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Finite deformations of polar media in angular coordinates

P. H. DŁUŻEWSKI (WARSZAWA)

IN THIS PAPER angular coordinates have been used to define new measures of finite deformation of elastoplastic polar medium. Euler angles are treated as an example of angular coordinates in orientation space. Angular and spatial-angular shifters are introduced to the description of polar deformation. A new definition of polar elastic medium is proposed. The angular deformation is described by three different tensors: the angular deformation gradient, the wryness tensor and the angular strain tensor.

Notations

$g_{\alpha\beta}$	angular metric tensor in the current configuration,
g_{kl}	metric tensor in the current configuration,
$\mathcal{G}_{\Theta\Phi}$	angular metric tensor in the reference configuration,
G_{KL}	metric tensor in the reference configuration,
g_k^K, g_K^k	shifters,
g_Θ^α, g_α^Θ	angular shifters,
$e_{\alpha kl}$, $E_{\Theta KL}$	alternating tensors,
ρ	mass density,
j^{kl}	tensor of inertia,
v^k	velocity vector,
ω^α	angular velocity vector,
f^k	external force density,
l^α	external angular force density,
k^α	angular momentum density,
t^{kl}	stress tensor,
$m^{\alpha k}$	couple stress tensor,
h	heat source density,
q^k	heat vector,
η	entropy density,
ε	internal energy density,
ψ	free energy density,
T	temperature,
$\mathbf{F}$	deformation gradient,
$\mathcal{F}$	angular deformation gradient,
$\mathfrak{E}$	strain tensor,
$\mathfrak{D}$	angular strain tensor,
$\mathfrak{T}$	stress tensor,
$\mathfrak{M}$	couple stress tensor,
$\mathfrak{C}$	Cosserat deformation tensor,
$\mathbf{\Gamma}$	wryness tensor,
$\mathbf{Q}$	rotation tensor,
$\mathbf{R}$	rotation tensor for non-polar medium,
$\mathbf{U}$	stretching tensor.

1. Introduction

THE POLAR MEDIUM is understood here as a continuum the configuration of which is described by two independent fields: the field of positions of its particles and the field of their orientations. Therefore, the absence of couple stress is irrelevant for identification of material as polar or nonpolar.

In 1909 E. and F. COSSERAT published the monograph [1] on the continuum mechanics of oriented bodies. The theory of polar continuum has been developed by TOUPIN [8], TRUESDELL and TOUPIN [9], ERINGEN and KAFADAR [3], NOWACKI [7], ERICKSEN [5] among many others. The motion of polar medium is often described with the help of the so-called directors.

In this paper the directors have been replaced by angular coordinates. The next section discusses a description of the position and orientation of a polar particle in spatial and angular coordinates, repectively. The Euler angles are used as an example of angular coordinates on the Riemannian space of orientations. The third section deals with field equations derived from the balance laws. The last section is devoted to the finite elasto-plastic deformations. A new definition of polar elastic medium is proposed. The paper concludes with an example of polar medium in which the couple stress is neglected.

2. Kinematics

Let $\mathbf{x}$ be a vector describing the position of a particle $\varkappa$ in the three-dimensional Euclidean space $\mathbb{R}$, where (x^1, x^2, x^3) denote components of the position in a curvi-linear coordinate system $\{x^k\}$. Let $\mathbf{e}_k$ denote the k-base vector tangent to the k-coordinate line at the point $\mathbf{x}$. Evidently, $\mathbf{x} \neq x^k\mathbf{e}_k$; however, for any vector $\mathbf{v}$ applied at $\varkappa$ we have $\mathbf{v} = v^k\mathbf{e}_k$. Similarly, let some coordinates, e.g. Euler angles (ϕ^1, ϕ^2, ϕ^3), describe the orientation $\boldsymbol{\phi}$ of $\varkappa$ in relation to the standard orthonormal basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ of the Euclidean space $\mathbb{R}$, Fig.1. So that, any configuration of polar medium can be fully described by determination of two fields (maps): the field $\mathbf{x}(\varkappa)$ of positions of polar particles and the field $\boldsymbol{\phi}(\varkappa)$ of their orientations. We will assume that each position on $\mathbb{R}$ can be occupied by only one particle. Therefore, for a given configuration we can also determine the mapping $\boldsymbol{\phi}(\mathbf{x})$.

