0,$$

$$m_t^0 = \frac{4\beta}{3(2 + \beta)} \operatorname{Tr}(K_t^0) I_0 + \frac{4}{3} K_t^0, \qquad K_t^0 = -\frac{\hat{\partial}}{\partial p_0} \frac{\partial \zeta_n^0}{\partial p_0},$$

$$p_t = g_t^+ + g_t^- + \int_{-1}^{1} f_t \, dz, \qquad M_t = g_t^+ - g_t^- + \int_{-1}^{1} z \, f_t \, dz$$

and $p_n = g_n^+ + g_n^- + \displaystyle\int_{-1}^{+1} f_n \, dz.$

P r o o f. The proof of this result is similar to the previous ones. Let us just give the main steps.

i) Problem $\mathcal{P}_5$ leads to the following expression of u^1 and V^1 which will be used later :

(6.3)
$$
\begin{cases}
u^1 = u^1(p_0) \\
V^1 = \underline{V}^1 - z\dfrac{\overline{\partial u^1}}{\partial p_0}
\end{cases}
$$

where $\underline{V}^1$ and $\underline{u}^2$ only depend on p_0.

ii) On the other hand, using (3.4)–(3.6), (6.1) and (6.2), we get:

(6.4) $$ E_t^4 = \Delta_t^0 + zK_t^0, \qquad \mathcal{H}_t^5 = \Sigma_t^4 = \frac{1}{2}(n_t^0 + 3zm_t^0) $$

where the expressions of n_t^0, m_t^0 Δ_t^0 and K_t^0 are those of Result 5.

iii) After a few calculations, the first equation of problem $\mathcal{P}_6$ leads to

$$
\mathcal{H}_s^6 = \frac{1}{2}\left(zp_t + g_t^+ - g_t^- + \int_z^1 f_t dz - \int_{-1}^z f_t dz + \frac{3(1-z^2)}{2}\overline{div(m_t^0)} + \Sigma_t^0\frac{\overline{\partial \zeta_n^0}}{\partial p_0}\right).
$$

Finally, the integration upon the thickness of the first equation of problem $\mathcal{P}_6$ and of the second equation of problem $\mathcal{P}_7$ leads to the equations of the Koiter's nonlinear shallow shell model of the Result 5.

6.3. Comments

Accordingly, the nonlinear Koiter's shallow shell model has been rigorously justified by asymptotic expansion, without any a priori assumption. On the contrary, the order of magnitude of the displacements has been directly deduced from the force level considered. We so justify the scaling assumptions on the displacements generally made in the literature [11][19].

Let us notice that the existence of a unique solution of Koiter's shallow shell model has been proved in [2] when the applied forces are weak enough.

We recall that there exists two other shallow shell models, the Marguerre-von Kármán and the Marguerre one, which are very close to the Koiter's one. These two models, which only differ by the boundary conditions on the lateral surface, have been obtained by asymptotic expansion in the case of a particular description of the middle surface in local coordinates [10][7].

At last, let us notice that Koiter's shallow shell model is a generalization to shallow shells of the usual nonlinear plate model whose justification by asymptotic expansions can be found in [9][21][34].

7. The linear Novozhilov-Donnell model

The Novozhilov-Donnell model is generally obtained from the linear three-dimensional elasticity by making a priori assumptions (Kirchhoff-Love assumptions) and neglecting the terms of second order with respect to the curvature [17]. Later, many authors have tried to justify rigorously this model by asymptotic expansion of the linear three-dimensional equations. One finds a justification in [11] for a particular parametrization of the middle surface in local coordinates, in [4][5] by using an intrinsic variational formulation and in [15][17] from the local equilibrium equations. In this section, we propose to justify rigorously the linear Novozhilov-Donnell model from asymptotic expansion of the *nonlinear* equilibrium equations. This will enable us to determine precisely its domain of validity.

7.1. Order of magnitude of displacements

For a low force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^4$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^5$, we obtain the same equations as in Result 5 without a right side. The associated minimization problem implies that $\Delta_t^0 = K_t^0 = 0$ and then that $(\zeta_t^0, \zeta_n^0) = (0,0)$ (see [6]).

Since we have proved that $(V^0, u^0) = (0,0)$, the reference scales of the displacements $V_r = h_0 \varepsilon$ and $u_r = h_0$ don't correspond to the low force level considered here. We have to make a new dimensional analysis of equilibrium equations (2.7)–(2.9) with $V_r = \varepsilon^2 h_0$ and $u_r = \varepsilon h_0$ as the new reference scales. We then obtain the same dimensionless Eqs. (3.7)–(3.9) where V and u must be changed into εV and εu in the previous expression (6.1) of the components of G. Hence we have now:

(7.1)
$$G_t = \varepsilon^4 \left(\frac{\hat{\partial} V}{\partial p_0} - u C_0 \right) \kappa^{-1} \qquad G_s = \varepsilon^3 \frac{\partial V}{\partial z},$$

$$G_s' = \varepsilon^3 \kappa^{-1} \left(\varepsilon^2 C_0 V + \overline{\frac{\partial u}{\partial p_0}} \right) \qquad G_n = \varepsilon^2 \frac{\partial u}{\partial z},$$

and the new expressions of E, Σ and H will be calculated using (3.4)–(3.6).

7.2. The associated linear asymptotic model

The asymptotic expansion method enables to write again the new dimensionless solution (V, u) corresponding to $V_r = \varepsilon^2 h_0$ and $u_r = \varepsilon h_0$ as a formal expan-

sion with respect to ε. This is equivalent to the change (V^i, u^i) into (V^{i-1}, u^{i-1}) for $i \geq 1$ in the results of the previous section. In particular (6.3) becomes

$$(7.2) \qquad u^0 = \zeta_n^0(p_0), \quad V^0 = \zeta_t^0(p_0) - z\overline{\frac{\partial \zeta_n^0}{\partial p_0}}$$

which proves that (V^0, u^0) is still a Kirchhoff-Love displacement. On the other hand, according to $(3.4)-(3.6)$ and (7.1), the first non-zero terms of the expansion of G, E, Σ and $\mathcal{H}$ are now given by:

$$\mathcal{H}_t^6 = \Sigma_t^5, \quad \mathcal{H}_s^6 = \Sigma_s^5, \quad \mathcal{H'}_s^6 = \Sigma_s^5, \quad \mathcal{H}_n^6 = \Sigma_n^5,$$

$$\Sigma_t^5 = \beta(\mathrm{Tr}(E_t^5) + E_n^5)I_0 + 2E_t^5, \quad \Sigma_s^5 = 2E_s^5,$$

$$\Sigma_n^5 = \beta\mathrm{Tr}(E_t^5) + (\beta + 2)E_n^5$$

$$2E_t^5 = \overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0} - 2u^0 C_0,$$

$$(7.3) \qquad 2E_s^5 = \frac{\partial V^1}{\partial z} + \overline{\frac{\partial u^1}{\partial p_0}},$$

$$2E_n^5 = 2\frac{\partial v^2}{\partial z},$$

$$G_t^4 = \frac{\hat{\partial} V^1}{\partial p_0} - u^1 C_0, \quad G_s^4 = \frac{\partial V^1}{\partial z},$$

$$G'^4_s = \overline{\frac{\partial u^1}{\partial p_0}}, \quad G_n^4 = \frac{\partial u^2}{\partial z}.$$

We then have the following result:

RESULT 6.

For a low force level such as $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^4$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^5$, the leading term (V^0, u^0) of the expansion of (V, u) is a Kirchhoff-Love displacement which verifies :

i) $\quad u^0 = \zeta_n^0(p_0) \;$ and $\; V^0 = \zeta_t^0(p_0) - z\overline{\frac{\partial \zeta_n^0}{\partial p_0}}.$

ii) $\zeta^0 = (\zeta_t^0, \zeta_n^0)$ is solution of the dimensionless Novozhilov-Donnell model:

$$\operatorname{div}\left(n_t^0\right) = -\overline{p_t} \qquad \text{in} \quad \omega_0,$$

$$\operatorname{div}(\overline{\operatorname{div} m_t^0}) + \operatorname{Tr}(n_t^0 C_0) = -p_n - \operatorname{div}(M_t) \quad \text{in} \quad \omega_0,$$

$$\zeta_n^0 = \frac{\partial \zeta_n^0}{\partial \nu_0} = 0 \qquad \text{and} \qquad \zeta_t^0 = 0 \qquad \text{on} \quad \gamma_0,$$

where ν_0 denotes the unit external normal along γ_0 and where

$$n_t^0 = \frac{4\beta}{2+\beta}\operatorname{Tr}(\Delta_t^0)\,I_0 + 4\Delta_t^0, \qquad \Delta_t^0 = \frac{1}{2}\left(\frac{\widehat{\partial \zeta_t^0}}{\partial p_0} + \frac{\hat{\partial}\zeta_t^0}{\partial p_0} - 2\zeta_n^0 C_0\right),$$

$$m_t^0 = \frac{4\beta}{3(2+\beta)}\operatorname{Tr}(K_t^0)I_0 + \frac{4}{3}K_t^0, \qquad K_t^0 = -\frac{\hat{\partial}}{\partial p_0}\frac{\overline{\partial \zeta_n^0}}{\partial p_0},$$

$$p_t = g_t^+ + g_t^- + \int\limits_{-1}^{1} f_t\,dz, \qquad M_t = g_t^+ - g_t^- + \int\limits_{-1}^{1} z\,f_t\,dz,$$

$$\text{and} \quad p_n = g_n^+ + g_n^- + \int\limits_{-1}^{1} f_n\,dz.$$

P r o o f.   The proof of this result is similar to the previous ones and is left to the reader.

Contrary to the existing justifications of the Novozhilov-Donnell model, the approach explained here enables us to deduce it directly from the *nonlinear three-dimensional elasticity*. This result is fundamental because it specifies its domain of validity.

Indeed, the linear Novozhilov-Donnell model is proved to be valid for weaker force levels as the nonlinear Koiter's shallow shell one. These forces lead to deflections of εh_0 order and not of h_0 order, as we could think according to the existing justifications of the Novozhilov-Donnell model from the linear elasticity [5][15].

On the other hand, let us notice that the so obtained Novozhilov-Donnell model is an extension of the linear Kirchhoff-Love plate model. Indeed, if the curvature operator C_0 takes the value zero, we find again the classical linear

Kirchhoff-Love model, which has been already justified by asymptotic expansion from the linear three-dimensional elasticity in [8][4][33]. One finds also in [12][35] a justification from the nonlinear three-dimensional elasticity. At least, we recall that the existence and the unicity of the solution of the Novozhilov-Donnell model has been proved in [4] for shallow shells.

7.3. Domain of validity of the Novozhilov-Donnell model

We have proved that the linear Novozhilov-Donnell model is valid for a low force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^4$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^5$ which leads to deflections of εh_0 order. In fact, the linear Novozhilov-Donnell is valid for lower force levels $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^p$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^{p+1}$, $p \geq 5$. These force levels lead to deflections of $\varepsilon^{p-3}h_0$ order.

Indeed, for a force level $\mathcal{G}_t = \mathcal{F}_t = \varepsilon^p$ and $\mathcal{G}_n = \mathcal{F}_n = \varepsilon^{p+1}$, $p \geq 5$, we obtain the Novozhilov-Donnell model of Result 6 without a right side, whose unique solution is $(V^0, u^0) = (0, 0)$. Therefore according to the same argument as previously, the reference scales (V_r, u_r) of the displacement must be chosen equal to $(h_0\varepsilon^{p-2}, h_0\varepsilon^{p-3})$. A new dimensional analysis of the equations with $V_r = h_0\varepsilon^{p-2}$, $u_r = h_0\varepsilon^{p-3}$ and a new asymptotic expansion lead again to the Novozhilov-Donnell model of Result 6.

It is important to notice that for sufficiently weak force levels (of ε^p order with $p \geq 4$), the problem becomes linear with respect to the displacements and the asymptotic model that we obtain is the Novozhilov-Donnell one. This result means that the linear Novozhilov-Donnell model can be used for sufficiently weak force levels of $\varepsilon^{p \geq 4}$ order, where the dimensionless numbers $\mathcal{G}_t, \mathcal{F}_t, \mathcal{G}_n, \mathcal{F}_n$ are known quantities of the problem.

8. Conclusion

The method of classification of asymptotic shell models developed in this paper is constructive. It leads to a classification from the level of applied forces without any a priori assumption[12]. On the contrary, the order of magnitude of the displacements (characterized by the reference scales V_r and u_r) and the corresponding two-dimensional model are directly deduced from the force levels. These force levels are characterized by the dimensionless numbers $\mathcal{F}_t$, $\mathcal{F}_n$, $\mathcal{G}_t, \mathcal{G}_n$ which are known data of the problem.

In this paper, we have studied only a combination of $(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n)$ for each value of $\tau = \text{Max}(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n)$. However, the study of the other combinations is not fundamental; it would lead to the same two-dimensional models with a

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[12] In the sense defined in the Introduction.

different right side. The following table resumes the so obtained classification with respect to τ:

τ	(V_r, u_r)	Shell model	$\Delta_t^0,\ K_t^0$
ε	(L_0, L_0)	**non linear membrane model** $\operatorname{div}\left(n_t^0\left(I_0 + \overline{\hat{\partial}V^0/\partial p_0}\right)\right) = -p_t$ $\operatorname{div}\left(n_t^0\ \overline{\partial u^0/\partial p^0}\right) = -p_n$ $V^0_{\mid\gamma_0} = u^0_{\mid\gamma_0} = 0$	$2\Delta_t^0 = \dfrac{\overline{\hat{\partial}V^0}}{\partial p_0} + \dfrac{\hat{\partial}V^0}{\partial p_0} + $ $\dfrac{\overline{\hat{\partial}V^0}}{\partial p_0}\dfrac{\hat{\partial}V^0}{\partial p_0} + \dfrac{\overline{\partial u^0}}{\partial p_0}\dfrac{\partial u^0}{\partial p_0}$
ε^2	(h_0, h_0)	**another membrane model** $\operatorname{div}(n_t^0) = -p_t$ $\operatorname{div}\left(n_t^0\ \overline{\partial u^0/\partial p^0}\right) + \operatorname{Tr}(n_t^0 C_0)$ $= -p_n - \operatorname{div}(M_t)$ $V^0_{\mid\gamma_0} = u^0_{\mid\gamma_0} = 0$	$2\Delta_t^0 = \dfrac{\overline{\hat{\partial}V^0}}{\partial p_0} + \dfrac{\hat{\partial}V^0}{\partial p_0}$
ε^3	$(\varepsilon h_0, h_0)$	**non linear Koiter's shallow shell model** $\operatorname{div}(n_t^0) = -p_t$ $\operatorname{div}(\operatorname{div} m_t^0) + \operatorname{div}\left(n_t^0\ \partial\zeta_n^0/\partial p^0\right)$ $+\operatorname{Tr}(n_t^0 C_0) = -p_n - \operatorname{div}(M_t)$ $\zeta_t^0_{\mid\gamma_0} = \zeta_n^0_{\mid\gamma_0} = \dfrac{\partial\zeta_n^0}{\partial\nu_0}_{\mid\gamma_0} = 0$	$2\Delta_t^0 = \dfrac{\overline{\hat{\partial}\zeta_t^0}}{\partial p_0} + \dfrac{\hat{\partial}\zeta_t^0}{\partial p_0} + $ $\dfrac{\overline{\hat{\partial}\zeta_n^0}}{\partial p_0}\dfrac{\hat{\partial}\zeta_n^0}{\partial p_0} - 2\zeta_n^0 C_0$ $K_t^0 = -\dfrac{\hat{\partial}}{\partial p_0}\dfrac{\overline{\hat{\partial}\zeta_n^0}}{\partial p_0}$
$\varepsilon^{p\geq 4}$	$h_0(\varepsilon^{p-2}, \varepsilon^{p-3})$	**linear Novozhilov-Donnell model** $\operatorname{div}(n_t^0) = -p_t$ $\operatorname{div}(\operatorname{div} m_t^0) + \operatorname{Tr}(n_t^0 C_0)$ $= -p_n - \operatorname{div}(M_t)$ $\zeta_t^0_{\mid\gamma_0} = \zeta_n^0_{\mid\gamma_0} = \dfrac{\partial\zeta_n^0}{\partial\nu_0}_{\mid\gamma_0} = 0$	$2\Delta_t^0 = \dfrac{\overline{\hat{\partial}\zeta_t^0}}{\partial p_0} + \dfrac{\hat{\partial}\zeta_t^0}{\partial p_0}$ $-2\zeta_n^0\, C_0$ $K_t^0 = -\dfrac{\hat{\partial}}{\partial p_0}\dfrac{\overline{\hat{\partial}\zeta_n^0}}{\partial p_0}$

where $n_t^0 = \dfrac{4\beta}{\beta+2}\operatorname{Tr}(\Delta_t^0)I_0 + 4\Delta_t^0$ and $m_t^0 = \dfrac{4\beta}{3(\beta+2)}\operatorname{Tr}(K_t^0)I_0 + \dfrac{4}{3}K_t^0$.

On the other hand, the classification deduced from the three-dimensional nonlinear elasticity enables us to specify the domain of validity of the obtained

two-dimensional shell models, thanks to the dimensionless numbers naturally introduced.

In particular we have proved that the usual linear Novozhilov-Donnell model is valid for applied force levels weaker than the ones for which the nonlinear Koiter's shallow shell model is obtained.

These forces lead to deflections of εh_0 order and not of h_0 order. This result is important and underlines the pathology of the results obtained from asymptotic expansion of linear three-dimensional equilibrium equations, which are already an "expansion at the first order" of nonlinear equilibrium equations. Indeed, when the linear Novozhilov-Donnell model is deduced from the linear three-dimensional [32], it seems to have the same domain of validity as the nonlinear Koiter's shallow shell model (deflections of h_0 order).

Finally, let us notice the constructive character of this approach. Indeed another membrane model, which has to our knowledge no equivalent in the literature, has been put in a prominent position for high force levels. This model cannot be obtained from the linear elasticity.

In the second part of this paper, we will study the strongly curved shells which have a different asymptotic behaviour. In this case, the classification is more complex : it depends not only on the force levels, but also on the existence of inextensional displacements which keep invariant the metric of the middle surface of the shell.

Appendix A. Intrinsic formalism of surface theory

We recall here the principal notations of the intrinsic formalism of surface theory used in this paper. It is inspired from the works of J.M. Souriau [45], R. Valid [47][48][25], J. Breuneval [3][4][5] and P. Destuynder [4][5].

Parametrized surface

Let U be an open set of $\mathbb{R}^2$ and

$$f \; : \; \begin{aligned} U & \to \mathbb{R}^3 \\ x = (u, v) & \mapsto p = f(x) \end{aligned}$$

an embedding in $\mathbb{R}^3$ (see Fig. 2). Then $\omega = f(U)$ is called a surface embedded in $\mathbb{R}^3$ and U the open set of reference of the system of local coordinates (f, U). We assume here that f is smooth enough ($C^2(U)$).

Local basis of ω

The independent vectors $a_1 = \dfrac{\partial f}{\partial u}$ and $a_2 = \dfrac{\partial f}{\partial v}$ span a vectorial space called tangent space at $p = f(x)$ to ω and denoted $T_p\omega$. We denote (a_1, a_2) the natural or the local basis of $T_p\omega$ and S the matrix defined by $S = (a_1, a_2)$.

Finally we define $N = \dfrac{a_1 \wedge a_2}{\| a_1 \wedge a_2 \|}$ the unit normal at p to ω. Therefore, each vector W of $\mathbb{R}^3$ can be split into

$$W = \Pi W + N \overline{N} W = V + uN$$

where $V = \Pi W$ and u denote respectively the orthogonal projection of W onto $T_p \omega$ and the normal N, the overbar the operator of transposition.
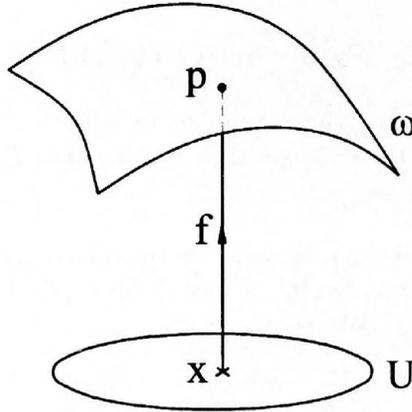
FIG. 2. Parametrization of the surface ω.

First fundamental form.

At each point p of ω, the scalar product of $\mathbb{R}^3$ implies a scalar product on $T_p \omega$:

$$\overline{dp}V = \overline{dx}(\overline{S}S)Y$$

where $dp = Sdx$ and $V = SY$ denote two tangent vectors of $T_p \omega$. We can so define, when p varies on ω, a field of covariant tensors $g \in T_p^* \omega \otimes T_p^* \omega$ where $T_p^* \omega$ denotes the dual space of $T_p \omega$.

DEFINITION 1. *The field of quadratic forms associated to g is called the first fundamental form of the surface ω. In the local or natural basis, it is represented by the matrix:* $G = \dfrac{\overline{\partial p}}{\partial x} \dfrac{\partial p}{\partial x} = \overline{S}S.$

Covariant derivative of a field of tangent vectors

Let $p \mapsto dp = Sdx$ and $p \mapsto V = SY$ be two fields of tangent vectors at p to ω. The derivative dV of the vector field V in the direction dp is not generally tangent to ω.

DEFINITION 2. *We define on ω a derivation ∇ for which the derivative of a tangent vector field is tangent :*

$$\nabla \quad : \quad T\omega \times T\omega \quad \to T_p\omega,$$
$$(dp, V) \qquad \mapsto \qquad \nabla_{dp}V \stackrel{\text{def}}{=} \Pi dV,$$

$\nabla_{dp}V$ *is the covariant derivative of the tangent vector field V in the direction dp, denoted also $\hat{d}V$. In the local basis, we have:*

$$\nabla_{dp}V = S[dY + \Gamma(dx, Y)].$$

Γ is the Christoffel operator whose components in the local basis are the Christoffel symbols $\Gamma_{\alpha\beta}^{\delta}$ and Π the orthogonal projection onto $T_p\omega$.