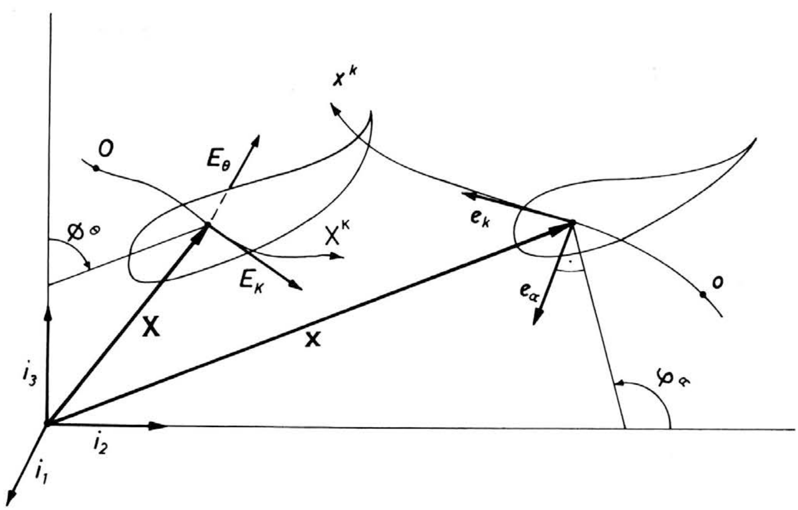

FIG. 1. Bases and coordinate lines.

It can be shown that for any geometric object we can determine the object orientation space $\mathcal{R}$ being the constant curvature Riemannian space. The angle between two different orientations of the object is simply the distance on $\mathcal{R}$, see DŁUŻEWSKI [2]. Thus, for any infinitesimal change of the object orientation through an angle $d\alpha$ follows

$$(d\alpha)^2 = g_{\beta\gamma} d\phi^\beta d\phi^\gamma \,. \tag{2.1}$$

An example of the angular coordinates are Euler angles, see Appendix A. The coordinate system $\{\phi^\alpha\}$ will be called the angular coordinate system on the object orientation space. It is worth to emphasize that the range of the Euler angles used to one-to-one description of polar particle orientations depends on the rotation symmetry of the particle.

According to Fig.1, the components (X^K) denote the reference position of the particle $\varkappa$ in terms of the curvilinear coordinate system $\{X^K\}$ on $\mathbb{R}$. Similarly, the components (Φ^Θ) determine the reference orientation of the particle $\varkappa$ in terms of the curvilinear coordinate system $\{\Phi^\Theta\}$ on $\mathcal{R}$.

Many tensorial quantities, as for example the angular velocity vector $\boldsymbol{\omega}$, can be considered in terms of the components ω^α (on $\mathcal{R}$), where

$$\omega^\alpha \equiv \dot{\phi}^\alpha \,, \tag{2.2}$$

as well as in terms of the generally known components ω^k on $\mathbb{R}$. The components are correlated by

$$\boldsymbol{\omega} = \omega^\alpha \mathbf{e}_\alpha = \omega^k \mathbf{e}_k, \tag{2.3}$$

where the angular base vector $\mathbf{e}_\alpha$ is parallel to the axis of the instantaneous rotation $d\phi^\alpha$ and is directed according to the right-hand screw rule, see Fig.2. The absolute values of the vectors $\mathbf{e}_\alpha$ are determined by the angular metric tensor as $|\mathbf{e}_\alpha| = \sqrt{g_{\alpha\alpha}}$. In the case of Euler angles, the angular base vectors $\mathbf{e}_\alpha$ are the unit vectors, see Appendix A.

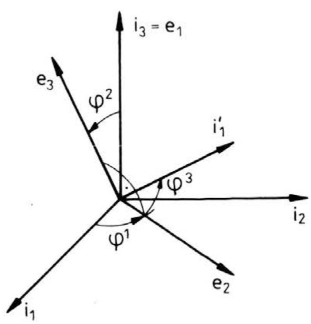

FIG. 2. Euler angles $\varphi^1, \varphi^2, \varphi^3$ understood as the angular coordinates, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ - the angular base vectors.