Second fundamental form

The normal part of the derivative dV in the direction dp of the tangent vector field $p \mapsto V$ can be represented in the local basis by the bilinear symmetric form F such as $\overline{N}dV = \overline{dx}FY$. We have

$$F = -\frac{\overline{\partial N}}{\partial x}\frac{\partial p}{\partial x} = -\frac{\overline{\partial p}}{\partial x}\frac{\partial N}{\partial x}$$

where $\dfrac{\partial N}{\partial x} : \mathbb{R}^2 \to \mathbb{R}^3$ denotes the linear tangent mapping to the field $x \mapsto N$.

DEFINITION 3. *The quadratic form associated to F is called the second fundamental form of the surface ω.*

Curvature operator

The linear tangent mapping $\dfrac{\partial N}{\partial p} : dp \mapsto dN = \dfrac{\partial N}{\partial p}dp$ of the field $p \mapsto N(p)$ defines an endomorphism of the tangent plane $T_p\omega$. Indeed, as $\overline{N}N = 1$, we have $\overline{N}dN = 0$ which implies that $dN \in T_p\omega$.

DEFINITION 4. *The endomorphism $C = -\dfrac{\partial N}{\partial p}$ is called curvature operator of the surface ω. It is symmetric with respect to the scalar product.*

Let us notice that in the local basis associated to the system of local coordinates (f, U), the operator C is represented by the matrix $G^{-1}F$. Indeed, we have:

$$F = \frac{\overline{\partial p}}{\partial x}C\frac{\partial p}{\partial x} = \overline{S}CS \qquad \text{and} \qquad S^{-1}CS = (\overline{S}S)^{-1}\overline{S}CS = G^{-1}F.$$

Derivative of a field of tangent vectors

Let $p \mapsto dp$ and $p \mapsto V$ be two tangent vector fields. Then the derivative dV of the tangent vector field V in the direction dp can be written in the intrinsic form :

$$dV = \hat{d}V + (\overline{dp}CV)N$$

where C denotes the curvature operator and $\hat{d}V = \dfrac{\hat{\partial}V}{\partial p}\,dp$ — the covariant derivative of V in the direction dp. We have also :

$$\frac{\partial V}{\partial p} = \frac{\hat{\partial}V}{\partial p} + N\overline{V}C$$

Derivative of a vector field of $\mathbb{R}^3$ defined on a surface

Let $p \mapsto W = V + uN$ be a vector field defined in ω which takes its values in $\mathbb{R}^3$ and $p \mapsto dp$ a field of tangent vectors. We then define the derivative dW of the vector field $p \mapsto W$ in the direction dp as : $dW = dV + duN + udN$. The associated tangent linear mapping

$$\frac{\partial W}{\partial p} \quad : \quad T_p\omega \quad \to \mathbb{R}^3,$$
$$dp \quad \mapsto \quad \frac{\partial W}{\partial p}dp = dW,$$

can be written:

$$\frac{\partial W}{\partial p} = \left[\frac{\hat{\partial}V}{\partial p} - uC\right] + N\left[\overline{V}C + \frac{\partial u}{\partial p}\right],$$

Classical two-dimensional divergence

The divergence of a tangent vector field V defined on a surface ω is given by:

$$\operatorname{div}(V) = \operatorname{Tr}\left(\frac{\hat{\partial}V}{\partial p}\right)$$

where Tr denotes the trace operator and $\dfrac{\hat{\partial}}{\partial p}$ the covariant derivative on ω.

The divergence of a field of endomorphisms A_t of the tangent plane $T_p\omega$ can be defined as follows

$$\operatorname{div}(A_t)V = \operatorname{div}(A_tV) - \operatorname{Tr}\left(A_t\frac{\hat{\partial}V}{\partial p}\right)$$

for all tangent vector field V defined on ω.

Particular divergence div_{t3}

It is possible to generalize the classical two-dimensional divergence of a field of endomorphisms of $T_p\omega$ to a field of operators $A_{t_3} : \omega \mapsto \mathcal{L}(\mathbb{R}^3, T_p\omega)$, denoted div_{t_3} as follows (see [47][49]) :

$$\text{div}_{t_3}(A_{t_3})W = \text{div}(A_{t_3}W) - \text{Tr}\left(A_t \frac{\hat{\partial}V}{\partial p}\right)$$

for all vector field $W : \omega \mapsto \mathbb{R}^3$ of ω.

The divergence div_{t_3} enables to write equations in a more compact form and to simplify the calculations. However, it can be linked to the classical two-dimensional divergence as follows:

LEMMA 2. Let A_{t3} be a field of operator defined on ω which takes its values in $\mathcal{L}(\mathbb{R}^3, T_p\omega)$. Then the field A_{t3} can be split as follows: $A_{t3} = A_t + A_s\overline{N}$ where $A_t = A_{t3}\Pi$ is a field of endomorphisms of $T_p\omega$ and $A_s = A_{t3}N$ a field of tangent vectors to ω. Moreover, it can be proved that:

$$\text{div}_{t3}(A_{t3}) = \text{div}(A_t) - \overline{A_s}C + (\text{div}(A_s) + \text{Tr}(A_tC))\overline{N}.$$

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Classification of thin shell models deduced from the nonlinear three-dimensional elasticity. Part II : the strongly curved shells

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IN THE FIRST part of this paper we have deduced a classification of asymptotic shallow shell models with respect to the level of applied forces, from the non-linear three-dimensional elasticity. We have used a constructive approach based on a dimensional analysis of the non-linear three-dimensional equilibrium equations, which naturally makes appear dimensionless numbers characterizing the applied forces ($\mathcal{F}$ and $\mathcal{G}$) and the geometry of the shell (ε and $\mathcal{C}$). To limit our study to one-scale problems, these dimensionless numbers are expressed in terms of the relative thickness ε of the shell, considered as the perturbation parameter. In the first part, we have studied the case of shallow shells corresponding to $\mathcal{C} = \varepsilon^2$. In the second part of this paper, we will study the case of strongly curved shells for which $\mathcal{C} = \varepsilon$. The classification that we obtain is then more complex. It depends not only on the force levels, but also on the existence of inextensional displacements which keep invariant the metric of the middle surface of the shell.

Key words: Nonlinear elasticity, Shell theory, Dimensional analysis, Asymptotic methods

1. Introduction

THIS PAPER is a continuation of [10] to which we will refer for the definitions and notations not explained here.

We recall that in the first part of this paper we have developed a constructive approach which enables us to deduce a classification of asymptotic shell models from the three-dimensional nonlinear elasticity. This approach is based on a dimensional analysis of nonlinear equilibrium equations which naturally makes appear dimensionless numbers, ε and $\mathcal{C}$ which reflect the geometry of the shell, $\mathcal{F}$ and $\mathcal{G}$ which characterize the applied forces. The reduction to a one-scale problem leads us to link $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{G}$ to the small reference parameter ε. In the first part, we have established a classification of shallow shells models (corresponding to

$\mathcal{C} = \varepsilon^2$) with respect to the level forces, from asymptotic expansion of the three-dimensional equations of nonlinear elasticity. In the second part of this paper, we propose to apply the same approach for strongly curved shells for which $\mathcal{C} = \varepsilon$. The classification obtained also depends on the geometric rigidity of the middle surface of the shell. However, contrary to the first part of this paper, the shell is now assumed to be clamped only on a part of the lateral surface and free on the other part.

The geometric rigidity of the shell is characterized by the existence of inextensional displacements which keep invariant the metric of the middle surface, in the linear and the nonlinear case. As the shell is assumed to be clamped only on a part of its lateral surface, such inextensional displacements are possible. Thus, in what follows, we will use the following terminology :

– a non-inhibited or inhibited shell in the nonlinear range (or just non-inhibited/ inhibited shell) will characterize a shell whose middle surface admits or not nonlinear inextensional mappings or displacements[1] (see (5.2) for the mathematical definition).

– a non-inhibited or inhibited shell in the linear range (or linearly non-inhibited/ inhibited shell) will characterize a shell whose middle surface admits or not linear inextensional displacements[2] (see (5.64) for the mathematical definition).

Let us notice that the definition of a non-inhibited shell in the nonlinear range used here is different from the one of "bendable surface" according to the terminology of SZWABOWICZ [24]. It is to be reminded that the importance of such inextensional deformations in shell theory is known since a long time (see for example LOVE [12], NOVOZHILOV [17], GOLDENVEIZER [9]). However, whereas the study of inextensional displacements in linear theory has been systematized in[2][8][19][20][26], only a few works on nonlinear inextensional displacements exist [24].

Moreover, to our knowledge there is no work which studies the link between linear and nonlinear inextensional displacements. In many practical cases, if the shell is inhibited (respectively non-inhibited) in the nonlinear range, then it is linearly inhibited (respectively non-inhibited). However, some examples exist which refute this observation. Indeed, let us consider half a sphere clamped on its lateral surface. If it is deformed so as to obtain the symmetric configuration with respect to the base, the transformation is inextensional in the nonlinear range, whereas it is well known that half a sphere completely clamped on its lateral surface is linearly inhibited (see [2]).

---

[1]The nonlinear inextensional mappings keep invariant the *nonlinear metric* of the middle surface.

[2]The linear inextensional displacements keep invariant the *linearized metric* of the middle surface.

2. Decomposition of the three-dimensional problem

As in the first part, we consider a shell of $2h_0$ thickness, whose middle surface is ω_0^*, which occupies the domain $\overline{\Omega}_0^*$ in its reference configuration, where $\Omega_0^* = \omega_0^* \times] - h_0, h_0[$ is an open set of $\mathbb{R}^3$. We recall that ω_0^* denotes a connected surface embedded in $\mathbb{R}^3$, whose diameter is L_0, with a "smooth enough" boundary γ_0^*. We note N_0 the unit normal to ω_0^*, C_0^* its curvature operator, q_0^* a generic point of $\overline{\Omega}_0^*$ and $\Gamma_0^{*\pm} = \overline{\omega}_0^* \times \{\pm h_0\}$ the upper and lower faces of the shell. Contrary to the first part of this paper, the shell is now assumed to be clamped only on a portion $\Gamma_0^{1*} = \gamma_0^{1*} \times [-h_0, h_0]$ of the lateral surface $\Gamma_0^* = \gamma_0^* \times [-h_0, h_0]$, and free on the other portion $\Gamma_0^{2*} = \gamma_0^{2*} \times [-h_0, h_0]$, where $(\gamma_0^{1*}, \gamma_0^{2*})$ denotes a partition of γ_0^*. Thus inextensional displacements are possible.
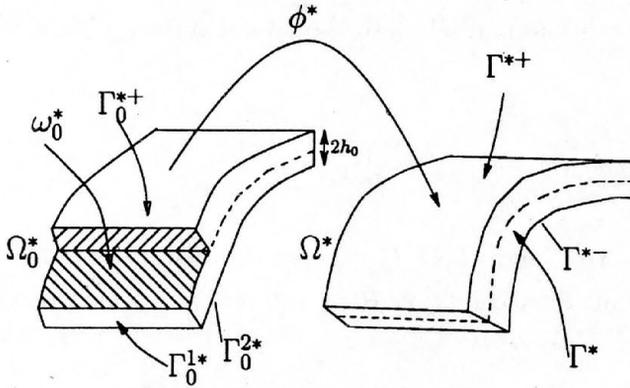
FIG. 1. Initial and final shell configuration.

Within the framework of nonlinear elasticity, the unknown mapping $\phi^*: \overline{\Omega}_0^* \to \mathbb{R}^3$ and the second Piola-Kirchhoff tensor Σ^* solve the equilibrium equations :

(2.1)
$$
\begin{aligned}
\text{Div}^*(\mathcal{H}^*) &= -\overline{f^*} && \text{in} \quad \Omega_0^*, \\
\overline{\mathcal{H}^*} N_0 &= \pm g^{*\pm} && \text{on} \quad \Gamma_0^{*\pm}, \\
\phi^* &= i_d && \text{on} \quad \Gamma_0^{1*}, \\
\overline{\mathcal{H}^*} n_0 &= 0 && \text{on} \quad \Gamma_0^{2*},
\end{aligned}
$$

with $\mathcal{H}^* = \Sigma^* \overline{F^*}$, where $F^* = \dfrac{\partial \phi^*(q_0^*)}{\partial q_0^*} = I_3 + \dfrac{\partial U^*}{\partial q_0^*}$ denotes the linear tangent mapping to ϕ^*, n_0 the unit external normal to Γ_0^*, $f^* : \overline{\Omega}_0^* \to \mathbb{R}^3$ and $g^{*\pm} : \Gamma_0^{*\pm} \to \mathbb{R}^3$ the applied body and surface forces, and i_d the identity mapping of $\mathbb{R}^3$. Let us recall that in the framework of Saint-Venant Kirchhoff materials,

Σ^* is linked to the nonlinear Green-Lagrange strain tensor $E^* = (\overline{F}^* F^* - I_3)/2$ by the constitutive relation $\Sigma^* = \lambda \operatorname{Tr}(E^*)I_3 + 2\mu E^*$, where I_3 denotes the identity of $\mathbb{R}^3$, λ and μ the Lamé constants of the material.

To make the expansion of the boundary condition $\overline{\mathcal{H}}^* n_0 = 0$ on Γ_0^{2*}, we must have an explicit expression of the normal n_0 with respect to the unit normal ν_0 to γ_0^*. We have the following proposition which has been proved in [6] :

PROPOSITION 1. Let ω_0^* be a connected surface embedded in $\mathbb{R}^3$. Let us consider the shell of $2h_0$ thickness which occupies the domain

$$\overline{\Omega_0^*} = \left\{ q_0^* = p_0^* + z^* N_0 \quad \text{where} \quad p_0^* \in \overline{\omega_0^*} \quad \text{and} \quad z^* \in [-h_0, +h_0] \right\}.$$

Then the unit external normal n_0 to the lateral surface Γ_0^* is given by:

$$(2.2) \qquad n_0 = \frac{1}{||\kappa_0^{*-1} \nu_0||} \kappa_0^{*-1} \nu_0$$

with $\kappa_0^* = I_0^* - z^* C_0^*$ and where I_0^* denotes the identity on $T\omega_0^*$.

Thus, the boundary condition $\overline{\mathcal{H}}^* n_0 = 0$ on Γ_0^{2*} can be written as :

$$(2.3) \qquad \mathcal{H}^* \Pi_0 \kappa_0^{*-1} \nu_0 = 0 \qquad \text{on} \quad \Gamma_0^{2*}.$$

In the case of strongly curved shells, it is not necessary to decompose completely the equilibrium Eqs. (2.1) onto $T\omega_0^* \oplus \mathbb{R} N_0$ as in the first part. To simplify the calculations, we will use only a partial decomposition. To do this, we introduce the two-dimensional divergence $\operatorname{div}_{t3}^*$ defined as follows[3] :

Let A be an operator field defined on ω_0^* which takes its values in $\mathcal{L}(\mathbb{R}^3, T\omega_0^*)$. Let us set $\mathcal{A}_t = \mathcal{A}\Pi_0$ and $\mathcal{A}_s = \mathcal{A}N_0$. Then we have :

$$\operatorname{div}_{t3}^*(\mathcal{A}) = \operatorname{div}^*(\mathcal{A}_t) - \overline{\mathcal{A}_s} C_0^* + (\operatorname{div}^*(\mathcal{A}_s) + \operatorname{Tr}(\mathcal{A}_t C_0^*)) \overline{N_0}$$

where div^* denotes the two-dimensional divergence on ω_0^*.

---

[3] This definition is similar to the one introduced in [25] by the author.

Thus, if we partially decompose $\mathcal{H}^*$ as follows : $\mathcal{H}^* = \Pi_0\mathcal{H}^* + N_0\overline{N_0}\mathcal{H}^*$, the equilibrium Eq. (2.1) can be written :

$$
(2.4) \quad
\begin{cases}
\text{div}^*_{t3}(\kappa_0^{*-1}\Pi_0\mathcal{H}^*) - \text{div}^*(\kappa^{*-1})\Pi_0\mathcal{H}^* \\
\qquad\qquad -\text{Tr}(\kappa_0^{*-1}C_0^*)\overline{N_0}\mathcal{H}^* + \dfrac{\partial\overline{N_0}\mathcal{H}^*}{\partial z^*} = -\overline{f^*} \quad \text{in} \quad \Omega_0^*, \\[2mm]
\overline{\mathcal{H}}^{*\pm}N_0 = \pm g^{*\pm} \quad \text{on} \quad \Gamma_0^{*\pm}, \\[2mm]
\phi^* = i_d \quad \text{on} \quad \Gamma_0^{1*}, \\[2mm]
\overline{\mathcal{H}}^*\Pi_0\kappa_0^{*-1}\nu_0 = 0 \quad \text{on} \quad \Gamma_0^{2*}.
\end{cases}
$$

3. Dimensional analysis and one-scale problem

As in the first part, we define the following dimensionless physical data and unknowns of the problem :

$$
(3.1) \quad
\begin{aligned}
&p_0 = \frac{p_0^*}{L_0}, \qquad q_0 = \frac{q_0^*}{L_0}, \qquad \phi = \frac{\phi^*}{\phi_r}, \qquad U = \frac{U^*}{U_r}, \qquad z = \frac{z^*}{h_0}, \\[2mm]
&C_0 = \frac{C_0^*}{C_r}, \qquad f_n = \frac{f_t^*}{f_{tr}}, \qquad g_n = \frac{f_n^*}{f_{nr}}, \qquad g_n = \frac{g_t^*}{g_{tr}}, \qquad g_t = \frac{g_n^*}{g_{nr}},
\end{aligned}
$$

where the variables with subscript r are the reference ones. The new variables which appear without an asterisk are dimensionless. To avoid any assumptions concerning the order of magnitude of the displacements, the reference scales ϕ_r and U_r are firstly assumed to be equal to L_0. If necessary, it will always be possible to define new reference scales for the displacement.

On the other hand, we will use as in the first part, the following notations to simplify the calculations :

$$
(3.2) \quad F = \varepsilon F^*, \quad E = \varepsilon^2 E^*, \quad \Sigma = \frac{\varepsilon^2}{\mu}\Sigma^* \quad \text{and} \quad \mathcal{H} = \frac{\varepsilon^3}{\mu}\mathcal{H}^*.
$$

Then the dimensionless expressions of F, E, Σ and $\mathcal{H}$ are given by:

$$
(3.3) \quad F = \varepsilon\frac{\partial\phi}{\partial p_0}\kappa_0^{-1} + \frac{\partial\phi}{\partial z}\overline{N_0}.
$$

$$
(3.4) \quad 2E = \overline{F}F - \varepsilon^2 I_3, \qquad \Sigma = \beta\,\text{Tr}(E)I_3 + 2E, \qquad \mathcal{H} = \beta\,\text{Tr}(E)\overline{F} + 2E\overline{F}
$$

and can be calculated from the mapping ϕ.

With these notations, the dimensional analysis of Eq. (2.4) leads to the dimensionless equilibrium equations:

$$\varepsilon \mathrm{div}_{t3}(\kappa_0^{-1}\Pi_0\mathcal{H}) - \varepsilon\mathrm{div}(\kappa_0^{-1})\Pi_0\mathcal{H} - \mathcal{C}\mathrm{Tr}(\kappa_0^{-1}C_0)\overline{N_0}\mathcal{H} + \frac{\partial\overline{N_0}\mathcal{H}}{\partial z}$$

$$= -\varepsilon^3\mathcal{F}\overline{f} \quad \text{in} \quad \Omega_0,$$

(3.5)

$$\overline{\mathcal{H}^{\pm}}N_0 = \pm\varepsilon^3\mathcal{G}g \quad \text{on} \quad \Gamma_0^{\pm},$$

$$\phi = i_d \quad \text{on} \quad \Gamma_0^1,$$

$$\overline{\mathcal{H}}\Pi_0\kappa_0^{-1}\nu_0 = 0 \quad \text{on} \quad \Gamma_0^2,$$

and naturally introduces the same dimensionless numbers ε, $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{G}$ as for shallow shells [10]. We recall that the two shape factors $\varepsilon = \dfrac{h_0}{L_0}$ and $\mathcal{C} = h_0 C_r$ characterize the geometry of the shell (relative thickness and curvature), whereas the force ratios $\mathcal{F} = \mathcal{F}_t = \mathcal{F}_n = \dfrac{h_0 f_{tr}}{\mu} = \dfrac{h_0 f_{nr}}{\mu}$ and $\mathcal{G} = \mathcal{G}_t = \mathcal{G}_n = \dfrac{g_{tr}}{\mu} = \dfrac{g_{nr}}{\mu}$ characterize the forces applied to the shell[4].

To apply the standard technique of asymptotic expansions, the problem must be reduced to a one-scale problem. To do this, ε is chosen as the reference perturbation parameter and the other dimensionless numbers must be linked to ε. In the first part of this paper, we have studied shallow shells which correspond to $\mathcal{C} = \varepsilon^2$. In the second part, we will consider strongly curved shells for which $\mathcal{C} = \varepsilon$.

On the other hand, as in the first part, the study of all the force levels can be reduced without loss of generality to the particular choices $\mathcal{F}_t = \mathcal{G}_t$ and $\mathcal{F}_n = \mathcal{G}_n$. Moreover, as in the case of strongly curved shells the tangential and the normal direction play a symmetrical role, we will only consider force levels such as $\mathcal{F}_t = \mathcal{F}_n = \mathcal{G}_t = \mathcal{G}_n$. However, to separate body forces from surface forces in the equations, we have set $\mathcal{F} = \mathcal{F}_t = \mathcal{F}_n$ and $\mathcal{G} = \mathcal{G}_t = \mathcal{G}_n$, even if we always consider force levels such as $\mathcal{F} = \mathcal{G}$.

Finally, the classification of asymptotic shell models will be deduced for decreasing force levels, from severe ($\mathcal{F} = \mathcal{G} = \varepsilon$) to low ($\mathcal{F} = \mathcal{G} = \varepsilon^{n\geq 4}$).