For polar medium many vectors can be shifted on the position space $\mathbb{R}$ as well as on the orientation space $\mathcal{R}$. The following shifters are defined on $\mathbb{R}$:

$$\begin{aligned} g^k_K &\equiv \mathbf{e}^k \cdot \mathbf{E}_K \,, \\ g^K_k &\equiv \mathbf{E}^K \cdot \mathbf{e}_k \,. \end{aligned} \tag{2.4}$$

We define also the mixed shifters

$$(2.5)\qquad \begin{aligned} g^{\alpha}_{k} &\equiv \mathbf{e}^{\alpha}\cdot\mathbf{e}_k\,,\\ g^{k}_{\alpha} &\equiv \mathbf{e}^{k}\cdot\mathbf{e}_\alpha\,,\\ g^{K}_{\Theta} &\equiv \mathbf{E}^{K}\cdot\mathbf{E}_\Theta\,,\\ g^{\Theta}_{K} &\equiv \mathbf{E}^{\Theta}\cdot\mathbf{E}_K\,. \end{aligned}$$

On the other hand, the following shifters can be also defined

$$(2.6)\qquad \begin{aligned} g^{\alpha}_{\Theta} &\equiv \mathbf{e}^{\alpha}\cdot\mathbf{E}_\Theta\,,\\ g^{\Theta}_{\alpha} &\equiv \mathbf{E}^{\Theta}\cdot\mathbf{e}_\alpha\,. \end{aligned}$$

Using the vectors $\mathbf{e}_\alpha$ and $\mathbf{E}_\Theta$ we can define the handedness of the angular coordinate systems.

DEFINITION. *Angular coordinate system* $(\phi^\alpha,\phi^\beta,\phi^\gamma)$ *is a right or left-handed angular coordinate system if the respective vectors* $(\mathbf{e}_\alpha,\mathbf{e}_\beta,\mathbf{e}_\gamma)$ *compose right or left-handed set, respectively.*

Excluding the origin of coordinates composed of Euler angles, the angles may be used as a left-handed angular coordinate system on any three-dimensional orientation space.

Alternating tensors are defined at the points $\mathbf{x}$ and $\mathbf{X}$ on $\mathbb{R}$ as follows,

$$(2.7)\qquad e_{klm}\equiv\pm\epsilon_{klm}\sqrt{\det(\mathbf{g})}\,,\qquad E_{KLM}\equiv\pm\epsilon_{KLM}\sqrt{\det(\mathbf{G})}\,,$$

where ϵ_{klm} is the permutation symbol and the sign $\pm$ is positive for right-handed coordinate system and negative for left-handed one. Similarly, at the points $\boldsymbol{\phi}$ and $\boldsymbol{\Phi}$ on $\mathcal{R}$ we may define, respectively

$$(2.8)\qquad e_{\alpha\beta\gamma}\equiv\pm\epsilon_{\alpha\beta\gamma}\sqrt{\det(g)}\,,\qquad \mathcal{E}_{\Theta\Phi\Lambda}\equiv\pm\epsilon_{\Theta\Phi\Lambda}\sqrt{\det(\mathcal{G})}\,.$$

Note that the shifters introduced by (2.4) and (2.5) lead to the following important relationships:

$$(2.9)\qquad \begin{aligned} g_{kl} &= g^{K}_{k}g^{L}_{l}G_{KL} = g^{\Theta}_{k}g^{\Phi}_{l}\mathcal{G}_{\Theta\Phi} = g^{\alpha}_{k}g^{\beta}_{l}g_{\alpha\beta}\,,\\ e_{klm} &= g^{K}_{k}g^{L}_{l}g^{M}_{m}E_{KLM} = g^{\Theta}_{k}g^{\Phi}_{l}g^{\Lambda}_{m}\mathcal{E}_{\Theta\Phi\Lambda} = g^{\alpha}_{k}g^{\beta}_{l}g^{\gamma}_{m}e_{\alpha\beta\gamma}\,. \end{aligned}$$

The angular velocity tensor is defined here by

$$(2.10)\qquad \omega_{kl}\equiv -e_{\alpha kl}\omega^{\alpha}\,,$$

where $e_{\alpha kl}$ is two-point tensor, $e_{\alpha kl}\equiv e_{mkl}g^{m}_{\alpha}$.