---

[4] More precisely, $\mathcal{F}_t$ and $\mathcal{F}_n$ (respectively $\mathcal{G}_t$ and $\mathcal{G}_n$) represent the ratio of the resultant on the thickness of the body forces (respectively the ratio of the surface forces) to μ considered as a reference stress.

4. The nonlinear membrane model

In this section, we begin the classification with severe force levels. We will show that the asymptotic expansion of equations naturally leads to the nonlinear membrane model.

4.1. Asymptotic expansion of equations

We consider a strongly curved shell $(\mathcal{C} = \varepsilon)$ subjected to a severe force level $\mathcal{G} = \mathcal{F} = \varepsilon$. Once reduced to a one-scale problem, we postulate that the displacement U or equivalently, the mapping $\phi = i_d + U$ admits a formal expansion with respect to ε:

$$U = U^0 + \varepsilon U^1 + \varepsilon^2 U^2 + ...$$

$$\phi = \phi^0 + \varepsilon \phi^1 + \varepsilon^2 \phi^2 + ...$$

with $\phi^0 = i_{\omega_0} + U^0$, $\phi^1 = U^1 + zN_0$ and $\phi^i = U^i$ for $i \geq 2$. If necessary, it will be possible to decompose U into $T\omega_0 \oplus \mathbb{R}N$ as follows : $U = V + uN_0$.

The expansion of ϕ implies via (3.3) and (3.4) an expansion of F, E, Σ and $\mathcal{H}$ whose terms will be calculated when necessary. Let us just notice that we now have :

$$\kappa^{-1} = (I_0 - \varepsilon z C_0)^{-1} = I_0 + z\varepsilon C_0^1 + z^2 \varepsilon^2 C_0^2 + ...$$

Then the asymptotic expansion of equations leads to the following result:

RESULT 1.

For applied forces such as $\mathcal{G} = \mathcal{F} = \varepsilon$, the leading term ϕ^0 of the expansion of ϕ depends only on p_0 and is a solution of the following nonlinear membrane model:

$$\text{div}_{t3} \left(n_t^0 \frac{\overline{\partial \phi^0}}{\partial p_0} \right) = -\overline{p} \quad \text{in} \quad \omega_0,$$

$$\phi^0 = i_{\omega_0} \quad \text{on} \quad \gamma_0^1,$$

$$n_t^0 \nu_0 = 0 \quad \text{on} \quad \gamma_0^2$$

where ν_0 denotes the unit external normal to γ_0 and where

$$n_t^0 = \frac{4\beta}{2+\beta} \text{Tr}(\Delta_t^0) I_0 + 4\Delta_t^0, \quad 2\Delta_t^0 = \frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial \phi^0}{\partial p_0} - I_0 \quad \text{and} \quad p = g^+ + g^- + \int\limits_{-1}^{+1} f \, dz.$$

P r o o f. The proof of this result is similar to the one of the nonlinear membrane model of the first part of this paper [10]. Let us just recall the intermediate

results which will be used in what follows. On one hand, the second term ϕ^1 of the expansion on ϕ can be written as:

$$(4.1) \qquad \phi^1 = U^1(p_0) + z\theta_0 N \qquad \text{with} \qquad \theta_0 = \sqrt{1 - \frac{2\beta}{\beta + 2}\mathrm{Tr}(\Delta_t^0)}$$

where N denotes the unit vector orthogonal to the surface $\omega = \phi^0(\omega_0)$ oriented so as θ_0 to be positive. On the other hand, according to (3.3)–(3.4), we get :

$$(4.2) \qquad F^1 = \theta^0 N \overline{N_0} + \frac{\partial \phi^0}{\partial p_0}.$$

$\square$

4.2. Comparison with existing results

To compare the nonlinear membrane model obtained in Result 1 to other existing models, we must explain its associated weak formulation. To do this, let us define the space of admissible displacements :

$$V(\omega_0) = \left\{ U \ : \ \omega_0 \to \mathbb{R}^3, \text{ "smooth", } U = 0 \text{ on } \gamma_0^1 \right\}$$

and the space of admissible mappings :

$$Q(\omega_0) = \left\{ \phi \ : \ \omega_0 \to \mathbb{R}^3, \text{ "smooth", } \phi = i_{\omega_0} \text{ on } \gamma_0^1 \right\}$$

Then the two-dimensional equations of Result 1 can be written in the following weak formulation:

RESULT 2.

The mapping $\phi^0 \in Q(\omega_0)$ satisfies the following weak problem :

$$(4.3) \qquad \int_{\omega_0} \mathrm{Tr}\,(n_t^0 \delta \Delta_t^0) d\omega_0 = \int_{\omega_0} \overline{p} \delta \phi^0 d\omega_0, \qquad \forall\, \delta\phi^0 \in V(\omega_0)$$

with

$$n_t^0 = \frac{4\beta}{2+\beta}\,\mathrm{Tr}\,(\Delta_t^0) I_0 + 4\Delta_t^0, \qquad 2\Delta_t^0 = \frac{\overline{\partial \phi^0}}{\partial p_0}\frac{\partial \phi^0}{\partial p_0} - I_0,$$

where $\delta \Delta_t^0$ denotes the virtual variation of Δ_t^0 due to the virtual displacement $\delta \phi^0$ associated to ϕ^0.

The proof of this result is classical and is based on the Stokes formula. It will not be detailed here. Let us notice that the non-linear membrane model has been also deduced by asymptotic expansion in [13] using a description of the shell in local coordinates. The equations obtained are the same as the ones of Result 2.

5. Non-inhibited shells in the nonlinear range

5.1. Nonlinear model coupling membrane-bending effects

In this section, we consider a shell non-inhibited in the nonlinear range subjected to a high force level of ε^2 order. First, using the previous results, we will specify the expressions of ϕ^0 and ϕ^1. Then we will continue the asymptotic expansion of the equilibrium Eq. (3.5).

5.1.1. Characterization of ϕ^0. For a level force such as $\mathcal{G} = \mathcal{F} = \varepsilon^2$, the Results 1 and 2 are still valid. We then obtain the same nonlinear membrane model without a right-hand side with the following associated minimization problem:

Find ϕ^0 which minimizes the functional $\mathcal{J} = \displaystyle\int_{\omega_0} \alpha \, d\omega_0$ on $Q(\omega_0)$, where

$$\alpha = \frac{2\beta}{\beta + 2} \text{Tr}(\Delta)^2 + 2\text{Tr}[(\Delta)^2] \quad and \quad 2\Delta = \frac{\overline{\partial\phi}}{\partial\phi} \frac{\partial\phi^0}{\partial p_0} - I_0.$$

As the density of energy α is positive and is equal to zero if and only if $\Delta = 0$, the solutions ϕ^0 of this minimization problem satisfy $\Delta = 0$ or equivalently:

$$(5.1) \qquad \frac{\overline{\partial\phi^0}}{\partial p_0} \frac{\partial\phi^0}{\partial p_0} = I_0.$$

As the shell is assumed to be non-inhibited, Eq. (5.1) admits other solutions as rigid mappings. Let us denote by $I_{\text{inex}}(\omega_0)$ the space of inextensional mappings:

$$(5.2) \quad I_{\text{inex}}(\omega_0)$$

$$= \left\{ \phi : \omega_0 \to \mathbb{R}^3, \text{ "smooth"}, \frac{\overline{\partial\phi}}{\partial p_0} \frac{\partial\phi}{\partial p_0} = I_0 \text{ in } \omega_0, \phi = i_{\omega_0} \text{ on } \gamma_0^1 \right\}$$

Thus we have $\phi^0 \in I_{\text{inex}}(\omega_0)$ and the expression (4.1) of ϕ^1 then becomes:

$$(5.3) \qquad \phi^1 = U^1(p_0) + zN$$

In the same way, the expression (4.2) of F^1 reduces to:

$$(5.4) \qquad F^1 = \frac{\partial\phi^0}{\partial p_0} + N\overline{N_0}$$

which implies that $\overline{F^1}F^1 = I_0 + N_0\overline{N_0} = I_3$. On the other hand, the expansion of the equation of continuity[5] $\det F^* \geq a > 0$ in Ω_0^* leads to $\det F^1 > 0$. Thus

---

[5] See condition (2) in the first part [10].

F^1 is a rotation of $\mathbb{R}^3$ and we have :

$$\overline{F^1}F^1 = F^1\overline{F^1} = I_3 \quad \text{and} \quad (F^1)^{-1} = \overline{F^1}.$$

Then replacing the expression (5.4) of F^1 in $F^1\overline{F^1} = I_3$, we get:

$$(5.5) \qquad\qquad \frac{\partial\phi^0}{\partial p_0}\frac{\overline{\partial\phi^0}}{\partial p_0} + N\overline{N} = I_3.$$

Using the decomposition $I_3 = I + N\overline{N}$ on $T\omega \oplus \mathbb{R}N$, where $\omega = \phi^0(\omega_0)$ and I denotes the identity on $T\omega$, we obtain

$$(5.6) \qquad\qquad \frac{\partial\phi^0}{\partial p_0}\frac{\overline{\partial\phi^0}}{\partial p_0} = I.$$

This relation will be used later to simplify the calculations.

Finally, using (3.3), (3.4), (5.1) and (5.3), we can calculate the first non-zero terms of the expansions of F, E, Σ and $\mathcal{H}$. On the one hand F^1 is given by (5.4), and on the other hand we have :

$$(5.7) \qquad F^2 = \frac{\partial\phi^2}{\partial z}\overline{N_0} + \frac{\partial U^1}{\partial p_0} - z\frac{\partial\phi^0}{\partial p_0}K_t^0, \qquad 2E^3 = \overline{F^1}F^2 + \overline{F^2}F^1,$$

$$\Sigma^3 = \beta\mathrm{Tr}(E^3)I_3 + 2E^3, \qquad\qquad \mathcal{H}^4 = \Sigma^3\overline{F^1},$$

with $K_t^0 = \tilde{C} - C_0$ and where $\tilde{C} = -\dfrac{\overline{\partial\phi^0}}{\partial p_0}\dfrac{\partial N}{\partial p_0} = -\dfrac{\overline{\partial\phi^0}}{\partial p_0}C\dfrac{\partial\phi^0}{\partial p_0}$ denotes the pull-back on ω_0 of the curvature operator C of the surface $\omega = \phi^0(\omega^0)$. Here $K_t^0 = \tilde{C} - C_0$ represents the classical nonlinear change of curvature.

5.1.2. Asymptotic expansion. Taking into account (5.1), we continue the asymptotic expansion of equations. We then have the following result:

RESULT 3.

For a non-inhibited shell in the nonlinear range, subjected to a high level of forces $\mathcal{G} = \mathcal{F} = \varepsilon^2$, the leading terms ϕ^0 and ϕ^1 of the expansion of ϕ satisfy:

i) ϕ^0 depends only on p_0 and $\phi^0 \in I_{\mathrm{inex}}(\omega_0)$.

ii) $\phi^1 = U^1 + zN$, where U^1 depends only on p_0 and N denotes the normal to the deformed configuration $\phi^0(\omega_0)$.

iii) ϕ^0 and U^1 are solutions of the following nonlinear equations:

$$\begin{cases} \text{div}_{t3}\left(n_t^1 \overline{\dfrac{\partial \phi^0}{\partial p_0}}\right) = -\overline{p} & \text{in} \quad \omega_0 \\[2mm] U^1 = 0 & \text{on} \quad \gamma_0^1 \\[2mm] n_t^1 \nu_0 = 0 & \text{on} \quad \gamma_0^2 \end{cases}$$

and

$$\begin{cases} \text{div}_{t3}\left((\chi - C_0 m_t^0)\overline{\dfrac{\partial \phi^0}{\partial p_0}} + n_t^1 \overline{\dfrac{\partial U^1}{\partial p_0}} - \overline{\text{div}}(m_t^0)\overline{N}\right) = -\overline{P} & \text{in} \quad \omega_0 \\[3mm] \phi^0 - i_{\omega_0} = \Theta^0 = 0 & \text{on} \quad \gamma_0^1 \\[3mm] \chi\nu_0 - m_t^0 C_0\nu_0 = m_t^0\nu_0 = \overline{M}\,\overline{\dfrac{\partial \phi^0}{\partial p_0}}\nu_0 - \text{div}\,(m_t^0)\nu_0 = 0 & \text{on} \quad \gamma_0^2 \end{cases}$$

where χ is a field of symmetrical tensors which depends only on ϕ^0, ϕ^1 and ϕ^2, and where:

$$n_t^1 = \frac{4\beta}{2+\beta}\text{Tr}\,(\Delta_t^1)I_0 + 4\Delta_t^1, \qquad\qquad 2\Delta_t^1 = \overline{\frac{\partial \phi^0}{\partial p_0}}\frac{\partial U^1}{\partial p_0} + \overline{\frac{\partial U^1}{\partial p_0}}\frac{\partial \phi^0}{\partial p_0},$$

$$m_t^0 = \frac{4\beta}{3(\beta+2)}\text{Tr}(K_t^0)I_0 + \frac{4}{3}K_t^0, \qquad\qquad K_t^0 = \tilde{C} - C_0,$$

$$\tilde{C} = -\overline{\frac{\partial \phi^0}{\partial p_0}}\frac{\partial N}{\partial p_0}, \qquad\qquad\qquad \Theta^0 = -\overline{\frac{\partial \phi^0}{\partial p_0}}N_0,$$

$$p = g^+ + g^- + \int\limits_{-1}^{1} f\,dz, \qquad\qquad\qquad M = g^+ - g^- + \int\limits_{-1}^{1} zf\,dz,$$

$$P = \overline{\text{div}}_{t3}\left(\overline{\frac{\partial \phi^0}{\partial p_0}}\,M\overline{N}\right) - \text{Tr}(C_0)M.$$

Before giving the proof of this result which is rather technical, let us notice that the model obtained here is not easy to interpret in this local formulation. Contrary to the asymptotic models previously obtained, this one takes into account the two unknowns ϕ^0 and U^1, where ϕ^0 is an inextensional mapping generating the curvature variation K_t^0, and U^1 is a displacement generating the membrane strain Δ_t^1.

On the other hand, let us remark that the expression of the field of symmetrical tensors χ, which is complex and depends on ϕ^0, ϕ^1 and ϕ^2, is not given explicitly. It is not necessary because it will vanish in the associated weak formulation which is given in the next result. For an interpretation of this model the reader can be referred to Result 4.

P r o o f. The proof can be split into five steps, from $i)$ to $v)$.

$i)$ *Determination of ϕ^2*

Problem $\mathcal{P}^4$ reduces to:

$$\frac{\partial \overline{\mathcal{H}^4} N^0}{\partial z} = 0 \quad \text{in} \quad \Omega_0,$$

$$\overline{\mathcal{H}^{4\pm}} N^0 = 0 \quad \text{on} \quad \Gamma_0^\pm,$$

which leads to $\overline{\mathcal{H}^4} N^0 = 0$ in Ω_0. Using (5.6) we get $F^1 \Sigma^3 N_0 = 0$ or equivalently

$$(5.8) \qquad\qquad \Sigma^3 N_0 = 0 \quad \text{in} \quad \Omega_0$$

because F^1 is inversible.

Then replacing Σ^3 by its expression (5.7), Eq. (5.8) becomes:

$$\left[(\beta + 2) \overline{N} \frac{\partial \phi^2}{\partial z} + \beta \operatorname{Tr}(\Delta_t^1 - z K_t^0) \right] N_0 + \frac{\overline{\partial U^1}}{\partial p_0} N_0 \frac{\partial \phi^0}{\partial p_0} \frac{\partial \phi^2}{\partial z} = 0$$

where $2\Delta_t^1 = \dfrac{\overline{\partial \phi^0}}{\partial p_0} \dfrac{\partial U^1}{\partial p_0} + \dfrac{\overline{\partial U^1}}{\partial p_0} \dfrac{\partial^0 \phi^0}{\partial p_0}$.

Now, let us project the last equation onto $T\omega_0$ and N_0. We get:

$$\frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial \phi^2}{\partial z} = -\frac{\overline{\partial U^1}}{\partial p_0} N \qquad \text{and} \qquad \overline{N} \frac{\partial \phi^2}{\partial z} - \frac{\beta}{\beta + 2} \operatorname{Tr}(\Delta_t^1 - z K_t^0)$$

or equivalently, using (5.6):

$$(5.9) \qquad I \frac{\partial \phi^2}{\partial z} = \frac{\partial \phi^0}{\partial p_0} \frac{\overline{\partial U^1}}{\partial p_0} N \qquad \text{and} \qquad \overline{N} \frac{\partial \phi^2}{\partial z} = -\frac{\beta}{\beta + 2} \operatorname{Tr}(\Delta_t^1 - z K_t^0).$$

As $I + N\overline{N} = I_3$, the two Eq. (5.9) are the projections onto $T\omega$ and N of the vector $\dfrac{\partial \phi^2}{\partial z}$.

Then we have:

$$(5.10) \qquad \frac{\partial \phi^2}{\partial z} = -\frac{\partial \phi^0}{\partial p_0} \frac{\overline{\partial U^1}}{\partial p_0} N - \frac{\beta}{\beta + 2} \operatorname{Tr}(\Delta_t^1 - z K_t^0) N.$$

A integration with respect to z then leads to the following expression of ϕ^2 :

$$(5.11) \qquad \phi^2 = \underline{U}^2 - z \frac{\partial \phi^0}{\partial p_0} \frac{\overline{\partial U^1}}{\partial p_0} N - z \frac{\beta}{2(\beta + 2)} \operatorname{Tr}(2\Delta_t^1 - z K_t^0) N.$$

where $\underline{U}^2$ depends only on p_0.

Now let us calculate the expressions of E^3, Σ^3 and $\mathcal{H}^4$. First, using (5.11), the expression (5.7) of F^2 becomes:

$$(5.12) \quad F^2 = -\frac{\partial \phi^0}{\partial p_0} \frac{\partial U^1}{\partial p_0} N \overline{N_0} + \frac{\partial U^1}{\partial p_0} - z \frac{\partial \phi^0}{\partial p_0} K_t^0 - \frac{\beta}{\beta+2} \operatorname{Tr}(\Delta_t^1 - z K_t^0) N \overline{N_0}.$$

Then multiplying the last equation by $\overline{F^1}$, and using the relations

$$F^1 = \frac{\partial \phi^0}{\partial p_0} + N \overline{N_0}, \quad \frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial \phi^0}{\partial p_0} = I_0 \quad \text{and} \quad \overline{N} \frac{\partial \phi^0}{\partial p_0} = 0,$$

we get:

$$\overline{F^1} F^2 = -\frac{\overline{\partial U^1}}{\partial p_0} N \overline{N_0} + N_0 \overline{N} \frac{\partial U^1}{\partial p_0} + \frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial U^1}{\partial p_0} - z K_t^0 - \frac{\beta}{\beta+2} \operatorname{Tr}(\Delta_t^1 - z K_t^0) N_0 \overline{N_0}.$$

Finally, in view of (5.7), E^3, Σ^3 and $\mathcal{H}^4$ can be expressed as follows :

$$(5.13) \quad \begin{aligned} E^3 &= \Delta_t^1 - z K_t^0 - \frac{\beta}{\beta+2} \operatorname{Tr}(\Delta_t^1 - z K_t^0) N_0 \overline{N_0}, \\ \Sigma^3 &= \frac{1}{2}(n_t^1 - 3 z m_t^0), \\ \mathcal{H}^4 &= \frac{1}{2}(n_t^1 - 3 z m_t^0) \frac{\overline{\partial \phi^0}}{\partial p_0}, \end{aligned}$$

where $n_t^1 = \dfrac{4\beta}{\beta+2} \operatorname{Tr}(\Delta_t^1) I_0 + 4\Delta_t^1$ and $m_t^0 = \dfrac{4\beta}{3(\beta+2)} \operatorname{Tr}(K_t^0) I_0 + \dfrac{4}{3} K_t^0$.

ii) *First equation of Result 3*

In view of (5.13), the cancellation of the factor of ε^5 in the expansion of equilibrium Eq. (3.5) leads to problem $\mathcal{P}^5$ which reduces to:

$$\operatorname{div}_{t3}(\Pi_0 \mathcal{H}^4) + \frac{\partial \overline{N_0} \mathcal{H}^5}{\partial z} = -\overline{f} \quad \text{in} \quad \Omega_0,$$

$$\overline{\mathcal{H}^{5\pm}} N_0 = \pm g^\pm \quad \text{on} \quad \Gamma_0^\pm,$$

Using (5.13) we get:

$$(5.14) \quad \frac{1}{2} \operatorname{div}_{t3}\left(n_t^1 \frac{\overline{\partial \phi^0}}{\partial p_0}\right) - \frac{3}{2} z \operatorname{div}_{t3}\left(m_t^0 \frac{\overline{\partial \phi^0}}{\partial p_0}\right) + \frac{\partial \overline{N_0} \mathcal{H}^5}{\partial z} = -\overline{f} \quad \text{in} \quad \Omega_0.$$

$$\overline{\mathcal{H}^{5\pm}} N_0 = \pm g^\pm \quad \text{on} \quad \Gamma_0^\pm.$$

Let us integrate the above equation upon the thickness. We then obtain:

$$\mathrm{div}_{t3}\left(n_t^1\frac{\overline{\partial U^1}}{\partial p_0}\right) = -\overline{p} \quad \text{in} \quad \omega_0$$

where $p = g^+ + g^- + \displaystyle\int_{-1}^{1} f \, dz$, which constitutes the first equation of Result 3.

On the other hand, an integration of (5.14) with respect to z leads to:

$$(5.15) \quad 2\overline{N_0}\mathcal{H}^5 = z\overline{p} + \overline{g^+} - \overline{g^-} + \int_z^1 \overline{f}\,dz - \int_{-1}^z \overline{f}\,dz - \frac{3}{2}(1-z^2)\mathrm{div}_{t3}\left(m_t^0\frac{\overline{\partial\phi^0}}{\partial p_0}\right).$$

In what follows, to simplify the calculations, we set:

$$(5.16) \quad \overline{A} = \overline{N_0}\mathcal{H}^5$$

$$= \frac{1}{2}\left(z\overline{p} + \overline{g^+} - \overline{g^-} + \int_z^1 \overline{f}\,dz - \int_{-1}^z \overline{f}\,dz - \frac{3}{2}(1-z^2)\mathrm{div}_{t3}\left(m_t^0\frac{\overline{\partial\phi^0}}{\partial p_0}\right)\right).$$

iii) Computation of $\mathcal{H}^5$:

Before writing problem $\mathcal{P}^6$, let us decompose $\mathcal{H}^5$ as follows:

$$(5.17) \qquad\qquad \mathcal{H}^5 = \Pi_0\mathcal{H}^5 + N_0\overline{N_0}\mathcal{H}^5 = \Pi_0\mathcal{H}^5 + N_0\overline{A}$$

according to (5.16). On the other hand, the expression of $\mathcal{H}^5$ reduces to:

$$(5.18) \qquad\qquad \mathcal{H}^5 = \Sigma^4\overline{F^1} + \Sigma^3\overline{F^2}$$

and Eq. (5.17) can be written as:

$$\mathcal{H}^5 = \Pi_0\Sigma^4\overline{F^1} + \Pi_0\Sigma^3\overline{F^2} + N_0\overline{A}.$$

Now, let us decompose also Σ^4 and Σ^3 as follows : $\Sigma^4 = \Sigma^4\Pi_0 + \Sigma^4 N_0\overline{N_0}$ and $\Sigma^3 = \Sigma^3\Pi_0 + \Sigma^3 N_0\overline{N_0}$. Then using (5.4), (5.8) and (5.12), the expression of $\mathcal{H}^5$ becomes:

$$(5.19) \quad \mathcal{H}^5 = \Pi_0\Sigma^4\Pi_0\frac{\overline{\partial\phi^0}}{\partial p_0} + \Pi_0\Sigma^4 N_0\overline{N} + \Pi_0\Sigma^3\Pi_0\left(\frac{\overline{\partial U^1}}{\partial p_0} - zK_t^0\frac{\overline{\partial\phi^0}}{\partial p_0}\right) + N_0\overline{A}.$$

On the other hand, let us multiply (5.18) by $\overline{N_0}$ on the left and by $\dfrac{\partial \phi^0}{\partial p_0}$ on the right. Using (5.4) and (5.8), we get $\overline{N_0}\,\Sigma^4\Pi_0 = \overline{A}\,\dfrac{\partial \phi^0}{\partial p_0}$ or equivalently

$$(5.20) \qquad\qquad \Pi_0 \Sigma^4 N_0 = \overline{\dfrac{\partial \phi^0}{\partial p_0}}\,A$$

because Σ^4 is symmetrical.

Finally, in view of (5.8), (5.13) and (5.20), the expression (5.19) of $\mathcal{H}^5$ becomes:

$$(5.21) \quad \mathcal{H}^5 = \Pi_0 \Sigma^4 \Pi_0 \overline{\dfrac{\partial \phi^0}{\partial p_0}} + \overline{\dfrac{\partial \phi^0}{\partial p_0}} A\overline{N} + \dfrac{1}{2}(n_t^1 - 3zm_t^0)\left(\overline{\dfrac{\partial U^1}{\partial p_0}} - zK_t^0 \overline{\dfrac{\partial \phi^0}{\partial p_0}}\right) + N_0\overline{A}.$$

Let us notice that the calculation of $\Pi_0 \Sigma^4 \Pi_0$ with respect to the displacements is not necessary. As already noticed, this term will vanish in the weak associated formulation.

iv) Second equation of Result 3

Problem $\mathcal{P}^6$ can be written as:

$$\mathrm{div}_{t3}(\Pi_0 \mathcal{H}^5 + zC_0\mathcal{H}^4) - \mathrm{Tr}\,(C_0)\overline{N_0}\mathcal{H}^5 - z\mathrm{div}(C_0)\Pi_0\mathcal{H}^4$$

$$(5.22) \qquad\qquad\qquad\qquad\qquad +\dfrac{\partial \overline{N_0}\mathcal{H}^6}{\partial z} = 0 \quad \text{in} \quad \Omega_0,$$

$$\overline{\mathcal{H}^{6\pm}}N_0 = 0 \quad \text{on} \quad \Gamma_0^{\pm}.$$

Using the expressions (5.13) and (5.21) of $\mathcal{H}^4$ and $\mathcal{H}^5$, an integration upon the thickness of Eq. (5.22) leads to:

$$(5.23) \quad \mathrm{div}_{t3}\left[\tilde{\chi}\overline{\dfrac{\partial \phi^0}{\partial p_0}} + n_t^1 \overline{\dfrac{\partial U^1}{\partial p_0}} - \overline{\dfrac{\partial \phi^0}{\partial p_0}}\,\mathrm{div}_{t3}\left(m_t^0 \overline{\dfrac{\partial \phi^0}{\partial p_0}}\right)\overline{N} - C_0 m_t^0 \overline{\dfrac{\partial \phi^0}{\partial p_0}}\right]$$

$$+ \mathrm{Tr}(C_0)\mathrm{div}_{t3}\left[m_t^0 \overline{\dfrac{\partial \phi^0}{\partial p_0}}\right] + \mathrm{div}\,(C_0)m_t^0 \overline{\dfrac{\partial \phi^0}{\partial p_0}} = -\overline{P} \quad \text{in} \quad \omega_0$$

where the expressions of P and M are those of Result 3 and where

$$\tilde{\chi} = \int_{-1}^{1}\Pi_0\Sigma^4\Pi_0\,dz + m_t^0 K_t^0.$$

In the last expression, $\tilde{\chi}$ is symmetrical because m_t^0 and K_t^0 are symmetrical and commute.

Now using the following property

$$\text{Tr}\,(C_0)\text{div}_{t3}\left(m_t^0\overline{\frac{\partial\phi^0}{\partial p_0}}\right) + \text{div}(C_0)m_t^0\overline{\frac{\partial\phi^0}{\partial p_0}} = \text{div}_{t3}\left(\text{Tr}\,(C_0)m_t^0\overline{\frac{\partial\phi^0}{\partial p_0}}\right),$$

Eq. (5.23) becomes:

$$(5.24)\quad \text{div}_{t3}\left[(\chi - C_0 m_t^0)\overline{\frac{\partial\phi^0}{\partial p_0}} + n_t^1\overline{\frac{\partial U^1}{\partial p_0}} - \overline{\frac{\partial\phi^0}{\partial p_0}}\text{div}_{t3}\left(m_t^0\overline{\frac{\partial\phi^0}{\partial p_0}}\right)\overline{N}\right]$$

$$= -\overline{P}\quad\text{in}\quad\omega_0$$

with:

$$(5.25)\qquad \chi = \tilde{\chi} + \text{Tr}\,(C_0)m_t^0 = \int_{-1}^{1}\Pi_0\Sigma^4\Pi_0\,dz + m_t^0 K_t^0 + \text{Tr}\,(C_0)m_t^0.$$

Let us just notice that χ is a field of symmetrical tensors.

Finally, as $\overline{\dfrac{\partial\phi^0}{\partial p_0}}\dfrac{\partial\phi^0}{\partial p_0} = I_0$, it is possible to prove that :

$$(5.26)\qquad\qquad \text{div}_{t3}\left(m_t^0\overline{\frac{\partial\phi^0}{\partial p_0}}\right)\frac{\partial\phi^0}{\partial p_0} = \text{div}\,(m_t^0)$$

where div denotes the classical two-dimensional divergence on ω_0. Thus Eq. (5.24) constitutes the second equation of Result 3.

v) Boundary conditions

To conclude the proof, let us examine the boundary conditions. The expansion of the clamping condition $\phi(q_0) = q_0$ on Γ_0^1 leads to $U^0 = 0$, $U^1 = 0$ and $N = N_0$ on γ_0^1. The last condition $N = N_0$ can also be written $\Theta^0 = 0$ on γ_0^1, where $\Theta^0 = -\dfrac{\partial\phi^0}{\partial p_0}N_0$ characterizes the rotation of the normal N_0 to the middle surface ω_0.

The boundary conditions on the portion γ_0^2 of the lateral surface γ_0 can be obtained formally from the three-dimensional boundary conditions as follows. As we have

$$\overline{\mathcal{H}}\kappa_0^{-1}\nu_0 = \varepsilon^4\overline{\mathcal{H}^4}\nu_0 + \varepsilon^5(z\overline{\mathcal{H}^4}C_0 + \overline{\mathcal{H}^5})\nu_0 + \ldots = 0\quad\text{on}\quad\Gamma_0^2,$$

using (5.13) and (5.21), we get:

$$(5.27)\qquad\qquad \frac{\partial\phi^0}{\partial p_0}(n_t^1\nu_0 - 3z\,m_t^0\nu_0) = 0\quad\text{on}\quad\Gamma_0^2,$$

(5.27)
[cont.]
$$\frac{\partial \phi^0}{\partial p_0} \Pi_0 \Sigma^4 \Pi_0 \nu_0 + z \frac{\partial \phi^0}{\partial p_0} (n_t^1 - 3zm_t^0) C_0 \nu_0 + N \overline{A} \frac{\partial \phi^0}{\partial p_0} \nu_0$$

$$+ \frac{1}{2} \left(\frac{\partial U^1}{\partial p_0} - z \frac{\partial \phi^0}{\partial p_0} K_t^0 \right) (n_t^1 - 3zm_t^0) \nu_0 = 0 \quad \text{on} \quad \Gamma_0^2.$$

The first equation of (5.27) leads to:

(5.28)
$$n_t^1 \nu_0 = 0 \quad \text{and} \quad m_t^0 \nu_0 = 0 \quad \text{on} \quad \gamma_0^2.$$

Now, multiplying the second equation of (5.27), on the one hand by $\overline{\dfrac{\partial \phi^0}{\partial p_0}}$ and on the other hand by $\overline{N}$, using (5.1) and (5.28), we get:

$$\Pi_0 \Sigma^4 \Pi_0 \nu_0 + \frac{1}{2} z (n_t^1 - 3zm_t^0) C_0 \nu_0 = 0 \quad \text{and} \quad \overline{A} \frac{\partial \phi^0}{\partial p_0} \nu_0 = 0 \quad \text{on} \quad \Gamma_0^2.$$

Then using (5.16), the integration upon the thickness of the above equations leads to:

(5.29)
$$\int_{-1}^{1} \Pi_0 \Sigma^4 \Pi_0 \nu_0 \, dz - m_t^0 C_0 \nu_0 = 0 \qquad \text{on} \qquad \gamma_0^2,$$

$$\overline{M} \frac{\partial \phi^0}{\partial p_0} \nu_0 - \text{div}_{t3} \left(m_t^0 \overline{\frac{\partial \phi^0}{\partial p_0}} \right) \frac{\partial \phi^0}{\partial p_0} \nu_0 = 0 \qquad \text{on} \qquad \gamma_0^2,$$

where $M = g^+ - g^- + \displaystyle\int_{-1}^{1} zf \, dz$.
According to (5.25) and (5.28), the first equation of (5.29) becomes:

$$\chi \nu_0 - m_t^0 C_0 \nu_0 = 0.$$

Finally using (5.26), the second equation of (5.29) reduces to

$$\overline{M} \frac{\partial \phi^0}{\partial p_0} \nu_0 - \text{div} \, (m_t^0) \nu_0 = 0 \qquad \text{on} \quad \gamma_0^2$$

which concludes the proof of Result 3. □

5.1.3. Nonlinear model with coupling effects. The model obtained in Result 3 is not usable numerically. It contains three unknowns ϕ^0, ϕ^1 and ϕ^2 coupled together in the tensor χ. However, its associated weak formulation enables to reduce the numbers of unknowns. Indeed, let us define the following admissible spaces of mappings and displacements :

$$V(\omega_0) = \left\{ U :\ \omega_0 \to \mathbb{R}^3, \text{ "smooth"},\ U = 0 \quad \text{on} \quad \gamma_0^1 \right\},$$

(5.30)
$$V_{\text{inex}}^{\phi^0}(\omega_0) = \left\{ U \in V(\omega_0),\ \frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial U}{\partial p_0} + \frac{\overline{\partial U}}{\partial p_0} \frac{\partial \phi^0}{\partial p_0} = 0 \quad \text{in} \quad \omega_0 \right\},$$

$$Q_{\text{inex}}(\omega_0) = \left\{ \phi \in I_{\text{inex}},\ \frac{\overline{\partial \phi}}{\partial p_0} N_0 = 0 \quad \text{on} \quad \gamma_0^1 \right\},$$

where I_{inex} is defined by (5.2).

Thus, the two-dimensional equations of Result 3 can be written in the following weak formulation:

RESULT 4.

$(\phi^0, U^1) \in Q_{\text{inex}}(\omega_0) \times V(\omega_0)$ is solution of the weak problem:

$$\int_{\omega_0} \text{Tr}\left(n_t^1 \delta \Delta_t^1 + m_t^0 \delta K_t^0 \right) d\omega_0 = \int_{\omega_0} (\overline{p} \delta U^1 - \text{Tr}(C_0) \overline{M} \delta \phi^0 + \overline{M} \delta N) d\omega_0$$

$$\forall\ (\delta \phi^0, \delta U^1) \in V_{\text{inex}}^{\phi^0}(\omega_0) \times V(\omega_0)$$

with:

$$n_t^1 = \frac{4\beta}{2+\beta} \text{Tr}\left(\Delta_t^1 \right) I_0 + 4\Delta_t^1, \qquad 2\Delta_t^1 = \frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial U^1}{\partial p_0} + \frac{\overline{\partial U^1}}{\partial p_0} \frac{\partial \phi^0}{\partial p_0},$$

$$m_t^0 = \frac{4\beta}{3(\beta+2)} \text{Tr}(K_t^0) I_0 + \frac{4}{3} K_t^0, \qquad K_t^0 = \tilde{C} - C_0 \quad \text{and} \quad \tilde{C} = -\frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial N}{\partial p_0},$$

$$p = g^+ + g^- + \int_{-1}^{1} f\, dz, \qquad M = g^+ - g^- + \int_{-1}^{1} zf\, dz.$$

The proof of this result is long and technical, hence will not be reported. It is based on the successive use of the Stokes formula.   □

5.1.4. Interpretation of this coupling model. In Result 4, we have obtained a two-dimensional shell model which couples membrane and bending effects. In this model, the resultant mapping of the middle surface of the shell is:

$$\tilde{\phi} = \phi^0 + \varepsilon U^1$$

and the resultant displacement of a point p_0 is represented in the following figure:
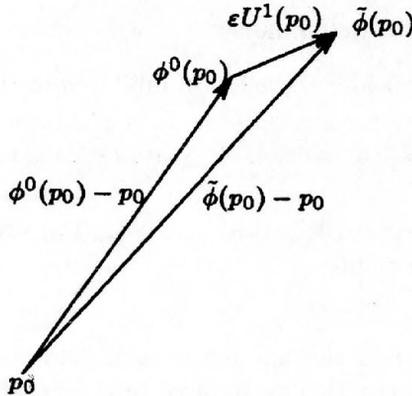
FIG. 2. Decomposition of the displacement at a material point p_0 of ω_0.

Thus, the displacement can be split into:
- an inextensional mapping ϕ^0.
- a small displacement εU^1.

On the other hand, in the coupling model of Result 4, the unknowns ϕ^0 and U^1 generate two kind of strain :
- a nonlinear pure bending strain K_t^0 due to ϕ^0
- a membrane strain Δ_t^1 due to the displacement U^1.

In fact, the strain Δ_t^1 can be written as

(5.31)
$$\Delta_t^1 = \overline{\frac{\partial \phi^0}{\partial p_0}} \Delta_{\phi^0}^1 \frac{\partial \phi^0}{\partial p_0}$$

where $\Delta_{\phi^0}^1 = \dfrac{1}{2}\left(\Pi \dfrac{\partial U^1}{\partial p} + \overline{\Pi \dfrac{\partial U^1}{\partial p}}\right)$ is the linear strain due to U^1 and calculated

at the point $p = \phi^0(p_0)$ of the deformed surface $\phi^0(\omega_0)$. Thus Δ_t^1 corresponds to the pull-back on ω_0 of the linear strain $\Delta_{\phi^0}^1$ due to U^1.

This coupled model is to our knowledge a new nonlinear shell model which couples membrane and bending effects. For a non-inhibited shell it is possible to prove formally that this model and the nonlinear Koiter's one have the same limit when ε tends towards zero. Thus, this new coupling model is an approximation of the nonlinear Koiter's one for non-inhibited shells. In the linear case, an asymptotic analysis of Koiter's model has been made in [19][20]. However, the only two models which are obtained are the linear membrane and the pure bending ones.

5.2. The nonlinear pure bending model

In this section we consider a shell, still inhibited in the nonlinear range, but subjected to a moderate force level $\mathcal{G} = \mathcal{F} = \varepsilon^3$. Then we prove that for this force level, the asymptotic expansion of equations leads to the classical nonlinear pure bending model.

We recall that the spaces $V_{\text{inex}}^{\phi^0}(\omega_0)$ and $Q_{\text{inex}}(\omega_0)$ are defined in (5.30). We then have the following result:

Result 5.

For a shell inhibited in the nonlinear range and subjected to a moderate force level $\mathcal{G} = \mathcal{F} = \varepsilon^3$, the leading term ϕ^0 of the expansion of the mapping ϕ depends only on p_0 and is solution of the nonlinear pure bending model:

$$\phi^0 \in Q_{\text{inex}}(\omega_0),$$

$$\int_{\omega_0} \text{Tr}\left(m_t^0 \delta K_t^0\right) d\omega_0 = \int_{\omega_0} \overline{p}\delta\phi^0 \, d\omega_0 \qquad \forall \, \delta\phi^0 \in V_{\text{inex}}^{\phi^0}(\omega_0)$$

where:

$$m_t^0 = \frac{4\beta}{3(\beta+2)}\text{Tr}\left(K_t^0\right) I_0 + \frac{4}{3}K_t^0, \qquad K_t^0 = \tilde{C} - C_0, \qquad \tilde{C} = -\frac{\overline{\partial\phi^0}}{\partial p_0}\frac{\partial N}{\partial p_0},$$

$$p = \int_{-1}^{+1} f \, dz + g^+ + g^-,$$

and where N denotes the normal to the deformed configuration $\phi^0(\omega_0)$.

Proof. For the moderate force level considered here, following the proof of Result 4, we obtain the same weak formulation with $\int_{-1}^{1} \overline{p} \, \delta\phi^0 \, d\omega_0$ as the right side:

$(\phi^0, U^1) \in Q_{\text{inex}}(\omega_0) \times V(\omega_0)$ satisfies:

$$(5.32) \quad \int_{\omega_0} \text{Tr}\left(n_t^1 \delta \Delta_t^1 + m_t^0 \delta K_t^0\right) d\omega_0 = \int_{\omega_0} \overline{p} \delta \phi^0 \, d\omega_0$$

$$\forall \, (\delta\phi^0, \delta U^1) \in V_{\text{inex}}^{\phi^0}(\omega_0) \times V(\omega_0).$$

Now, if we choose $\delta\phi^0 = 0$ in this weak formulation, we obtain

$$\int_{\omega_0} \text{Tr}\left(n_t^1 \delta \Delta_t^1\right) d\omega_0 = 0 \qquad \forall \, \delta U^1 \in V(\omega_0),$$

which leads to

$$(5.33) \qquad 2\Delta_t^1 = \overline{\frac{\partial \phi^0}{\partial p_0}} \frac{\partial U^1}{\partial p_0} + \overline{\frac{\partial U^1}{\partial p_0}} \frac{\partial \phi^0}{\partial p_0} = 0$$

according to the definition of n_t^1 (see Result 4). Finally, as $\Delta_t^1 = 0$ we have $n_t^1 = 0$ and the weak formulation (5.32) leads to the classical pure bending model.

$\square$

Thus we have justified the nonlinear pure bending model for a non-inhibited shell subjected to a moderate force level. The intrinsic approach used here makes clearly appear the curvature change $K_t^0 = \tilde{C} - C_0$, difference between the pull-back of the final curvature and the initial curvature. This nonlinear pure bending model has been justified also in [11] using a description of the middle surface of the shell in local coordinates. However, in this case the expression of K_t^0 which is obtained is difficult to interpret.

Finally let us notice that the existence of solutions of the pure bending model has recently been studied in [3]. However, the eventual uniqueness of the solution is still to be proved.

5.3. The linear pure bending model for linearly non-inhibited shells

We now consider a shell, still non-inhibited in the nonlinear range, but subjected to a low force level $\mathcal{G} = \mathcal{F} = \varepsilon^4$. It is then necessary to distinguish the linearly non-inhibited from the linearly inhibited shells as well.

We will prove here that for linearly non-inhibited shells[6] subjected to the low force level considered here, the displacements are of the thickness order and the asymptotic model that we obtain is the linear pure bending one.

---

[6]Still non-inhibited in the nonlinear range.

5.3.1. New reference scales for the displacement field. We begin to prove that the leading term U^0 of the expansion of the displacement vector is equal to zero. Indeed, for a low force level $\mathcal{G} = \mathcal{F} = \varepsilon^4$, we obtain the nonlinear pure bending model of Result 5 without a right side whose associated minimization problem is the following one:

Find ϕ^0 which minimizes in $Q_{inex}(\omega_0)$ the functional $\mathcal{J}(\phi) = \displaystyle\int_{\omega_0} \alpha \, d\omega_0$,

with

$$\alpha = \frac{2\beta}{3(\beta+2)}\mathrm{Tr}(K)^2 + \frac{2}{3}\mathrm{Tr}(K^2), \quad K = \tilde{C} - C_0, \quad \tilde{C} = -\frac{\overline{\partial\phi}}{\partial p_0}\frac{\partial N}{\partial p_0},$$

were N denotes the unit normal to $\phi(\omega_0)$.