A motion of polar continuum will be described in relation to the reference configuration by two maps $\mathbf{x}(\mathbf{X},t)$ and $\boldsymbol{\phi}(\mathbf{X},t)$. In addition, we shall assume that for the reference configuration the mapping $\boldsymbol{\phi}(\mathbf{X})$ is given. The gradient of the displacement deformation is defined as [(1)]

$$(2.11)\qquad F^{k}{}_{K}\equiv x^{k}{}_{,K}=\frac{\partial x^k}{\partial X^K}\,,$$

[(1)] Except for the differentiation over time $\left(\rho_{,t}\equiv\dfrac{\partial\rho}{\partial t}\right)$, a comma denotes the covariant differentiation, e.g. $\omega^{\alpha}{}_{,k}=\dfrac{\partial\omega^\alpha}{\partial x^k}+\dfrac{\omega^\beta\,\partial\mathbf{e}_\beta}{\partial x^k}\cdot\mathbf{e}^\alpha$, where the connection coefficients $\Gamma_{\beta}{}^{\alpha}_{k}=\dfrac{\partial\mathbf{e}_\beta}{\partial x^k}\cdot\mathbf{e}^\alpha$ depend on the mapping $\boldsymbol{\phi}(\mathbf{x})$.

see Eq. (19.1) in ERICKSEN [4]. Using the angular coordinates we can also define the angular deformation gradient:

$$\mathcal{F}^{\alpha}{}_{K} \equiv \phi^{\alpha}{}_{,K} = \frac{\partial \phi^{\alpha}}{\partial X^{K}} . \tag{2.12}$$

The gradients $\mathbf{F}$ and $\mathcal{F}$ are mutually independent measures what can be shown by considering integrability conditions of displacements and rotations. The conditions take the form

$$F^{k}{}_{K,L} = F^{k}{}_{L,K} \quad \text{and} \quad \mathcal{F}^{\alpha}{}_{K,L} = \mathcal{F}^{\alpha}{}_{L,K} . \tag{2.13}$$

In many cases the rotation of microstructure can not be determined on the basis of the so-called polar decomposition $\mathbf{F} = \mathbf{RU}$. In other words $\mathbf{Q} \neq \mathbf{R}$, where $\mathbf{Q}$ denotes the local rotation of microstructure, e.g. the rotation of crystal lattice, see Fig.3.

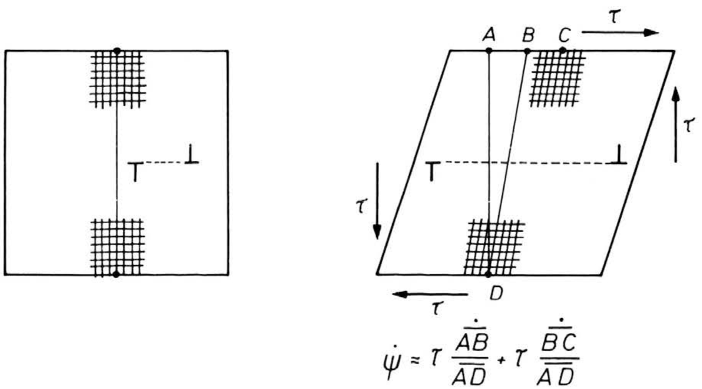

FIG. 3. Elastic strain of crystal with dislocation loop, schematic drawing.

In such a case we may use the following polar decompositions of deformation gradients into the rotation of microstructure and deformation:

$$\mathbf{F} = \mathbf{Q}\mathfrak{C} , \qquad \mathcal{F} = \mathbf{Q}\boldsymbol{\Gamma} . \tag{2.14}$$

The corrotational gradients $\mathfrak{C}$ and $\boldsymbol{\Gamma}$ are often called the Cosserat deformation tensor and the wryness tensor, respectively. Let us assume that

$$\mathfrak{C} = \mathfrak{C}_0 + \mathfrak{E} , \qquad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 + \mathfrak{D} , \tag{2.15}$$

where $\Gamma_0{}^{\Theta}{}_{K} \equiv \Phi^{\Theta}{}_{,K}$ and $\mathfrak{C}_0{}^{K}{}_{L} \equiv X^{K}{}_{,L} = \delta^{K}{}_{L}$. It corresponds to the following defintions:

$$\begin{aligned} \mathfrak{E}^{K}{}_{L} &\equiv Q_{k}{}^{K} x^{k}{}_{,L} - X^{K}{}_{,L} , \\ \mathfrak{D}^{\Theta}{}_{L} &\equiv Q_{\alpha}{}^{\Theta} \phi^{\alpha}{}_{,L} - \Phi^{\Theta}{}_{,L} . \end{aligned} \tag{2.16}$$