The solutions of this problem are the mappings ϕ^0 which satisfy

$$K_t^0 = \tilde{C} - C_0 = 0.$$

As ϕ^0 is an inextensional mapping which satisfies

$$\frac{\overline{\partial\phi^0}}{\partial p_0}\frac{\partial\phi^0}{\partial p_0} - I_0 = 0 \quad \text{in} \quad \omega_0$$

the rigid motion lemma implies that $\phi^0 = i_{\omega_0}$. We have in particular $\dfrac{\partial\phi^0}{\partial p_0} = I_0$ and $N = N_0$. Thus, the leading term of the expansion of the displacement satisfies $U^0 = \phi^0 - i_{\omega_0} = 0$. Moreover, according to (5.11) and (5.33), we get:

(5.34) $$\frac{\overline{\hat{\partial}U^1}}{\partial p_0} + \frac{\hat{\partial}U^1}{\partial p_0} = 0 \quad \text{and} \quad U^2 = \phi^2 = \underline{U}^2 - z\frac{\overline{\partial U^1}}{\partial p_0}N_0$$

where U^1 and $\underline{U}^2$ only depend on p_0 and where $\dfrac{\hat{\partial}}{\partial p_0} = \Pi_0\dfrac{\partial}{\partial p_0}$ denotes the covariant derivative on ω_0.

As we have proved that $U^0 = 0$, we get

$$U = \frac{U^*}{U_r} = \frac{U^*}{L_0} = \varepsilon U^1 + \varepsilon^2 U^2 + \cdots$$

which is equivalent to :

$$\tilde{U} = \frac{U^*}{\varepsilon U_r} = \frac{U^*}{h_0} = U^1 + \varepsilon U^2 + \cdots = \widetilde{U^0} + \varepsilon\widetilde{U^1} + \varepsilon^2\widetilde{U^2} + \cdots$$

Accordingly, for this low force level, the reference scale $U_r = L_0$ of the displacement is not properly chosen. We must consider $U_r = h_0$ for the leading term of the displacement to be different from zero. So the dimensionless equilibrium equations must be written again with $U_r = h_0$ as the new reference scale. The dimensionless displacement will still be noted with U. This new dimensional analysis does not modify the dimensionless equations (3.5) but only the components of F, E, Σ and H, where U must be changed into εU. In particular, the expression (3.3) of the tangent mapping F becomes:

$$(5.35) \qquad F = \varepsilon I_3 + \varepsilon^2 \frac{\partial U}{\partial p_0} \kappa^{-1} + \varepsilon \frac{\partial U}{\partial z} \overline{N_0}.$$

A new expansion of the displacement is then equivalent to change U^i into U^{i-1} for $i \geq 1$ in the previous results. In particular, expressions (5.34) become:

$$(5.36) \qquad 2\Delta_t^0 = \frac{\overline{\hat{\partial} U^0}}{\partial p_0} + \frac{\hat{\partial} U^0}{\partial p_0} = 0 \quad \text{and} \quad U^1 = \underline{U}^1 - z\frac{\overline{\partial U^0}}{\partial p_0} N_0$$

where U^0 and $\underline{U}^1$ only depend on p_0.

On the other hand, with this new reference scale of the displacement, the first non-zero terms of the expansion of F, E, Σ and $\mathcal{H}$ can be calculated from (3.3), (3.4) and (5.35) as follows:

$$(5.37) \qquad \begin{aligned} F^1 &= I_3, \quad F^2 = \frac{\partial U^0}{\partial p_0} + \Theta^0 \overline{N_0}, \\ F^3 &= \frac{\partial U^2}{\partial z} \overline{N_0} + \frac{\partial \underline{U}^1}{\partial p_0} + z\left(\frac{\partial \Theta^0}{\partial p_0} + \frac{\partial U^0}{\partial p_0} C_0\right), \\ 2E^3 &= F^3 + \overline{F^3} + \overline{F^2} F^2 \quad \text{and} \quad \Sigma^4 = \mathcal{H}^5 = \beta \mathrm{Tr}(E^4) I_3 + 2E^4 \end{aligned}$$

where $\Theta^0 = -\dfrac{\overline{\partial U^0}}{\partial p_0} N_0$.

5.3.2. Asymptotic expansion of equations. For the low force level considered here, the displacement is of the thickness order and we have the following result:

RESULT 6.

For a shell non-inhibited in the nonlinear and in the linear range, subjected to a low force level $\mathcal{G} = \mathcal{F} = \varepsilon^4$, the leading term U^0 of the new expansion of U depends only on p_0 and satisfies the conditions:

i) U^0 is a linearly inextensional mapping which verifies:

$$2\Delta_t^0 = \frac{\overline{\hat{\partial} U^0}}{\partial_0} + \frac{\hat{\partial} U^0}{\partial_0} = 0 \quad \text{in} \quad \omega_0,$$

ii) U^0 is solution to the problem:

$$\text{div}_{t3} \left(\chi + C_0 m_t^0 + \overline{\text{div}}(m_t^0)\overline{N_0} \right) = -\overline{p} \quad \text{in} \quad \omega_0,$$

$$U^0 = \Theta^0 = 0 \quad \text{on} \quad \gamma_0^1,$$

$$\chi \nu_0 + m_t^0 C_0 \nu_0 = m_t^0 \nu_0 = \text{div}(m_t^0)\nu_0 = 0 \quad \text{on} \quad \gamma_0^2,$$

where χ is a field of symmetrical tensor which depends on U^0, U^1 and U^2, where N_0 denotes the normal to the initial configuration ω_0, and where :

$$m_t^0 = \frac{4\beta}{3(\beta + 2)} \text{Tr}(K_t^0) I_0 + \frac{4}{3} K_t^0,$$

$$2K_t^0 = \frac{\hat{\partial}\Theta^0}{\partial p_0} + \overline{\frac{\hat{\partial}\Theta^0}{\partial p_0}} + \frac{\hat{\partial}U^0}{\partial p_0} C_0 + C_0 \overline{\frac{\hat{\partial}U^0}{\partial_0}},$$

$$\Theta^0 = -\frac{\overline{\partial U^0}}{\partial p_0} N_0, \quad p = \int\limits_{-1}^{1} f \; dz + g^+ + g^-.$$

Before giving the proof of this result, let us notice that the expression of the field of symmetrical tensors χ, which is complex and depends on U^0, U^1 and U^2, is not given explicitly. As in Result 3, it is not necessary because it will vanish in the associated weak formulation (see the next result).

P r o o f. The proof of this result is similar to the previous one. It can also be split into five steps.

i) Computation of $\mathcal{H}^5$

By using (5.37), problem $\mathcal{P}^5$ reduces to :

$$\frac{\partial \overline{N_0}\mathcal{H}^5}{\partial z} = 0 \quad \text{in} \quad \Omega_0 \qquad \text{and} \qquad \overline{\mathcal{H}^{5\pm}} N_0 = \pm g^\pm \quad \text{on} \quad \Gamma_0^\pm$$

which implies that

(5.38) $$\overline{N_0}\mathcal{H}^5 = 0 \quad \text{in} \quad \Omega_0$$

Equivalently, according to (5.37) we get:

(5.39) $$\overline{N_0}\Sigma^4 = \beta \text{Tr}(E^4)\overline{N_0} + 2\overline{N_0}E^4 = 0 \quad \text{in} \quad \Omega_0$$

where E^4 is given by:

(5.40) $$2E^4 = F^3 + \overline{F^3} + \overline{F^2}F^2 = \frac{\partial U^2}{\partial_z} \overline{N_0} + N_0 \overline{\frac{\partial U^2}{\partial z}} + 2(\Delta^1 + zK^0)$$

with

$$2\Delta^1 = \overline{\frac{\partial U^1}{\partial p_0}} + \frac{\partial U^1}{\partial p_0} + \overline{\frac{\partial U^0}{\partial p_0}}\frac{\partial U^0}{\partial p_0} + \overline{\frac{\partial U^0}{\partial p_0}}\Theta^0\overline{N_0}$$

(5.41)

$$+ N_0\overline{\Theta^0}\frac{\partial U^0}{\partial p_0} + ||\Theta^0||^2 N_0\overline{N_0},$$

$$2K^0 = \frac{\partial\Theta^0}{\partial p_0} + \overline{\frac{\partial\Theta^0}{\partial p_0}} + \frac{\partial U^0}{\partial p_0}C_0 + C_0\overline{\frac{\partial U^0}{\partial p_0}}.$$

Thus equation (5.39) enables us to calculate $\partial U^2/\partial z$. Indeed we get:

(5.42) $$\frac{\partial U^2}{\partial z} = -\frac{1}{\beta+2}\Big[\beta\mathrm{Tr}(\Delta^1 + zK^0)$$

$$+ 2\overline{N_0}(\Delta^1 + zK^0)N_0\Big]N_0 - 2\Pi_0(\Delta^1 + zK^0)N_0.$$

Replacing the expression (5.42) of $\partial U^2/\partial z$ in (5.40), and decomposing Δ^1 and K^0 into $T\omega_0 \oplus \mathbb{R}N_0$, we obtain :

(5.43) $$E^4 = -\frac{\beta}{\beta+2}\mathrm{Tr}\,(\Delta_t^1 + zK_t^0)N_0\overline{N_0} + (\Delta_t^1 + zK_t^0)$$

with :

$$2\Delta_t^1 = 2\Pi_0\Delta^1\Pi_0 = \overline{\frac{\hat\partial U^1}{\partial p_0}} + \frac{\hat\partial U^1}{\partial p_0} + \overline{\frac{\partial U^0}{\partial p_0}}\frac{\partial U^0}{\partial p_0},$$

(5.44)

$$2K_t^0 = 2\Pi_0 K^0\Pi_0 = \frac{\hat\partial\Theta^0}{\partial p_0} + \overline{\frac{\hat\partial\Theta^0}{\partial p_0}} + \frac{\hat\partial U^0}{\partial p_0}C_0 + C_0\overline{\frac{\hat\partial U^0}{\partial p_0}}.$$

Hence the expression of $\mathcal{H}^5$ becomes:

(5.45) $$\mathcal{H}^5 = \Sigma^4 = \frac{1}{2}(n_t^1 + 3zm_t^0)$$

where:

$$n_t^1 = \frac{4\beta}{\beta+2}\mathrm{Tr}(\Delta_t^1)I_0 + 4(\Delta_t^1),$$

(5.46)

$$m_t^0 = \frac{4\beta}{3(\beta+2)}\mathrm{Tr}(K_t^0)I_0 + \frac{4}{3}(K_t^0).$$

ii) Characterization of $\underline{U}^1$

Using (5.37), problem $\mathcal{P}^6$ can be reduced to :

(5.47)
$$\text{div}_{t3}(\Pi_0 \mathcal{H}^5) + \frac{\partial \overline{N_0} \mathcal{H}^6}{\partial z} = 0 \quad \text{in} \quad \Omega_0,$$

$$\overline{\mathcal{H}^6}^{\pm} N_0 = 0 \quad \text{on} \quad \Gamma_0^{\pm}.$$

Then, according to (5.45), an integration upon the thickness leads to

(5.48)
$$\text{div}_{t3}(n_t^1) = 0 \quad \text{in} \quad \omega_0$$

whose solutions verify

(5.49)
$$2\Delta_t^1 = \frac{\partial \underline{U}^1}{\partial p_0} + \frac{\overline{\partial \underline{U}^1}}{\partial p_0} + \frac{\overline{\partial U^0}}{\partial p_0} \frac{\partial U^0}{\partial p_0} = 0.$$

Finally, taking into account (5.49), expression (5.45) reduces to:

(5.50)
$$\mathcal{H}^5 = \Sigma^4 = \frac{3}{2} z \, m_t^0.$$

iii) Expression of $\mathcal{H}^6$

Now let us integrate the Eq. (5.47) of problem $\mathcal{P}^6$ with respect to z. We get:

(5.51)
$$\overline{N_0} \mathcal{H}^6 = \frac{3(1 - z^2)}{4} \, \text{div}_{t3}(m_t^0).$$

Thus $\mathcal{H}^6$ can be written as :

(5.52)
$$\mathcal{H}^6 = \Pi_0 \mathcal{H}^6 + N_0 \overline{N_0} \mathcal{H}^6 = \Pi_0 \mathcal{H}^6 + \frac{3(1 - z^2)}{4} N_0 \, \text{div}_{t3}(m_t^0).$$

On the other hand, according to (5.37), we have:

(5.53)
$$\mathcal{H}^6 = \Sigma^5 + \Sigma^4 \overline{F^2}.$$

Hence (5.52) can be written as :

(5.54) $\mathcal{H}^6 = \Pi_0 \Sigma^5 + \Pi_0 \Sigma^4 \overline{F^2} + \dfrac{3(1 - z^2)}{4} N_0 \, \text{div}_{t3}(m_t^0)$

$$= \Pi_0 \Sigma^5 \Pi_0 + \Pi_0 \Sigma^5 N_0 \overline{N_0} + \frac{3}{2} z m_t^0 \frac{\overline{\partial U^0}}{\partial p_0} + \frac{3(1 - z^2)}{4} N_0 \, \text{div}_{t3}(m_t^0),$$

where F^2 and Σ^4 have been replaced by their expressions (5.37) and (5.50).

On the other hand, multiplying (5.53) by $\overline{N_0}$ and using (5.39), we get:

$$\Sigma^5 N_0 = \frac{3(1-z^2)}{4}\, \overline{\mathrm{div}_{t3}}(m_t^0)$$

because Σ^5 is symmetrical.

Eventually, the expression (5.54) of $\mathcal{H}^6$ becomes:

$$(5.55) \quad \mathcal{H}^6 = \Pi_0 \Sigma^5 \Pi_0 + \frac{3}{2} z\, m_t^0 \frac{\overline{\partial U^0}}{\partial p_0}$$

$$+ \frac{3(1-z^2)}{4}\left(\Pi_0 \overline{\mathrm{div}_{t3}}(m_t^0)\overline{N_0} + N_0\, \mathrm{div}_{t3}(m_t^0)\right).$$

iv) Equilibrium equations

The cancellation of the factor of ε^7 in the expansion of Eq. (3.5) leads to problem $\mathcal{P}^7$:

$$\mathrm{div}_{t3}(\Pi_0 \mathcal{H}^6 + z\, C_0 \mathcal{H}^5) - \mathrm{Tr}(C_0)\overline{N_0}\mathcal{H}^6 - z\,\mathrm{div}\,(C_0)\Pi_0\mathcal{H}^5 + \frac{\partial \overline{N_0}\mathcal{H}^7}{\partial z}$$

$$(5.56) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\overline{f} \quad \text{in} \quad \Omega_0,$$

$$\overline{\mathcal{H}^7}^{\pm} N_0 = \pm g^{\pm} \quad \text{in} \quad \Gamma_0^{\pm}.$$

By using (5.50) and (5.55), an integration upon the thickness leads to:

$$(5.57) \quad \mathrm{div}_{t3} \left(\int\limits_{-1}^{1} \Pi_0 \Sigma^5 \Pi_0 \; dz \; + C_0 m_t^0 + \Pi_0 \overline{\mathrm{div}_{t3}}(m_t^0)\overline{N_0} \right)$$

$$- \mathrm{Tr}(C_0)\mathrm{div}_{t3}\,(m_t^0) - \mathrm{div}(C_0)m_t^0 = -\overline{p} \quad \text{in} \quad \omega_0$$

where $p = \int_{-1}^{+1} f\, dz + g^+ + g^-$.

Finally, using the following properties of the divergence div_{t3}:

$$\mathrm{div}_{t3}(m_t^0)\Pi_0 = \mathrm{div}(m_t^0)$$

and

$$\mathrm{Tr}(C_0)\mathrm{div}_{t3}\,(m_t^0) + \mathrm{div}(C_0)m_t^0 = \mathrm{div}_{t3}\,(\mathrm{Tr}(C_0)m_t^0)$$

we transform the last equation into:

$$\text{(5.58)} \qquad \operatorname{div}_{t3}\left(\chi + C_0 m_t^0 + \overline{\operatorname{div}}(m_t^0)\overline{N_0}\right) = -\overline{p} \quad \text{in} \quad \omega_0$$

which constitutes the equilibrium equation of Result 6 with:

$$\text{(5.59)} \qquad \chi = \int_{-1}^{1} \Pi_0 \Sigma^5 \Pi_0 \, dz \; - \operatorname{Tr}(C_0)m_t^0.$$

We recall that χ is a field of symmetrical tensors.

   v) Boundary conditions

To conclude the proof, let us examine the boundary conditions. The clamped condition $U = 0$ on Γ_0 easily leads to :

$$\text{(5.60)} \qquad U^0 = \Theta^0 = 0 \quad \text{on} \quad \gamma_0^1.$$

The boundary conditions on γ_0^2 can be obtained from the expansion of the condition $\overline{\mathcal{H}}\kappa_0^{-1}\nu_0 = 0$. Taking into account expressions (5.50) and (5.55) of $\mathcal{H}^5$ and $\mathcal{H}^6$, we get:

$$z\, m_t^0 \nu_0 = 0 \quad \text{on} \quad \Gamma_0^2,$$

$$\text{(5.61)} \quad \Pi_0 \Sigma^5 \Pi_0 \nu_0 + \frac{3(1-z^2)}{4} N_0 \operatorname{div}_{t3}(m_t^0)\nu_0$$

$$+ \frac{3z}{2}\frac{\partial U^0}{\partial p_0}m_t^0 \nu_0 + \frac{3z^2}{2}m_t^0 C_0 \nu_0 = 0 \quad \text{on} \quad \Gamma_0^2.$$

The first equation of (5.61) directly leads to

$$\text{(5.62)} \qquad m_t^0 \nu_0 = 0 \quad \text{on} \quad \gamma_0^2.$$

Let us project the second equation onto $T\omega_0$ and the normal N_0, and integrate the two equations obtained upon the thickness. Taking into account (5.62), we obtain

$$\text{(5.63)} \quad \int_{-1}^{1} \Pi_0 \Sigma^5 \Pi_0 \, dz \, \nu_0 + m_t^0 C_0 \nu_0 = 0 \quad \text{and} \quad \operatorname{div}(m_t^0)\nu_0 = 0 \qquad \text{on} \quad \gamma_0^2,$$

where we have used the property $\operatorname{div}_{t3}(m_t^0)\nu_0 = \operatorname{div}_{t3}(m_t^0)\Pi_0\nu_0 = \operatorname{div}(m_t^0)\nu_0.$

Finally, taking into account (5.62), the first equation of (5.63) is equivalent to:

$$\chi \nu_0 + m_t^0 C_0 \nu_0 = 0 \quad \text{on} \quad \gamma_0^2$$

where χ is given by (5.59). This concludes the proof of Result 6.

$\square$

REMARK 1.

Let us notice that if we decompose U^0 on $T\omega_0 \oplus \mathbb{R}N_0$ as follows:

$$U^0 = V^0 + u^0 N_0,$$

then Δ_t^0 and K_t^0 can be written as:

$$\Delta_t^0 = \frac{1}{2}\left(\overline{\frac{\hat{\partial} V^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0}\right) - u^0 C_0$$

and

$$K_t^0 = \frac{1}{2}\left(\frac{\hat{\partial}\Theta^0}{\partial p_0} + \overline{\frac{\hat{\partial}\Theta^0}{\partial p_0}} + \frac{\hat{\partial} V^0}{\partial p_0}C_0 + C_0\overline{\frac{\hat{\partial} V^0}{\partial p_0}} - u^0 C_0^2\right)$$

with $\Theta^0 = -\overline{\dfrac{\partial u^0}{\partial p_0}} - C_0 V^0$. We then recognize the classical expressions of the linear membrane strain Δ_t^0 and of the linear curvature change K_t^0.

5.3.3. The linear pure bending model. Let us define the space of linear inextensional displacements :

$$(5.64) \quad V_{\text{inex}}(\omega_0) = \left\{ U : \omega_0 \to \mathbb{R}^3 \text{ "smooth"}, \quad \frac{\hat{\partial} U}{\partial p_0} + \overline{\frac{\hat{\partial} U}{\partial p_0}} = 0 \quad \text{in} \quad \omega_0 \right.$$

$$\left. \text{and} \quad U = \overline{\frac{\partial U}{\partial p_0}}N_0 = 0 \quad \text{on} \quad \gamma_0^1 \right\}.$$

Then equations of Result 6 can be written in the following weak formulation:

RESULT 7.

For a shell non-inhibited in the nonlinear and in the linear range, subjected to a low force level $\mathcal{F} = \mathcal{G} = \varepsilon^4$, the leading term U^0 of the expansion of the displacement is a solution of the linear pure bending model:

$$U^0 \in V_{\text{inex}}(\omega_0),$$

$$(5.65) \quad \int_{\omega_0} \text{Tr}(m_t^0 \delta K_t^0)d\omega_0 = \int_{\omega_0} \overline{p}\delta U^0 d\omega_0 \quad \forall\, \delta U^0 \in V_{\text{inex}}(\omega_0),$$

where

$$2K_t^0 = \frac{\hat{\partial}\Theta^0}{\partial p_0} + \frac{\overline{\hat{\partial}\Theta^0}}{\partial p_0} + \frac{\hat{\partial}U^0}{\partial p_0}C_0 + C_0\frac{\overline{\hat{\partial}U^0}}{\partial p_0}, \quad \Theta^0 = -\frac{\overline{\partial U^0}}{\partial p_0}N_0,$$

$$m_t^0 = \frac{4\beta}{3(\beta+2)}\mathrm{Tr}(K_t^0)I_0 + \frac{4}{3}K_t^0 \quad \text{and} \quad p = \int\limits_{-1}^{1} f\, dz + g^+ + g^-.$$

The proof of this result is similar to the one of Result 4. It is based on the successive use of the Stokes formula. We just need the restriction $\delta U^0 \in V_{\mathrm{inex}}(\omega_0)$ to eliminate χ in the weak formulation of the Result 6.