Note that

$$Q^{k}{}_{K,M} = Q^{k}{}_{K,\alpha} \phi^{\alpha}{}_{,M} + Q^{k}{}_{K,\Theta} \phi^{\Theta}{}_{,M} = -\phi^{\alpha}{}_{,M} e_{\alpha}{}^{k}{}_{l} Q^{l}{}_{K} - \phi^{\Theta}{}_{,M} E_{\Theta}{}^{L}{}_{K} Q^{k}{}_{L} \tag{2.17}$$

(see Appendix B); it follows that

$$\mathfrak{D}^{\Theta}{}_{M} = \frac{1}{2} E^{\Theta K L} Q_{lK,M} Q^{l}{}_{L} ; \tag{2.18}$$

Similarly, for co-rotational gradient $\mathbf{\Gamma}$ we obtain

$$\Gamma^{\Theta}{}_{M} = \frac{1}{2} E^{\Theta KL} \chi_{lK,M} \chi^{l}{}_{L}\,, \tag{2.19}$$

where $\chi = \mathbf{Q}^{\Phi}$, see Appendix A, cf. ERINGEN and KAFADAR [3].

3. Balance laws

For polar medium the balance laws have been discussed in many papers, e.g. KAFADAR and ERINGEN [6], NOWACKI [7]. The mass balance, momentum balance, moment of momentum balance and energy balance are stated respectively by:

$$\begin{aligned}
&\frac{d}{dt}\int_V \rho dV = 0\,,\\
&\frac{d}{dt}\int_V \rho\mathbf{v} dV = \int_S \mathbf{t}_{(n)} dS + \int_V \rho\mathbf{f} dV\,,\\
&\frac{d}{dt}\int_V (\mathbf{x}\times\rho\mathbf{v} + \rho\mathbf{k}) dV = \int_S (\mathbf{x}\times\mathbf{t}_{(n)} + \mathbf{m}_{(n)}) dS + \int_V (\mathbf{x}\times\rho\mathbf{f} + \rho\mathbf{l}) dV,\\
&\frac{d}{dt}\int_V \left(\rho\varepsilon + \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\rho\mathbf{k}\cdot\boldsymbol{\omega}\right) dV = \int_S (\mathbf{t}_{(n)}\cdot\mathbf{v} + \mathbf{m}_{(n)}\cdot\boldsymbol{\omega}) dS\\
&\qquad + \int_V (\rho\mathbf{f}\cdot\mathbf{v} + \rho\mathbf{l}\cdot\boldsymbol{\omega}) dV - \int_S \mathbf{q}\cdot d\mathbf{S} + \int_V \rho h dV\,,
\end{aligned} \tag{3.1}$$

where $\mathbf{t}_{(n)} = t^{kl} n_l \mathbf{e}_k$, $\mathbf{m}_{(n)} = m^{\alpha k} n_k \mathbf{e}_\alpha$, $\mathbf{n}$ denotes the unit normal to the surface S bounding the polar body region V, $\times$ denotes the vector product. The above laws give the following field equations

$$\begin{gathered}
\rho_{,t} + (\rho v^k)_{,k} = 0\,,\\
t^{kl}{}_{,l} + \rho f^k = \rho \dot{v}^k\,,\\
m^{\alpha k}{}_{,k} + 2t^\alpha + \rho l^\alpha = \rho \dot{k}^\alpha\,,\\
\rho\dot{\varepsilon} - \frac{1}{2}\rho\omega_\alpha \dot{j}^{\alpha\beta}\omega_\beta = t^{kl} v_{k,l} - 2t^\alpha\omega_\alpha + m^{\alpha k}\omega_{\alpha,k} - q^k{}_{,k} + \rho h\,,
\end{gathered} \tag{3.2}$$

where

$$t^\alpha \equiv -\frac{1}{2} e^{\alpha mn} t_{mn} \quad \text{and} \quad k^\alpha \equiv j^{\alpha k}\omega_k\,.$$

Conservation of inertia

After Eringen we assume the conservation of inertia in the form $\mathbf{j} = \mathbf{Q}\mathbf{j}_{|t=0}\mathbf{Q}^T$. Hence, the corrotational derivative of the inertia tensor is equal to zero,

$$j^{kl}{}_{,t} + j^{kl}{}_{,m} v^m + \omega_m{}^k j^{ml} + \omega_m{}^l j^{km} = 0\,. \tag{3.3}$$

In this case the term $\frac{1}{2}\boldsymbol{\omega}\cdot\dot{\mathbf{j}}\cdot\boldsymbol{\omega}$ vanishes in $(3.2)_4$.

Entropy inequality

The entropy inequality is stated for polar body by

$$\frac{d}{dt}\int_V \rho\eta dV \geq -\int_S \frac{\mathbf{q}\cdot d\mathbf{S}}{T} + \int_V \frac{\rho h}{T}dV\,. \tag{3.4}$$

It leads to the field equation

$$\rho\dot{\eta} + \left(\frac{q^k}{T}\right)_{,k} - \frac{\rho h}{T} \geq 0\,. \tag{3.5}$$

Using $(3.2)_4$ and (3.3), the last inequality can be expressed as

$$-\rho(\dot{\psi} + \eta\dot{T}) + t^{kl}v_{k,l} - 2t^{\alpha}\omega_{\alpha} + m^{\alpha k}\omega_{\alpha,k} - \frac{q^k}{T}T_{,k} \geq 0\,, \tag{3.6}$$

where $\psi = \varepsilon - \eta T$.

4. Polar elasticity based on strain energy

Let us assume the following definition of polar elastic medium.

DEFINITION. *A medium is polar elastic if the free energy density has the form*

$$\psi = \psi(\mathbf{X}, \mathfrak{C}, \mathfrak{D}, T)\,. \tag{4.1}$$

To define the polar elastic medium Eringen and Kafadar used the tensors $\mathfrak{C}$ and $\mathbf{\Gamma}$. They assumed that the initial configuration was the reference one and that $\mathbf{\Gamma}_{|t=0} = 0$. Otherwise, their equation $\psi = \widehat{\psi}(\mathbf{X}, \mathfrak{C}, \mathbf{\Gamma}, T)$ does not take into account the orientation distribution of the particle orientations in the reference configuration. In such a case the general dependence should have the form $\psi = \psi(\mathbf{X}, \mathfrak{C}, \mathbf{\Gamma}, \mathbf{\Gamma}_0, T)$. In our case

$$\dot{\psi} = \frac{\partial\psi}{\partial\mathfrak{C}}\dot{\mathfrak{C}} + \frac{\partial\psi}{\partial\mathfrak{D}}\dot{\mathfrak{D}} + \frac{\partial\psi}{\partial T}\dot{T}\,, \tag{4.2}$$

where

$$\dot{\mathfrak{C}}^K{}_L = -\omega^{\alpha}e_{\alpha k}{}^{l}Q_l{}^K x^k{}_{,L} + Q_k{}^K v^k{}_{,l}x^l{}_{,L} \tag{4.3}$$

and

$$\dot{\mathfrak{D}}^{\Theta}{}_L = \omega^{\alpha}{}_{,k}x^k{}_{,L}Q_{\alpha}{}^{\Theta}\,. \tag{4.4}$$

So, the inequality (3.6) can be rewritten as

$$\begin{aligned}(4.5)\qquad -\rho\left(\frac{\partial\psi}{\partial T} + \eta\right)\dot{T} &+ \left(t_k{}^l - \rho\frac{\partial\psi}{\partial\mathfrak{C}^K{}_L}Q_k{}^K x^l{}_{,L}\right)v^k{}_{,l} \\ &- \left(2t_{\alpha} - \rho\frac{\partial\psi}{\partial\mathfrak{C}^K{}_L}e_{\alpha k}{}^{l}Q_l{}^K x^k{}_{,L}\right)\omega^{\alpha} \\ &+ \left(m_{\alpha}{}^k - \rho\frac{\partial\psi}{\partial\mathfrak{D}^{\Theta}{}_L}Q_{\alpha}{}^{\Theta}x^k{}_{,L}\right)\omega^{\alpha}{}_{,k} - \frac{q^k}{T}T_{,k} \geq 0\,.\end{aligned}$$

Assuming that the constitutive equation for heat flux, e.g. $\mathbf{q} = \mathbf{Q}\cdot\widehat{\mathbf{q}}(\mathbf{X}, \mathfrak{C}, \mathfrak{D}, T, \mathbf{Q}^T\nabla T)$, satisfies the condition $\frac{q^k}{T}T_{,k} \geq 0$ we conclude that

$$\eta = -\frac{\partial\psi}{\partial T},$$
$$t_k{}^l = \rho\frac{\partial\psi}{\partial\mathfrak{C}^K{}_L}Q_k{}^K x^l{}_{,L}, \tag{4.6}$$
$$m_\alpha{}^k = \rho\frac{\partial\psi}{\partial\mathfrak{D}^\Theta{}_L}Q_\alpha{}^\Theta x^k{}_{,L}.$$