□

Thus we have justified the linear pure bending model for a non-inhibited shell in the nonlinear and in the linear range, subjected to a low force level of ε^4 order. For this force level, the displacements are of the thickness order ($U_r = h_0$). This linear pure bending model has been also justified by asymptotic expansion of the three-dimensional equations of linear elasticity in [14][15][21]. But contrary to these works, the linear pure bending model is deduced here from the nonlinear three-dimensional elasticity.

5.4. Domain of validity of the linear pure bending model

It is possible to prove that for a non-inhibited shell in the nonlinear and in the linear range, the linear pure bending model is valid for force levels lower than ε^4. Indeed, for a force level $\mathcal{F} = \mathcal{G} = \varepsilon^5$, we would obtain the weak formulation (5.65) without a right side whose solutions satisfy $K_t^0 = 0$. As U^0 is an inextensional displacement in the linear range, the linear version of the rigid motion lemma implies that $U^0 = 0$. Following the same reasoning as in the previous sections, we find out that the reference scale of the displacement is not properly chosen. We have to consider $U_r = \varepsilon h_0$. Then, a new dimensional analysis and a new asymptotic expansion of equations lead again to the linear pure bending model. For the low force level considered here, the problem becomes linear with respect to the displacement. In fact, with a recurrence on n, we can prove the following result:

RESULT 8.

For a non-inhibited shell, in the linear and the nonlinear range, subjected to low force levels of $\varepsilon^{n\geq 4}$ order, the order of magnitude of the displacement is $U_r = \varepsilon^{n-4}h_0$. Moreover, the leading term U^0 of the expansion of the displacement satisfies equations of the pure bending model of Result 7.

5.5. The linear membrane model for linearly inhibited shells

We see that for a non-inhibited shell in the nonlinear range subjected to low force levels of ε^4 order and lower, we have to distinguish the linearly non-inhibited from the linearly inhibited shells. In Subsecs. 5.3 and 5.4, we proved that for a linearly non-inhibited shell, we obtain the linear pure bending model. For a linearly inhibited shell, the following result is obtained:

RESULT 9.

For a non-inhibited shell in the nonlinear range but inhibited in the linear range, and subjected to low force levels $\mathcal{F} = \mathcal{G} = \varepsilon^{n \geq 4}$, the magnitude of the displacement is $U_r = \varepsilon^{n-2} h_0$. Moreover, the leading term U^0 of the expansion of the displacement is a solution of the following linear membrane model:

$$\operatorname{div}(n_t^0) = -\overline{p_t} \quad \text{in} \quad \omega_0, \qquad \operatorname{Tr}(n_t^0 C_0) = -p_n \quad \text{in} \quad \omega_0,$$

$$U^0 = 0 \quad \text{on} \quad \gamma_0^1, \qquad n_t^0 \nu_0 = 0 \quad \text{on} \quad \gamma_0^2,$$

where

$$n_t^0 = \frac{4\beta}{2+\beta} \operatorname{Tr}(\Delta_t^0) \, I_0 + 4\Delta_t^0, \quad 2\Delta_t^0 = \frac{\overline{\hat{\partial} V^0}}{\partial p_0} + \frac{\hat{\partial} V^0}{\partial p_0} - 2u^0 C_0,$$

$$p_t = g_t^+ + g_t^- + \int_{-1}^{1} f_t dz \quad \text{and} \quad p_n = g_n^+ + g_n^- + \int_{-1}^{1} f_n dz.$$

For the proof of this result, we refer the reader to the next section where the study is similar.

6. Inhibited shells in the nonlinear range

It must be reminded that for a shell subjected to a severe force level of ε order, the asymptotic expansion of equations leads to the nonlinear membrane model whatever the nonlinear rigidity of the middle surface is (see Sec. 4). For a high force level of ε^2 order we had to distinguish the nonlinear inhibited from the nonlinear non-inhibited shells. In the last section we have completed the classification for non-inhibited shells, in the nonlinear range.

In this section, we will study the other branch of the classification which corresponds to inhibited shells in the nonlinear range. In order to do this, we resume the calculations after the nonlinear membrane model obtained at the Result 1.

6.1. The linear membrane model for a high force level

We consider a inhibited shell in the nonlinear range subjected to a high force level $\mathcal{G} = \mathcal{F} = \varepsilon^2$. We first prove that for this force level, the order of magnitude of the displacement is $U_r = h_0$ and not L_0. Then, a new dimensional analysis will lead to the linear membrane model.

6.1.1. New reference scale of the displacement. For a force level of ε^2 order, we obtain the weak formulation (4.3) of the Result 2 without a right-hand side, whose solutions are the inextensional mappings ϕ^0 which satisfy

$$(6.1) \qquad \frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial \phi^0}{\partial p_0} = I_0 \quad \text{in} \quad \omega_0.$$

As the shell is assumed to be inhibited in the nonlinear range, the space of inextensional mappings reduces to identity. Hence we have $\phi^0 = i_{\omega_0}$ or equivalently $U^0 = 0$. The expression of N introduced in (4.1) becomes $N = N_0$ and we still have:

$$(6.2) \qquad U^1 = U^1(p_0) \quad \text{in} \quad \omega_0.$$

Therefore, for this force level, we have to consider $U_r = h_0$ so as U^0 to be different from zero. So we make a new dimensional analysis of Eq. (2.4) with $U_r = h_0$ as the new reference scale, and we still denote $U = U^*/h_0$ the new dimensionless displacement. As in Sec. 5.3, this new dimensional analysis does not modify the dimensionless Equation (3.5) but only the components of F, E, Σ and H, where U must be changed into εU. The expression of the tangent mapping F that we now have to consider is given by (5.35):

$$(6.3) \qquad F = \varepsilon I_3 + \varepsilon^2 \frac{\partial U}{\partial p_0} \kappa^{-1} + \varepsilon \frac{\partial U}{\partial z} \overline{N_0}.$$

A new expansion of the displacement is then equivalent to change U^i into U^{i-1} for $i \geq 1$ in the previous results. In particular (6.2) gives us

$$(6.4) \qquad U^0 = U^0(p_0).$$

On the other hand, with this new reference scale for the displacement, we must calculate again the first non-zero terms of the expansions of F, E, Σ and $\mathcal{H}$. According to (3.3), (3.4) and (6.3), we have $F^1 = I_3$ and:

$$(6.5) \qquad F^2 = \frac{\partial U^1}{\partial_z} \overline{N_0} + \frac{\partial U^0}{\partial p_0}, \qquad 2E = \overline{F^2} + F^2,$$

$$\Sigma^3 = \beta \text{Tr} \, (E^3) I_3 + 2E^3, \qquad \mathcal{H}^4 = \Sigma^3.$$

6.1.2. Asymptotic expansion. The asymptotic expansion of the new dimensionless equations leads to the following result:

RESULT 10.

For a shell inhibited in the non-linear range and subjected to a high force level $\mathcal{G} = \mathcal{F} = \varepsilon^2$, the leading term $U^0 = (V^0, u^0)$ of the expansion of $U = (V, u)$ only depends on p_0 and satisfies the linear membrane model :

$$\text{div}(n_t^0) = -\overline{p_t} \quad \text{in} \quad \omega_0, \qquad\qquad \text{Tr}(n_t^0 C_0) = -p_n \quad \text{in} \quad \omega_0,$$

$$U^0 = 0 \quad \text{on} \quad \gamma_0^1, \qquad\qquad n_t^0 \nu_0 = 0 \quad \text{on} \quad \gamma_0^2.$$

where the expressions of n_t^0, Δ_t^0, p_t and p_n are those of Result 9.

P r o o f.   The proof is split into two steps.
i)   *Determination of U^1*

The cancellation of the factor of ε^4 in the new expansion of the dimensionless equilibrium Eq. (3.5) leads tothe new problem $\mathcal{P}^4$:

$$\frac{\partial \overline{\mathcal{H}}^4 N_0}{\partial z} = 0 \quad \text{in} \quad \omega_0,$$

$$\left[\overline{\mathcal{H}^4 N_0}\right]^{\pm} = 0 \quad \text{on} \quad \Gamma_0^{\pm},$$

which implies that $\overline{\mathcal{H}^4} N_0 = 0$ or equivalently that:

$$(6.6) \qquad\qquad \beta \text{Tr}(E^3) N_0 + 2E^3 N_0 = 0 \quad \text{in} \quad \omega_0$$

in view of (6.5). On the other hand, we have:

$$2E^3 = F^2 + \overline{F^2} = \frac{\partial U^1}{\partial z} \overline{N_0} + N_0 \frac{\overline{\partial U^1}}{\partial z} + \frac{\partial U^0}{\partial p_0} + \frac{\overline{\partial U^0}}{\partial p_0}.$$

Now if we decompose $\dfrac{\partial U^0}{\partial p_0}$ as follows:

$$\frac{\partial U^0}{\partial p_0} = \Pi \frac{\partial U^0}{\partial p_0} + N_0 \overline{N_0} \frac{\partial U^0}{\partial p_0}$$

we get:

$$\text{Tr}(E^3) = \overline{N_0} \frac{\partial U^1}{\partial z} + \text{Tr}(\Delta_t^0) \quad \text{and} \quad 2E^3 N_0 = \frac{\partial U^1}{\partial z} + \left(\overline{N_0} \frac{\partial U^1}{\partial z}\right) N_0 + \frac{\overline{\partial U^0}}{\partial p_0} U^0 N_0,$$

with $2\Delta_t^0 = \Pi_0 \overline{\dfrac{\partial U^0}{\partial p_0}} + \Pi_0 \dfrac{\partial U^0}{\partial p_0}$. According to the Remark 1, Δ_t^0 corresponds to the classical linear membrane strain.

Thus (6.6) can be written as:

$$\beta \left(\overline{N_0 \dfrac{\partial U^1}{\partial z}} + \mathrm{Tr}(\Delta_t^0) \right) N_0 + \dfrac{\partial U^1}{\partial z} + \left(\overline{N_0 \dfrac{\partial U^1}{\partial z}} \right) N_0 + \overline{\dfrac{\partial U^0}{\partial p_0}} N_0 = 0$$

and the projection of this equation onto $T\omega_0$ and N_0 gives us:

$$\Pi_0 \dfrac{\partial U^1}{\partial z} = -\overline{\dfrac{\partial U^0}{\partial p_0}} N_0 \quad \text{and} \quad \overline{N_0 \dfrac{\partial U^1}{\partial z}} = -\dfrac{\beta}{\beta+2} \mathrm{Tr}(\Delta_t^0)$$

which leads to:

$$\dfrac{\partial U^1}{\partial z} = -\overline{\dfrac{\partial U^0}{\partial p_0}} N_0 - \dfrac{\beta}{\beta+2} \mathrm{Tr}(\Delta_t^0) N_0.$$

As U^0 depends only on p_0 according to (6.4), we get finally:

$$(6.7) \qquad U^1 = \underline{U}^1 - z \left(\overline{\dfrac{\partial U^0}{\partial p_0}} N_0 + \dfrac{\beta}{\beta+2} \mathrm{Tr}(\Delta_t^0) N_0 \right)$$

where $\underline{U}^1$ only depends on p_0.

The expressions of E^3, Σ^3 and H^4 can also be calculated from (6.5). We get:

$$(6.8) \qquad E^3 = \Delta_t^0 - \dfrac{\beta}{\beta+2} \mathrm{Tr}(\Delta_t^0)\, N_0 \overline{N_0} \quad \text{and} \quad \Sigma^3 = H^4 = \dfrac{1}{2} n_t^0$$

where $n_t^0 = \dfrac{4\beta}{\beta+2} \mathrm{Tr}(\Delta_t^0) I_0 + 4\Delta_t^0$.

   ii)   *Linear membrane equations*

Problem $\mathcal{P}^5$ then reduces to:

$$\mathrm{div}_{t3}(\Pi_0 \mathcal{H}^4) + \dfrac{\overline{\partial \mathcal{H}^5 N_0}}{\partial z} = -\overline{f} \quad \text{in} \quad \Omega_0,$$

$$\left[\overline{\mathcal{H}^5 N_0} \right]^{\pm} = \pm g^{\pm} \quad \text{on} \quad \Gamma_0^{\pm}.$$

Using (6.8), an integration upon the thickness of the above equations leads to:

$$(6.9) \qquad \mathrm{div}_{t3}(n_t^0) = -\overline{p} \quad \text{in} \quad \omega_0$$

where $p = g^+ + g^- + \int_{-1}^{+1} f\,dz$. As n_t^0 is a field of endomorphims on $T\omega_0$, we can decompose easily $\mathrm{div}_{t3}(n_t^0)$ into $T\omega_0 \oplus \mathbb{R}N_0$. The last equation then becomes:

$$\mathrm{div}(n_t^0) + \mathrm{Tr}(n_t^0 C_0)\overline{N_0} = -p \quad \text{on} \quad \omega_0$$

which leads to the two classical equations of the membrane model of Result 10:

$$\mathrm{div}(n_t^0) = -\overline{p_t} \quad \text{and} \quad \mathrm{Tr}(n_t^0 C_0) = -p_n \quad \text{in} \quad \omega_0.$$

Finally, the boundary conditions on γ_0^1 and γ_0^2 can be obtained easily from the expansion of the three-dimensional boundary conditions on Γ_0^1 and Γ_0^2. This concludes the proof of Result 10.

$\square$

6.1.3. Weak formulation. Let us define the following space of admissible displacements :

$$V(\omega_0) = \left\{U \ : \ \omega_0 \to \mathbb{R}^3, \ \text{"smooth"}, \ U = 0 \ \text{on} \ \gamma_0^1\right\}$$

Then the linear membrane equations can be written in the following weak formulation :

RESULT 11.

The displacement $U^0 \in V(\omega_0)$ satisfies:

(6.10) $$\int_{\omega_0} \mathrm{Tr}(n_t^0 \delta\Delta_t^0)\,d\omega_0 = \int_{\omega_0} \overline{p}\delta U^0, \qquad \forall\, \delta U^0 \in V(\omega_0)$$

where $p = p_t + p_n N_0$.

This weak formulation is identical to the one obtained by asymptotic expansion from the linear three-dimensional elasticity, with an intrinsic approach [4][5][7] or with a description of the shell in local coordinates [14]. But contrary to these other justifications, the linear membrane model is deduced here from the nonlinear equilibrium three-dimensional equations without any assumption on the scalings concerning the displacements.

6.2. The linear membrane model still valid for linearly inhibited shells

For moderate and lower force levels, we have now to distinguish the linearly inhibited from the linearly non-inhibited shells. For linearly inhibited shells we have the following result:

RESULT 12.

For a shell inhibited in the linear and the nonlinear range, the linear membrane model is still valid for force levels of $\varepsilon^{n\geq3}$ order. For these force levels, the order of magnitude of the displacement is $U_r = h_0\varepsilon^{n-2}$.

The proof can be obtained with a recurrence on n. The main step is to solve the weak formulation (6.10) without a right side. Considering the associated minimization problem, we obtain that U^0 is an inextensional displacement in the linear range. As the shell is linearly inhibited, we have $U^0 = 0$. Following the proof of Result 10, a new dimensional analysis with $U_r = \varepsilon h_0$ and a new asymptotic expansion of equations lead again to the linear membrane model, with or without a right side, according to the considered force level. This operation can be repeated until we find $U^0 \neq 0$. Finally, using a recurrence on n, we find that for force levels of $\varepsilon^{n\geq3}$ order, the order of magnitude of the displacement is $U_r = h_0\varepsilon^{n-2}$, and the asymptotic model obtained is the linear membrane one.

<div align="right">□</div>

6.3. Domain of validity of the linear membrane model

We proved in Result 10 that the linear membrane model is valid for an inhibited shell in the non-linear range, subjected to a high force level of ε^2 order. For moderate and lower force levels of $\varepsilon^{n\geq3}$, this model is still valid if the shell is inhibited in the linear range as well.

We recall that in the Subsec. 5.5, we have proved that this linear membrane model is also obtained for a non-inhibited shell in the nonlinear range but linearly inhibited, subjected to low force levels of $\varepsilon^{n\geq4}$ order. Thus, the linear membrane model is valid for a linearly inhibited shell subjected to low force levels of $\varepsilon^{n\geq4}$ order, whatever the nonlinear geometric rigidity is.

6.4. Two other models for linearly non-inhibited shells

We study now the last case: a shell subjected to moderate and low force levels, linearly non-inhibited, but always inhibited in the nonlinear range. The asymptotic expansion of the three-dimensional equilibrium Equation (3.5) leads to calculations similar to the ones of the previous sections. Thus we only give here the asymptotic models which are obtained.

6.4.1. Another coupling model for a moderate force level.
For a moderate force level $\mathcal{F} = \mathcal{G} = \varepsilon^3$, the order of magnitude of the displacement is $U_r = h_0$ and the two first terms U^0 and U^1 of the expansion of the displacement are solution of a variational problem which couples membrane and bending effects.

Let us recall the definition of the following admissible spaces of displacements:

$$V(\omega_0) = \left\{ U : \omega_0 \to \mathbb{R}^3, \text{ "smooth"}, \ U = 0 \ \text{ on } \ \gamma_0^1 \right\},$$

$$V_{\text{inex}}(\omega_0) = \left\{ U \in V(\omega_0), \ \frac{\hat{\partial} U}{\partial p_0} + \frac{\overline{\hat{\partial} U}}{\partial p_0} = 0 \ \text{ in } \ \omega_0 \ \text{ and } \ \frac{\overline{\partial U}}{\partial p_0} N_0 = 0 \ \text{ on } \ \gamma_0^1 \right\}.$$

Then we have the following result :

RESULT 13.

For a shell inhibited in the nonlinear range but linearly non-inhibited, sub-jected to a moderate force level $\mathcal{F} = \mathcal{G} = \varepsilon^3$, $(U^0, U^1) \in V_{\text{inex}}(\omega_0) \times V(\omega_0)$ and satisfies the following weak problem:

$$\int_{\omega_0} \text{Tr} \left(n_t^1 \delta \Delta_t^1 + m_t^0 \delta K_t^0 \right) d\omega_0 = \int_{\omega_0} (\overline{p} \delta U^1 - \text{Tr}(C_0) \overline{M} \delta U^0 + \overline{M} \delta \Theta^0) d\omega_0$$

$$\forall \, (\delta U^0, \delta U^1) \in V_{\text{inex}}(\omega_0) \times V(\omega_0)$$

with:

$$n_t^1 = \frac{4\beta}{2 + \beta} \text{Tr}(\Delta_t^1) I_0 + 4 \Delta_t^1, \qquad 2 \Delta_t^1 = \frac{\hat{\partial} U^1}{\partial p_0} + \frac{\overline{\hat{\partial} U^1}}{\partial_0} + \frac{\overline{\partial U^0}}{\partial p_0} \frac{\partial U^0}{\partial p_0},$$

$$m_t^0 = \frac{4\beta}{3(\beta + 2)} \text{Tr}(K_t^0) I_0 + \frac{4}{3} K_t^0, \qquad K_t^0 = \frac{\hat{\partial} \Theta^0}{\partial p_0} + \frac{\overline{\hat{\partial} \Theta^0}}{\partial p_0} + \frac{\hat{\partial} U^0}{\partial p_0} C_0 + C_0 \frac{\overline{\hat{\partial} U^0}}{\partial p_0},$$

$$p = g^+ + g^- + \int_{-1}^{1} f \, dz, \qquad M = g^+ - g^- + \int_{-1}^{1} zf \, dz.$$

For the proof of this result, which is similar to the one of Result 4, we refer the reader to Sec. 5.1.

□

This coupling model is similar to the one obtained in Result 4, with different expressions of the strain mesures K_t^0 and Δ_t^1. Here K_t^0 is the linear classical variation of curvature. The coupling between U^0 and U^1 is contained in the non-classical membrane strain Δ_t^1, which is linear with respect to U^1 but nonlinear with respect to U^0.

The physical interpretation of this model is also similar to the one of Result 4. The solution of this model is the displacement $U^0 + \varepsilon U^1$, where U^0 is a linear inextensional displacement which generates the curvature variation K_t^0, and εU^1 a small displacement which generates with U^0 the nonlinear membrane strain Δ_t^1.

6.4.2. The linear pure bending model for low force levels. Let us consider to finish a shell subjected to low force levels $\mathcal{F} = \mathcal{G}_g = \varepsilon^{n \geq 4}$. Then we have the following result:

RESULT 14.

For a shell inhibited in the nonlinear range but linearly non-inhibited, and subjected to low force levels $\mathcal{F} = \mathcal{G} = \varepsilon^{n \geq 4}$, the order of magnitude of the displacement is $U_r = h_0 \varepsilon^{n-4}$. Moreover the leading term U^0 of the expansion of the displacement is a solution of the linear pure bending model of Result 7.

For the proof of this result, we refer the reader to Sec. 5.3 where the calculations are similar.

$\square$

Thus, according to Result 8, the linear pure bending model is valid for a linearly non-inhibited shell subjected to low force levels of $\varepsilon^{n \geq 4}$ order, whatever the non-linear geometric rigidity is. We find again the classical results obtained in [14][21][22] from linear elasticity.

7. Conclusion

In the second part of this paper we have established a classification of asymptotic models for strongly curved shells. The results are different from those obtained in the first part for shallow shells [10]. In particular, for the same force level, the obtained behaviour depend on the geometric rigidity of the middle surface of the shell, in the linear and in the nonlinear range.

As in the first part, we have studied only one combination of $(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n)$ for each value of $\tau = \mathrm{Max}(\mathcal{F}_t, \mathcal{F}_n, \mathcal{G}_t, \mathcal{G}_n)$. However, the study of the other combinations is not fundamental; it would lead to the same two-dimensional models with a right side slightly different. The following table resumes the so obtained classification with respect to τ, where the abbreviation $L.I.S.$(respectively $N.L.I.S$) means *linearly inhibited shell* (respectively nonlinearly inhibited shell): with:

$$n_t^0 = \frac{4\beta}{\beta+2}\mathrm{Tr}(\Delta_t^0)I_0 + 4\Delta_t^0, \qquad n_t^1 = \frac{4\beta}{\beta+2}\mathrm{Tr}(\Delta_t^1)I_0 + 4\Delta_t^1,$$

$$m_t^0 = \frac{4\beta}{3(\beta+2)}\mathrm{Tr}(K_t^0)I_0 + \frac{4}{3}K_t^0, \qquad \Theta^0 = -\frac{\overline{\partial U^0}}{\partial p_0}N_0,$$

$$p = \int_{\omega_0} f\,dz + g^+ + g^-, \qquad M = \int_{\omega_0} z\,f\,dz + g^+ - g^-.$$

Table 1. Non-inhibited shells in the nonlinear range.