The deformation measures $\mathfrak{C}$ and $\mathfrak{D}$ are conjugated with the following stress measures

$$\mathfrak{T} = \rho_0\frac{\partial\psi}{\partial\mathfrak{C}} \quad \text{and} \quad \mathfrak{M} = \rho_0\frac{\partial\psi}{\partial\mathfrak{D}}, \tag{4.7}$$

where

$$\mathfrak{T}_K{}^L \equiv \frac{\rho_0}{\rho}Q^k{}_K t_k{}^l X^L{}_{,l} \tag{4.8}$$

and

$$\mathfrak{M}_\Theta{}^L \equiv \frac{\rho_0}{\rho}Q^\alpha{}_\Theta m_\alpha{}^l X^L{}_{,l}, \tag{4.9}$$

thus the entropy inequality can be rewritten as

$$\rho_0\, d\psi \leq \mathfrak{T}\cdot d\mathfrak{C} + \mathfrak{M}\cdot d\mathfrak{D} + \eta_0 dT, \tag{4.10}$$

where $\eta_0 = \frac{\rho_0}{\rho}\eta$. So it has been shown that, from the viewpoint of balance laws, polar elastic media may be defined by (4.1).

Up to here we considered elastic bodies. For elasto-plastic media the following decomposition of deformation gradients may be proposed,

$$\begin{aligned} \mathbf{F} &= \mathbf{Q}\mathbf{F}^e\mathbf{F}^p = \mathbf{Q}(\mathbf{F}^e_0 + \mathfrak{C}^e)(\mathbf{F}^p_0 + \mathfrak{C}^p), \\ \mathcal{F} &= \mathbf{Q}(\mathcal{F}^e\mathbf{F}^p + \mathcal{F}^p) = \mathbf{Q}(\mathcal{F}^e_0 + \mathfrak{D}^e)(\mathbf{F}^p_0 + \mathfrak{C}^p) + \mathbf{Q}(\mathcal{F}^p_0 + \mathfrak{D}^p). \end{aligned} \tag{4.11}$$

where the indices e and p denote elastic and plastic deformation tensors, respectively, Fig.4.

In conclusion, let us assume that in an elastic body the quantities related to couple stresses are negligibly small, i.e.

$$\mathbf{m} = \mathbf{0}, \quad \mathbf{j} = \mathbf{0}, \quad \mathbf{l} = \mathbf{0}, \quad \frac{\partial\psi}{\partial\mathfrak{D}} = \mathbf{0}. \tag{4.12}$$

Similar assumptions are generally made in calculations of trusses where the torsional and bending moments acting on their members are often neglected. In the case of negligibly small rotation stiffness, the angular deformation tensor $\mathfrak{D}$ may not play any role in the particle energetic state, but for the body as a whole the minimum of free energy depends on the orientation field with regard to the dependence of $\mathfrak{C}$ on $\mathbf{Q}$. Therefore, the field of angular deformations can be determined from the global minimum of free energy. On the other hand, the mentioned constrains find their expression in compatibility conditions. On the basis of (2.13)–(2.16) we obtain the integrability equations

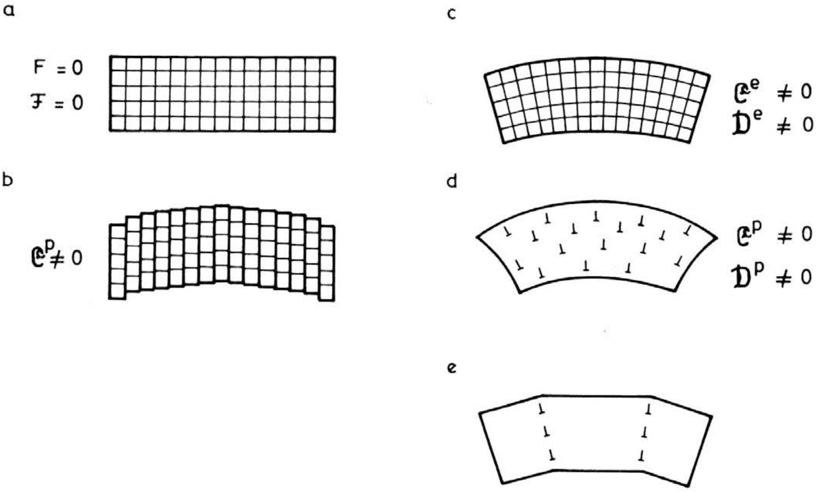

FIG. 4. Examples of crystal deformations: a)reference configuration, b)displacement deformation, c)angular-displacement elastic deformation, d)angular-displacement plastic deformation, e)discontinuous field of permanent deformations.