τ	U_r	Shell model	$\Delta_t^0,\ K_t^0$
ε	L_0	**nonlinear membrane model** $\phi^0 \in Q(\omega_0)$ and $\forall\,\delta\phi^0 \in V(\omega_0)$ $$\int_{\omega_0} \mathrm{Tr}(n_t^0 \delta\Delta_t^0)\,d\omega_0 = \int_{\omega_0} \overline{p}\delta\phi^0\,d\omega_0$$	$$2\Delta_t^0 = \overline{\frac{\partial\phi^0}{\partial p_0}}\frac{\partial\phi^0}{\partial p_0} - I_0$$
ε^2	L_0	**nonlinear coupling model** $(\phi^0, U^1) \in Q_{\text{inex}}(\omega_0) \times V(\omega_0)$ $$\int_{\omega_0} \mathrm{Tr}(n_t^1 \delta\Delta_t^1 + m_t^0 \delta K_t^0)\,d\omega^0 =$$ $$\int_{\omega_0} (\overline{p}\delta U^1 - \mathrm{Tr}(C_0)\overline{M}\delta\phi^0 + \overline{M}\delta N)\,d\omega_0$$ $$\forall\,(\delta\phi^0, \delta U^1) \in V_{\text{inex}}^{\phi^0}(\omega_0) \times V(\omega_0)$$	$$2\Delta_t^1 = \overline{\frac{\partial\phi^0}{\partial p_0}}\frac{\partial U^1}{\partial p_0} + \overline{\frac{\partial U^1}{\partial p_0}}\frac{\partial\phi^0}{\partial p_0}$$ $$K_t^0 = \tilde{C} - C_0$$ ϕ^0 is inextensional
ε^3	L_0	**nonlinear pure bending model** $\phi^0 \in Q_{\text{inex}}(\omega_0)$ and $\forall\,\delta\phi^0 \in V_{\text{inex}}^{\phi^0}(\omega_0)$ $$\int_{\omega_0} \mathrm{Tr}(m_t^0 \delta K_t^0)\,d\omega_0 = \int_{\omega_0} \overline{p}\delta U^0\,d\omega_0$$	$$K_t^0 = \tilde{C} - C_0$$ ϕ^0 is inextensional
$\varepsilon^{n\geq 4}$	$h_0\varepsilon^{n-2}$	**linear membrane model** **if L.I.S.** $U^0 \in V(\omega_0)$ and $\forall\,\delta U^0 \in V(\omega_0)$ $$\int_{\omega_0} \mathrm{Tr}(n_t^0 \delta\Delta_t^0) = \int_{\omega_0} \overline{p}\delta U^0\,d\omega_0$$	$$2\Delta_t^0 = \frac{\hat{\partial}U^0}{\partial p_0} + \overline{\frac{\hat{\partial}U^0}{\partial p_0}}$$
$\varepsilon^{n\geq 4}$	$h_0\varepsilon^{n-4}$	**linear pure bending model** **if N.L.I.S.** $U^0 \in V_{\text{inex}}(\omega_0)$ and $\forall\,\delta U^0 \in V_{\text{inex}}(\omega_0)$ $$\int_{\omega_0} \mathrm{Tr}(m_t^0 \delta K_t^0)\,d\omega_0 = \int_{\omega_0} \overline{p}\delta U^0\,d\omega_0$$	$$2K_t^0 = \frac{\hat{\partial}\Theta^0}{\partial p_0} + \overline{\frac{\hat{\partial}\Theta^0}{\partial p_0}} + \frac{\hat{\partial}U^0}{\partial p_0}C_0$$ $$+C_0\overline{\frac{\hat{\partial}U^0}{\partial p_0}}$$ U^0 is linearly inextensional

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Table 2. Inhibited shells in the nonlinear range.

τ	U_r	Shell model	Δ_t, K_t
ε	L_0	***nonlinear membrane model*** $\phi^0 \in Q(\omega_0)$ and $\forall\, \delta\phi^0 \in V(\omega_0)$ $\int_{\omega_0} \mathrm{Tr}(n_t^0 \delta\Delta_t^0)\, d\omega_0 = \int_{\omega_0} \overline{p}\delta\phi^0\, d\omega_0$	$2\Delta_t^0 = \dfrac{\overline{\partial\phi^0}}{\partial p_0} \dfrac{\partial\phi^0}{\partial p_0} - I_0$
ε^2	h_0	***linear membrane model*** $U^0 \in V(\omega_0)$ and $\forall\, \delta U^0 \in V(\omega_0)$ $\int_{\omega_0} \mathrm{Tr}(n_t^0 \delta\Delta_t^0)\, d\omega_0 = \int_{\omega_0} \overline{p}\delta U^0\, d\omega_0$	$2\Delta_t^0 = \dfrac{\overline{\partial U^0}}{\partial p_0} + \dfrac{\hat{\partial} U^0}{\partial p_0}$
$\varepsilon^{n\geq 3}$	$h_0\varepsilon^{n-2}$	***linear membrane model*** ***if L.I.S.***	...
ε^3	h_0	***second coupling model*** ***if N.L.I.S*** $(U^0, U^1) \in V_{\mathrm{inex}}(\omega_0) \times V(\omega_0)$ $\int_{\omega_0} \mathrm{Tr}(n_t^1 \delta\Delta_t^1 + m_t^0 \delta K_t^0)\, d\omega_0 =$ $\int_{\omega_0} (\overline{p}\delta U^1 - \mathrm{Tr}(C_0)\overline{M}\delta U^0 + \overline{M}\delta\Theta^0)\, d\omega_0$ $\forall\, (\delta U^0, \delta U^1) \in V_{\mathrm{inex}}(\omega_0) \times V(\omega_0)$	$2\Delta_t^1 = \dfrac{\overline{\hat{\partial} U^1}}{\partial_0} + \dfrac{\hat{\partial} U^1}{\partial p_0} + \dfrac{\overline{\partial U^0}}{\partial p_0}\dfrac{\partial U^0}{\partial p_0}$ $K_t^0 = \dfrac{\overline{\hat{\partial}\Theta^0}}{\partial p_0} + \dfrac{\hat{\partial}\Theta^0}{\partial p_0} + \dfrac{\hat{\partial} U^0}{\partial p_0}C_0$ $\qquad + C_0\dfrac{\overline{\hat{\partial} U^0}}{\partial p_0}$ U^0 is linearly inextensional
$\varepsilon^{n\geq 4}$	$h_0\varepsilon^{n-4}$	***linear pure bending model*** ***if N.L.I.S.*** $U^0 \in V_{\mathrm{inex}}(\omega_0)$ and $\forall\, \delta U^0 \in V_{\mathrm{inex}}(\omega_0)$ $\int_{\omega_0} \mathrm{Tr}(m_t^0 \delta K_t^0)\, d\omega_0 = \int_{\omega_0} \overline{p}\delta U^0\, d\omega_0$	$K_t^0 = \dfrac{\overline{\hat{\partial}\Theta^0}}{\partial p_0} + \dfrac{\hat{\partial}\Theta^0}{\partial p_0} + \dfrac{\hat{\partial} U^0}{\partial p_0}C_0$ $\qquad + C_0\dfrac{\overline{\hat{\partial}\hat{U}^0}}{\partial p_0}$ U^0 is linearly inextensional

[216]

We recall the definitions of the admissible spaces of mapping and displacements:

$$V(\omega_0) = \left\{ U : \omega_0 \to \mathbb{R}^3, \text{ "smooth"}, \quad U = 0 \quad \text{on} \quad \gamma_0^1 \right\},$$

$$Q(\omega_0) = \left\{ \phi : \omega_0 \to \mathbb{R}^3, \text{ "smooth"}, \quad \phi = i_{\omega_0} \quad \text{on} \quad \gamma_0^1 \right\},$$

$$V_{\text{inex}}(\omega_0) = \left\{ U \in V(\omega_0), \quad \frac{\hat{\partial} U}{\partial p_0} + \frac{\overline{\hat{\partial} U}}{\partial p_0} = 0 \quad \text{in} \quad \omega_0 \text{ and } \quad \frac{\overline{\partial U}}{\partial p_0} N_0 = 0 \quad \text{on} \quad \gamma_0^1 \right\},$$

$$V_{\text{inex}}^{\phi^0}(\omega_0) = \left\{ U \in V(\omega_0), \quad \frac{\overline{\partial \phi^0}}{\partial p_0} \frac{\partial U}{\partial p_0} + \frac{\overline{\partial U}}{\partial p_0} \frac{\partial \phi^0}{\partial p_0} = 0 \quad \text{in} \quad \omega_0 \right\},$$

$$Q_{\text{inex}}(\omega_0) = \left\{ \phi \in I_{\text{inex}}(\omega_0), \quad \frac{\overline{\partial \phi}}{\partial p_0} N_0 = 0 \quad \text{on} \quad \gamma_0^1 \right\},$$

where the space of inextensional mappings $I_{\text{inex}}(\omega_0)$ is defined as follows :

$$I_{\text{inex}}(\omega_0) = \left\{ \phi : \omega_0 \to \mathbb{R}^3, \text{ "smooth"}, \quad \frac{\overline{\partial \phi}}{\partial p_0} \frac{\partial \phi}{\partial p_0} = I_0 \text{ in } \omega_0, \quad \phi = i_{\omega_0} \text{ on } \gamma_0^1 \right\}.$$

With the approach developed in this paper, the obtained asymptotic shell models, even the linear ones, have been deduced from the nonlinear three-dimensional elasticity. This enables us to specify their domain of validity thanks to the dimensionless numbers naturally introduced. In particular, we proved in this second part that the linear membrane model (respectively the pure bending one) is valid for a linearly inhibited (respectively for a linearly non-inhibited) shell subjected to low force levels of $\varepsilon^{n \geq 4}$ order. We find again the classical results [14][21][22] obtained here from the *nonlinear elasticity*. This proves that for sufficiently low force levels, the membrane strain becomes linear and only the geometric rigidity in the linear range must be taken into account. However, the link between the linear and the nonlinear inextensional displacements is still to study.

On the other hand, in the literature only two nonlinear shell models are obtained by asymptotic expansion of three-dimensional elasticity: the nonlinear membrane model [13] and the pure bending one [11]. Contrary to these works, the systematic study of all the force levels has put here in a prominent position two other nonlinear shell models which couple the membrane and the bending effects. These models are different from the usual models of SANDERS [23], NAGHDI [16], SCHMIDT [1], PIETRASZKIEWICZ [18]. This constitutes the constructive character of the approach presented.

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"Bottom crystal" and possibility of water wave attenuation

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THE INFLUENCE of periodic bottom structure ("bottom crystal") on surface water waves is considered. The problem reduces to a two-dimensional Helmholtz operator with periodic potential. Zero-range potential method based on the theory of self-adjoint extensions of symmetric operators is used. It is shown that there is a gap in the spectrum. An application of this spectral property to the problem of wave attenuation is discussed.

1. Introduction

THE PROBLEM of surface water waves near a coastline, in harbours and channels, is very interesting both from the theoretical point of view and from the point of view of applications in ocean engineering. There is a number of works concerning edge waves and trapped modes (i.e. modes of oscillation which occur at discrete frequences below a certain cut-off frequency and consist of motion which is confined to some localized region of water near an obstacle or variable bottom topography). The oldest example of such a mode was provided by STOKES [1] who constructed an explicit expression for a wave travelling along a beach of constant slope (edge wave). Ursell extended the results of Stokes to show that there is a set of trapped modes for a beach of constant slope with the number of possible modes increasing as the beach angle becomes small. It has been shown that trapped modes can exist due to a submerged obstacle [2-7] or geometric properties of the system-form of the coastline, coupling apertures, crest at the bottom, etc. [8-13].

In the present paper we shall deal with periodic bottom structures (a periodic system of hills or crests at the bottom). It is convenient to use the term "bottom crystal" for such structures, because the corresponding system has properties which are analogous to that of a two-dimensional crystal. From a mathematical point of view the linearized problem of water waves reduces to the investigation of the two-dimensional Helmholtz equation. Cartesian coordinates are chosen with x, y in the undisturbed free surface and z directed vertically upwards. First suppose that the fluid is of uniform depth h and the usual assumption of classical

water-wave theory is made. Thus, we seek a time-harmonic velocity potential $\Phi(x, y, z, t)$ and we write

$$\Phi(x, y, z, t) = \Re\{\phi(x, y) \cosh\left(k(z + h)\right) \exp\left(-i\omega t\right)\}$$

to ensure that the velocity of the fluid normal to the bottom vanishes on $z = -h$. Here, in order to satisfy the convential linearized free-surface condition on $z = 0$, k is a positive root of the equation

$$(1.1) \qquad\qquad\qquad \omega^2 = gk \tanh\left(kh\right)$$

and ω is the radiation frequency, g is the gravitational acceleration. As a result, we get the two-dimensional Helmholtz equation for function ϕ:

$$(1.2) \qquad\qquad\qquad \Delta\phi(x, y) + k^2\phi(x, y) = 0.$$

Now, suppose that the depth is not uniform. For example, suppose there is system of small circles Ω_s on the plane (x, y) with the centres at the nodes of a doubly-periodic lattice on the plane. Suppose the depth to be equal to h for the points (x, y) outside the circles and to $h_1, h_1 < h$, for $(x, y) \in \Omega_s$, for some s (see Fig. 1).
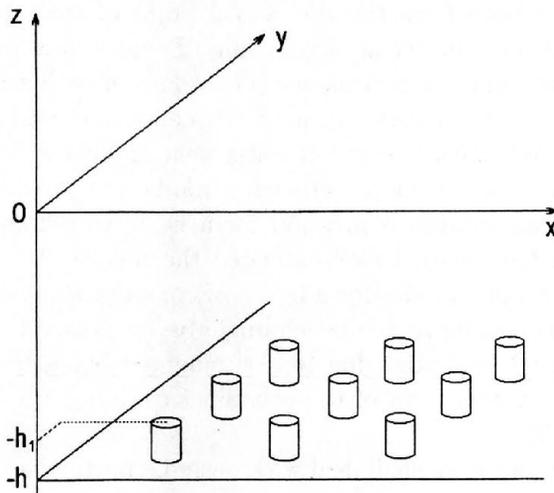

FIG. 1. Periodic bottom structure.

In this situation we have the Helmholtz Equation (1.2) with periodically varying coefficient k (see (1.1)). If the diameters of the circles are small, we can use a model in which the perturbations of k^2 are replaced by point-like ones – the zero-range potential approach. This method is widely used in quantum mechanics

[14, 15], diffraction theory [16], fluid mechanics [17]. The approach is based on the theory of self-adjoint extensions of symmetric operators. We deal with the Laplace operator perturbed by periodic array of zero-range potentials, k^2 is the spectral parameter. We analyse spectral properties of our periodic system in the framework of the method and show that there is a gap in the spectrum for some parameters of the "bottom crystal". It means that some wave frequencies are prohibited. This effect can be used for wave attenuation. Namely, for a concrete harbour some wave frequencies are the most dangerous and powerful. Suppose we make a "bottom crystal" in this harbour with such parameters that these frequencies lie in the gap. Hence, these frequencies are prohibited, and we get essential wave attenuation.

The dispersion equation for a "bottom crystal" is obtained. A "bottom crystal waveguide" (a system in which one or several lines of nodes of the lattice are empty) is considered. It can be used for wave concentration in some regions. The application of the model to the description of a system of thin submerged cylinders is discussed.

2. Spectral properties of a "bottom crystal"

Let Λ be the two-dimensional lattice

$$\Lambda = \{n_1 a_1 + n_2 a_2 \in \mathbf{R}^2; (n_1, n_2) \in \mathbf{Z}^2\},$$

where

$$a_j = (a_j^1, a_j^2), \quad j = 1, 2$$

are two linearly independent vectors in $\mathbf{R}^2$,

$$\Gamma = \{n_1 b_1 + n_2 b_2 \in \mathbf{R}^2; (n_1, n_2) \in \mathbf{Z}^2\}$$

is the dual lattice $(a_i b_{j'} = 2\pi\delta_{jj'}, j, j' = 1, 2)$, $\widehat{\Lambda}$ is the Brillouin zone,

$$\widehat{\Lambda} = \{s_1 b_1 + s_2 b_2 \in \mathbf{R}^2; s_j \in [-1/2, 1/2), j = 1, 2\}.$$

To construct an operator with periodic point-like interactions we start from the closure of the Laplace operator in $L_2(\mathbf{R}^2)$ restricted to the set of smooth functions which vanish at the nodes of the lattice Λ. It is a symmetric non-self-adjoint operator. To switch on the point-like interaction means to construct its self-adjoint extension. Taking into account the periodicity condition, one obtains a model operator $-\Delta_\Lambda$, more precisely, one-parameter (α) family of model operators (self-adjoint extensions). It is known [14] that the spectrum of $-\Delta_\Lambda$ is absolutely continuous and has a band structure. The dispersion equation has the form

$$g_k(0, \theta) = \alpha,$$

where

$$(2.1) \quad g_k(0, \theta) = -\frac{1}{2\pi}(\ln{(-ik/2)} - C_E) + \sum_{\lambda \in \Lambda, \lambda \neq 0} \frac{i}{4} H_0^{(1)}(k \mid \lambda \mid) \exp(-i\theta\lambda),$$

α is a model parameter which is related to "the strength" of the potentials (in our case to $h - h_1$ and diameters of the circles). The sum can be computed using the Poisson summation formula [14]. Note that one can consider the sum using arguments analogous to that for the three-dimensional lattice sum in [18]. As a result, we can describe the spectrum of $-\Delta_\Lambda$. Namely, if the basic cell contains only one centre, it is the following:

$$(2.2) \quad \sigma(-\Delta_\Lambda) = \sigma_{ac}(-\Delta_\Lambda) = [E_0^{\alpha, \Lambda}(0), E_0^{\alpha, \Lambda}(\theta_0)] \cup [E_1^{\alpha, \Lambda}, \infty),$$

where

$$\theta_0 = -\frac{1}{2}(b_1 + b_2),$$

$$E_1^{\alpha, \Lambda} = \min\left\{E_b^{\alpha, \Lambda}(0), \frac{1}{4} \mid b_- \mid^2 \right\}.$$

Here b_- is the member of the pair $\{b_1, b_2\}$ of least magnitude. $E_0^{\alpha, \Lambda}(\theta)$ is the first root $(E_{b_-}^{\alpha, \Lambda}(\theta)$ is the second root) of the equation

$$(2.3) \quad \alpha = (C_E + \ln 2)/(2\pi)$$

$$+ (2\pi)^{-2} \lim_{\omega \to \infty} \left[\sum_{\gamma \in \Gamma, |\gamma + \theta| \leq \omega} \frac{\mid \widehat{\Lambda} \mid}{\mid \gamma + \theta \mid^2 - k^2} - 2\pi \ln \omega \right], \qquad \theta \in \widehat{\Lambda},$$

θ is a quasimomentum, α is a model parameter (related to the "strength" of the potential).

The following inequalities are valid:

$$E_1^{\alpha, \Lambda} > 0, \alpha \in \mathbf{R},$$

$$E_0^{\alpha, \Lambda}(\theta_0) < 0 \iff \alpha \leq g_0(0, \theta_0).$$

Moreover,

$$E_0^{\alpha, \Lambda}(\theta_0) \to \begin{cases} \mid \theta_0 \mid^2, & \alpha \to \infty, \\ -\infty, & \alpha \to -\infty, \end{cases}$$

$$E_0^{\alpha, \Lambda}(0) \to \begin{cases} 0, & \alpha \to \infty, \\ -\infty, & \alpha \to -\infty, \end{cases}$$

$$E_1^{\alpha,\Lambda} \to \begin{cases} |b_-|^2/4, & \alpha \to \infty, \\ 0, & \alpha \to -\infty. \end{cases}$$

It means that in the generic case we have a gap in the spectrum $(E_0^{\alpha,\Lambda}(\theta_0), E_1^{\alpha,\Lambda})$, a part of which is on positive half-axis. But there exists a model parameter $\alpha_{1,\Lambda} \in \mathbf{R}$, such that there is no gap:

$$\sigma(-\Delta_\Lambda) = \sigma_{ac}(-\Delta_\Lambda) = [E_0^{\alpha,\Lambda}(0), \infty), \alpha \geq \alpha_{1,\Lambda}.$$

3. "Bottom crystal waveguide"

Consider a "bottom crystal" with one empty line of nodes. To study the spectral properties of the system it is convenient to investigate, firstly, a periodic chain of zero-range potentials in $\mathbf{R}^2$. Let Λ_1 be

$$\Lambda_1 = \{(0, na) \in \mathbf{R}^2; n \in \mathbf{Z}\},$$

where $a > 0$, $\widehat{\Lambda_1} = [-\pi/a, \pi/a)$, $\Gamma_1 = \{(0, 2\pi n/a) \in \mathbf{R}^2; n \in \mathbf{Z}\}$. Suppose, as earlier, that the basic cell contains only one centre. Using the "restriction- extension" procedure described above one obtains the spectrum of the corresponding operator $-\Delta_{\Lambda_1}$. The dispersion equation has a form analogous to Eq. (2.1):

$$(3.1) \quad \alpha = g_k(0, \theta) = -\frac{1}{2\pi}(\ln(-ik/2) - C_E) + \sum_{\lambda \in \Lambda_1, \lambda \neq 0} \frac{i}{4} H_0^{(1)}(k \mid \lambda \mid) \exp(-i\theta\lambda).$$

Using the Poisson summation formula [14], one can compute the lattice sum. The result is that the spectrum is absolutely continuous and has the following structure:

$$(3.2) \quad \sigma(-\Delta_{\Lambda_1})$$

$$= \sigma_{ac}(-\Delta_{\Lambda_1}) = \begin{cases} [E^{\alpha,\Lambda_1}(0), \infty), & \alpha \geq \alpha_{\Lambda_1}, \\ [E^{\alpha,\Lambda_1}(0), E^{\alpha,\Lambda_1}(-\pi/a)] \cup [0, \infty), & \alpha < \alpha_{\Lambda_1}, \end{cases}$$

where $E^{\alpha,\Lambda_1}(\theta)$, $E^{\alpha,\Lambda_1}(\theta) = k^2$, is the root of the equation

$$(3.3) \quad \alpha = (C_E + \ln 2)/(2\pi)$$

$$+ (2\pi)^{-2} \lim_{\omega \to \infty} \left[\sum_{\gamma \in \Gamma_1, |\gamma + \theta| \leq \omega} \frac{a}{\sqrt{|\gamma + \theta|^2 - k^2}} - 2\pi \ln \omega \right],$$

$\theta \in \widehat{\Lambda_1}, \Im k \geq 0, \Re k \geq 0.$

Moreover, $E^{\alpha,\Lambda_1}(0) < 0, \alpha \in \mathbf{R}, E^{\alpha,\Lambda_1}(0) < E^{\alpha,\Lambda_1}(-\pi/a) < 0$ for $\alpha < \alpha_{\Lambda_1}$, where

$$\alpha_{\Lambda_1} = (C_E + \ln 2)/(2\pi) + (2\pi)^{-2} \lim_{\omega\to\infty} \left[\sum_{\gamma\in\Gamma_1, |\gamma-\pi/a|\leq\omega} \frac{a}{|\gamma-\pi/a|} - 2\pi\ln\omega \right].$$

Hence, for some values of the parameter we have two bands and a gap.

To construct the model of a waveguide in a bottom crystal we deal with a lattice of zero-range potentials with one empty line of nodes $\Lambda \setminus \Lambda_1$. Following the described procedure, one obtains the dispersion equation in the form

$$-\frac{1}{2\pi}(\ln(k/2) + C_E) + i/4 + \sum_{\gamma\in\Lambda, \gamma\neq 0} \frac{i}{4} H_0^{(1)}(ik\,|\gamma\,|) \exp(-i\theta\gamma)$$

$$-\sum_{\gamma\in\Lambda_1, \gamma\neq 0} \frac{i}{4} H_0^{(1)}(ik\,|\gamma\,|) \exp(-i\theta\gamma) = \alpha,$$

where C_E is the Euler constant, $C_E = 0.5772....$ One can see that each term of the right-hand part has been considered earlier (3.1), (2.1), and we obtain the following form of the dispersion equation:

$$\alpha = (C_E + \ln 2)/(2\pi) + (2\pi)^{-2} \lim_{\omega\to\infty} \left[\sum_{\gamma\in\Gamma, |\gamma+\theta|\leq\omega} \frac{|\widehat{\Lambda}|}{|\gamma+\theta|^2 - k^2} - 2\pi\ln\omega \right]$$

$$-(2\pi)^{-2} \lim_{\omega\to\infty} \left[\sum_{\gamma\in\Gamma_1, |\gamma+\theta|\leq\omega} \frac{a}{\sqrt{|\gamma+\theta|^2 - k^2}} - 2\pi\ln\omega \right].$$

Taking into account known information about each term of the right-hand part (see above), we come to the conclusion that, generally speaking, there are two bands ("crystal" band and "waveguide" band), which may intersect. The position of bands depends on the correlation between $E^{\alpha,\Lambda_1}(0)$, $E^{\alpha,\Lambda_1}(-\pi/a)$, $E_0^{\alpha,\Lambda}(0)$, $E_0^{\alpha,\Lambda}(\theta_0)$, $E_1^{\alpha,\Lambda}$ (see (2.2), (3.2)). States corresponding to the "waveguide" band describe waves spreading along the empty line of nodes, i.e. waveguide effect.

Analogous consideration takes place in a case when there are several (for example, three) empty lines. Evidently, in this situation we have additional bands. The number of "waveguide bands" coincides with the number, n, of empty

lines of centres (of course, the bands can intersect), because in this situation we have a periodic chain with basic cell consisting of n centres.

One can consider more complicated structure – two coupled bottom crystal waveguides. Namely, one deals with a lattice with two empty layers of nodes and one additional empty node between them. Consider one centre Λ_0 in $\mathbf{R}^2$ as a simple preliminary stage. To introduce zero-range potential means to state a relation between coefficients of asymptotics of functions near the point Λ_0 [14]. Taking into account that the Green function for free two-dimensional space $\frac{i}{4}H_0^{(1)}(kr)$ has the following asymptotics near zero

$$\frac{i}{4}H_0^{(1)}(kr) = -\frac{1}{2\pi}(\ln r + \ln(-ik/2) - C_E/2) + o(r), r \to 0,$$

one obtains the following dispersion equation:

$$(3.4) \qquad\qquad \alpha - C_E/2 = \ln\frac{k}{2i}.$$

Here α is model parameter ("strength" of the potential). One can see that (3.4) has one imaginary root k. Hence, there is one bound state $k^2, k^2 < 0$.

Using the above arguments, one comes to the conclusion that there are two "waveguide" bands for the system of coupled waveguides, because the basic cell for 1D lattice consists of two centres. There is also an eigenvalue (bound state) which corresponds to "coupling aperture" (empty node). Note that if there are several (n) empty nodes ("coupling windows") then there are, generally speaking, n bound states.

4. Discussion

Let us discuss the problem of the choice of the model parameters. For this purpose we consider the problem for single "cylinder" of radius a. The solution of the corresponding two-dimensional problem of scattering of the plane wave u_0 should be continuous together with its derivative on the circle $r = a$:

$$(4.1) \qquad\qquad (u^+ - u^-)\,|_{r=a} = 0,$$

$$(4.2) \qquad\qquad \left(\frac{\partial}{\partial r}u^+ - \frac{\partial}{\partial r}u^-\right)|_{r=a} = 0.$$

The function u satisfies the following conditions:

$$\Delta u + k^2 u = 0, \quad r > a,$$

$$\Delta u + k_1^2 u = 0, \quad r < a.$$

We seek the solution in the form of series:

$$u = \begin{cases} \sum_{m=0}^{\infty} B_m J_m(k_1 r) \cos(m\varphi), & r \leq a, \\ \\ u_0 + \sum_{m=0}^{\infty} A_m H_m^{(1)}(kr) \cos(m\varphi), & r \geq a. \end{cases}$$

Due to the conditions on the circle one obtains the system:

$$(4.3) \quad A_m H_m^{(1)}(ka) - B_m J_m(k_1 a) = -(\pi(1 + \delta_{mo}))^{-1} \int_0^{2\pi} u_0 \mid_{r=a} \cos(m\varphi) d\varphi,$$

$$(4.4) \quad k A_m H_m^{(1)'}(ka) - k_1 B_m J_m'(k_1 a)$$

$$= -(\pi(1 + \delta_{mo}))^{-1} \int_0^{2\pi} \frac{\partial}{\partial r} u_0 \mid_{r=a} \cos(m\varphi) d\varphi.$$

Solving the system, one gets

$$(4.5) \quad A_m = (\pi(1 + \delta_{mo}) D_m)^{-1} \left(\int_0^{2\pi} u_0 \mid_{r=a} \cos(m\varphi) d\varphi k_1 J_m'(k_1 a) \right.$$

$$\left. - \int_0^{2\pi} \frac{\partial}{\partial r} u_0 \mid_{r=a} \cos(m\varphi) d\varphi J_m(k_1 a) \right),$$

$$(4.6) \quad D_m = J_m(k_1 a) k H_m^{(1)'}(ka) - J_m'(k_1 a) k_1 H_m^{(1)}(ka).$$

The model described above deals with the first term of the series only $(m = 0)$.
For $a \to 0$ integrals in (4.5) transform:

$$\int_0^{2\pi} u_0 \mid_{r=a} d\varphi = 2\pi u_0(0) + o(1), \quad a \to 0, \quad u_0(0) = u_0 \mid_{r=0} .$$

Using the Green formula, one obtains for small a

$$\int_0^{2\pi} \frac{\partial}{\partial r} u_0 \mid_{r=a} d\varphi = \frac{1}{a} \int_{r=a} \frac{\partial}{\partial r} u_0 \mid_{r=a} ds$$

$$= \frac{1}{a} \int\int_{r<a} \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) dxdy = -\frac{k^2}{a} \int\int_{r<a} u_0 dxdy$$

$$= -\frac{k^2}{a}(u_0(0) + o(1)) \int\int_{r<a} dxdy = -\pi k^2 u_0(0)a + o(a).$$

Thus, using the asymptotics of the cylindrical functions, one gets from (4.5), (4.6) under the condition $a \to 0$:

$$(4.7) \qquad A_0 = \frac{u_0(0)}{((k_1^2 - k^2)a^2)^{-1} - \ln a - \ln k - \ln(-i/2) - C_E/2}.$$

Compare the result with the corresponding result in the model. The solution of the scattering problem in the model has the following form:

$$u = u_0 + \tilde{A}_0 H_0^{(1)}(kr).$$

The solution has the following asymptotics in the neighbourhood of zero:

$$u = c_+ \ln r + c_- + o(r^0).$$

To construct the model one should assume a relation between the asymptotics coefficients:

$$c_- = \alpha c_+.$$

Taking into account the asymptotics of the Hankel function, one finds:

$$(4.8) \qquad \tilde{A}_0 = \frac{u_0(0)}{\alpha - \ln k - \ln(-i/2) - C_E/2}.$$

The comparison of (4.7) and (4.8) gives the following condition for choosing the model parameter:

$$\alpha = ((k_1^2 - k^2)a^2)^{-1} - \ln a.$$

Note that $k_1^2 - k^2$ is related with the vertical size of the cylinder $h - h_1$ (1.1). Using two parameters (k_1^2, a), one can choose α in such a way that it gives us the appropriate correlation between the model and the realistic solutions in the fixed range of frequencies. For example, for $ka = \text{const} = M$:

$$(4.9) \qquad \alpha = (k_1^2 a^2 - M^2)^{-1} - \ln a.$$

Taking into account the locality, one can believe that this choice of the parameters is appropriate in a more complicated problem of periodic system of cylinders.

Numerical analysis of the dispersion equation (3.1) is made. The results is in Fig. 2, where the dependence of the first roots of the equation of one of the quasi-momentum components is shown (for fixed second component). There are three curves for three fixed values of the second component on the figure for better seeing. Marked strip in the picture does not contain roots for any values of any components of the quasi-momentum – it is really a gap in the spectrum.
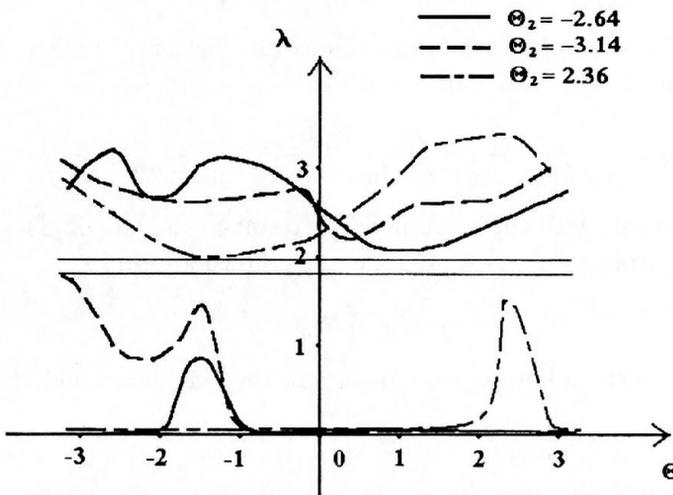

FIG. 2. The dependence of the first roots of the equation on one of the quasi-momentum components for fixed second component. The gap is marked.

The described effects can be applied to wave attenuation in harbours and channels and near some submerged or semi-submerged constructions. Namely, if dangerous frequencies for a concrete body (pier of bridge or derrick, etc.), that coincide with characteristic resonant frequencies of the object, are in the gap of the "bottom crystal". Then, these frequencies are prohibited, and, consequently, there will be attenuation of the surface wave. The same effect occurs for a channel. Moreover, one can use "bottom crystal waveguide" and coupled "bottom crystal waveguides" to concentrate waves in some regions and to create a waveguide effect in a part of the channel.

We use the periodicity condition in the model. It is an idealization. There is no periodic structure in reality. Every structure is finite, we have only a part

of a lattice. But calculations show that effects, analogous to those for periodic system appear when there are not very many of centres (30-40). Hence, it seems to be realistic to use the effect for engineering applications.

One can use the model for investigation of trapped modes for a system of thin submerged cylinders. In this case the problem reduces to the two-dimensional Helmholtz equation in a cross-section of the system [4, 6, 7]. It is easy to show that single zero-range potential in a strip gives us a mode (see above the description of zero-range potential in a free space). It corresponds to a trapped mode for single thin submerged cylinder [2]. Now, suppose there is a periodic chain of thin cylinders. Hence, in the model one has the two-dimensional Helmholtz problem in a strip with a chain of zero-range potentials. The dispersion equation is analogous. The only difference is that we should replace the Green function G_k, $G_k = \dfrac{i}{4} H_0^{(1)}(k \mid \lambda \mid)$, for free space by the Green function for the strip. The corresponding model operator has a band analogous to that in previous section ("bottom crystal waveguide").

Acknowledgement

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Degradations and Instabilities in Geomaterials *F. Darve (Grenoble, F), I. Vardoulakis (Athens, GR)*	**June 16-20**
Chemo-mechanical Couplings in Porous Media – Geomechanics and Biomechanics *B. Loret (Grenoble, F)*	**June 23-27**
Fluid Mechanics of Surfactant and Polymer Solutions *V. Starov (Loughborough, UK), I. Ivanov (Sofia, BG)*	**June 30-July 4**
Mechanics of Microstructured Materials *H.J. Boehm (Vienna, A)*	**July 7-11**
Bone Cell and Tissue Mechanics *S.C. Cowin (The City University of New York, USA)*	**July 14-18**
Mechanics of Solid Polymers: the Kinetics of Irreversible Processes *A. Dorfmann (Vienna, A), A. Drozdov (Beersheba, IL)*	**July 21-25**
Microfluidics: History, Theory and Applications *W.B.J. Zimmerman (Sheffield, UK)*	**Sept.1-5**
Walking-Biological and Technological Aspects *F. Pfeiffer (Garching, D), T. Zielińska (Warsaw, PL)*	**Sept. 15-19**
Dynamical Systems, Wave Based Computation and Neuro-inspired Robots *P. Arena (Catania, I)*	**Sept. 22-26**

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Parameter Identification of Materials and Structures   **October 6-9**
Z. Mróz (Warsaw, PL), G. Stavroulakis (Ioannina, GR)

Phenomenological and Mathematical Modelling   **October 20-24**
of Structural Instabilities
M. Pignataro (Roma, I), V. Gioncu (Timisoara, RO)

OTHER EVENTS

Procrustes Analysis and Geodetic Sciences   **October 15-16**
E.W. Grafarend (Stuttgart, D), F. Crosilla (Udine, I)

IU̅ᵀAM

**International Congress
of Theoretical and Applied Mechanics
August 15 - 21, 2004 ♦ Warsaw, Poland**

President:	*Co-Chairman:*	*Co-Chairman:*	*Secretary-General:*
Witold Gutkowski	Michał Kleiber	Włodzimierz Kurnik	Tomasz Kowalewski
PAN, Warszawa	*IPPT PAN, Warszawa*	*Politechnika Warszawska*	*IPPT PAN, Warszawa*

Preliminary Announcement

Scientific Program

The scientific program will start and end with opening and closing lectures, presented by prominent scientists. Titles of lectures and names of lecturers will be announced in October 2002, in the First Announcement and Call for Papers for the Congress.

The program will consist, moreover, of sectional lectures, mini-symposia and contributed papers presented in lecture and seminar presentation sessions. Invitations to present the contributed papers will be made on the recommendation of the International Paper Committee, based on their reviews of submitted abstracts and extended summaries.

Mini-Symposia

- Smart materials and structures.
- Tissue, cellular and molecular biomechanics.
- Mechanics of thin films and nanostructures.
- Microfluidics.
- Microgravity flow phenomena.
- Atmosphere and ocean dynamics.

Pre-Nominated Sessions

In Fluid Mechanics

Biological fluid dynamics • Boundary layers • Combustion and flames • Complex and smart fluids • Compressible flow • Computational fluid dynamics (jointly with IACM) • Convective phenomena • Drops and bubbles • Environmental fluid mechanics • Experimental methods in fluid mechanics • Flow control • Flow in porous media • Flow instability and transition • Flow in thin films • Fluid mechanics of materials processing • Granular flows • Low-Reynolds-number flow • Magnetohydrodynamics • Multiphase flows • Solidification and crystalgrowth • Stirring and mixing • Suspension mechanics • Topological fluid mechanics • Turbulence • Vortex dynamics • Waves

In Solid Mechanics

Computational solid mechanics (jointly with IACM) • Contact and friction problems (jointly with IAVSD)g • Control of multibody systems • Control of structures • Damage mechanics • Dynamic plasticity of structures • Elasticity • Experimental methods in solid mechanics • Fatigue • Fracture and crack mechanics (jointly with ICF) • Functionally graded materials • Impact and wave propagation • Material instabilities • Mechanics of composites • Mechanics of phase transformations (jointly with IACM) • Mechanics of porous materials • Multibody dynamics • Plasticity and viscoplasticity • Plates and shells (jointly with IACM) • Rock mechanics and geomechanics • Solid mechanics in manufacturing • Stability of structures • Stochastic micromechanics • Structural optimization (jointly with ISSMO) • Structural vibrations • Viscoelasticity and creep

Topics involving both fluid mechanics and solid mechanics

Acoustics • Chaos in fluid and solid mechanics • Continuum mechanics • Fluid-structure interaction • Mechanics of foams and cellular materials • Multiscale phenomena in mechanics

Details on preparation of manuscripts will be provided in the First Announcement and Call for Papers in October 2002.

It is planned to provide all future information mostly by e-mail. If you are interested in receiving future information, *please pre-register on the Congress World Wide Web site*:

<div align="center">

http://ictam04.ippt.gov.pl

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On this site, you can already find many interesting information concerning the Congress

Correspondence related to the Congress should be sent to:

<div align="center">

Prof. Tomasz Kowalewski, ICTAM04 Secretary-General

Institute of Fundamental Technological Research

Świętokrzyska 21, 00-049 Warszawa, Poland

e-mail: ictam04@ippt.gov.pl

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NATO Advanced Research Workshop
on
Polycrystals and Smart Materials

Nonlinear Homogenization and Its Application to Composites, Polycrystals and Smart Materials
http://ippt.gov.pl/NATO-ARW
June 23-26, 2003, Kazimierz Dolny, Poland

Organizers:

Józef Joachim Telega, Institute of Fundamental Technological Research, Warsaw, Poland (**chairman**)

Pedro Ponte Castañeda, University of Pennsylvania, Philadelphia, USA (**co-chairman**)

Tomaš Roubièek, Charles University, Prague, Czech Republic

Barbara Gambin, Institute of Fundamental Technological Research, Warsaw,

Key lectures:

1. Gregoire Allaire, *Optimal design of compliant mechanics by the homogenization methods.*
2. Kaushik Bhattacharya, *The effective behavior of polycrystals made of active materials. Asymptotic theories for thin films.*
3. Andrea Braides, *The passage from discrete to continuous variational problems: a nonlinear homogenization process.*
4. Andrej Cherkaev, *Structures of bi-stable inertial elements: waves, averaging, control.*
5. Tomasz Lewiński, *Homogenization and optimum design in structural mechanics.*
6. Robert Lipton, *Optimal design of Functionally Graded Structures for maximum strength and stiffness.*
7. François Murat, *Application of H-convergence to nonlinear homogenization. Junction problems between an elastic plate and an elastic beam.*
8. Pablo Pedregal, *Optimal design via variational principles.*
9. Pedro Ponte Castañeda, *Linear comparison methods in non-linear homogenization: Theory and applications.*
10. Tomas Roubièek, *Models of microstructure evolution in smart materials.*
11. Pierre Suquet, *Bounds and estimates for nonlinear composites in the light of numerical simulation.*
12. Józef Joachim Telega, *Stochastic homogenization.*

Besides the key lectures participants will present contributed papers. The participants are high-level specialists in the research field of the Workshop; 13 participants from USA, 36 from 9 different European countries and 1 from Turkey.

Total number of participants: 50.

Erratum

to the paper: J. Merodio and R.W. Ogden *"Material instabilities in fiber-reinforced nonlinearly elastic solids under plane deformation"*
Arch. Mech., **54**, 5-6, pp. 525–552, 2002

| p. 532 | line 2 | (from bottom) | written: (2.24) |
| | | | correct: (2.29) |

p. 533 line before Section 2.4, add: in general a unit vector.

| p. 534 | line 16 | (from top) | written: (2.29) |
| | | | correct: (2.34) |

| p. 537 | line 4 | (from bottom) | written: (2.39) |
| | | | correct: (2.33) |

| p. 538 | line 13 | (from top) | written: $(2.48)_3$ |
| | | | correct: $(2.46)_3$ |

pp. 539, 541 The labels in Figures 2 and 3 are misplaced.

| p. 543 | line 5 | (from top) | written: (2.45) |
| | | | correct: (3.8) |

| p. 544 | lines 18, 20 | (from top) | written: (3.17) |
| | | | correct: (3.18) |

p. 546 Eq. (3.23) line 3 (from top)

correct: $\sum_{i=1}^{2} (\mathbf{a}_i \cdot \mathbf{n})^2 \left[F_i'(I_4^{(i)}) + 2(\mathbf{a}_i \times \mathbf{n})^2 F_i''(I_4^{(i)}) \right]$

A pdf file with all corrections included is available from Professor Ogden:
rwo@maths.gla.ac.uk