$$\begin{aligned} \mathfrak{E}^M{}_{K,L} - \mathfrak{E}^M{}_{L,K} &= E_\Theta{}^M{}_N[(\mathfrak{E}^N{}_K + X^N{}_{,K})(\mathfrak{D}^\Theta{}_L + \Phi^\Theta{}_{,L}) \\ &\qquad - (\mathfrak{E}^N{}_L + X^N{}_{,L})(\mathfrak{D}^\Theta{}_K + \Phi^\Theta{}_{,K})], \\ \mathfrak{D}^\Theta{}_{K,L} - \mathfrak{D}^\Theta{}_{L,K} &= 0, \end{aligned} \tag{4.13}$$

cf. [6]. Let us turn back to the assumptions (4.12). The assumptions lead to the symmetry of the stress tensor **t**. In this case (4.1) transforms into

$$\psi = \psi(\mathbf{X}, \mathfrak{E}, T). \tag{4.14}$$

Note that the symmetry of the stress tensor **t** does not impose a symmetry on the strain tensor $\mathfrak{E}$, see Fig.3. It is worth to emphasize that the absence of couple stresses does not reduce the polar continuum to the classical one.

Appendix A

Let the Euler angles φ^α determine the orientation of the particle occupying the position **x**, Fig.1. The particle orientation can be determined also by the tensor $\mathbf{Q}^{\Phi}$ of turning the particle from the origin of the angular coordinate system $\{\phi^\alpha\}$ to the current orientation $\boldsymbol{\phi}$. It corresponds to

$$\mathbf{Q}^{\Phi} = \mathbf{Q}^{\phi^3}\mathbf{Q}^{\phi^2}\mathbf{Q}^{\phi^1}, \tag{A.1}$$

where

$$\mathbf{Q}^{\phi^\alpha} = \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha + \cos(\phi^\alpha)(\mathbf{g} - \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha) - \sin(\phi^\alpha)(\mathbf{e} \cdot \mathbf{e}_\alpha). \tag{A.2}$$

The symbol $\otimes$ denotes the dyadic product, **g**, **e** and $\mathbf{e}_\alpha$ denote, respectively, the metric tensor, alternating tensor and the angular base vectors:

$$\mathbf{e}_\alpha \equiv \begin{cases} \mathbf{e}_1 = \mathbf{i}_3, \\ \mathbf{e}_2 = \mathbf{i}_1 \cos(\phi^1) + \mathbf{i}_2 \sin(\phi^1), \\ \mathbf{e}_3 = \mathbf{i}_1 \sin(\phi^1)\sin(\phi^2) - \mathbf{i}_2 \cos(\phi^1)\sin(\phi^2) + \mathbf{i}_3 \cos(\phi^2). \end{cases} \tag{A.3}$$

It can be shown [2] that such determined angular coordinates correspond to the angular metric tensor

(A.4)
$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & \cos\phi^2 \\ 0 & 1 & 0 \\ \cos\phi^2 & 0 & 1 \end{bmatrix} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta .$$

Analogous relations are obtained for the reference orientation $\mathbf{\Phi}$. Assuming that in our case the origin of the angular coordinate system $\{\phi^\alpha\}$ coincides with the origin of the system $\{\Phi^\Theta\}$, the rotation tensor $\mathbf{Q}$ can be determined as

(A.5)
$$\mathbf{Q} = \mathbf{Q}^{\phi}(\mathbf{Q}^{\Phi})^T .$$

Appendix B

Exceptionally in this Appendix let us assume that the current as well as the reference configuration of a polar body change in time. Then the rotation rates of the unit directors $\mathbf{N}$ and $\mathbf{n} = \mathbf{QN}$ satisfy

(B.1)
$$\begin{aligned} \dot{\mathbf{n}} &= \boldsymbol{\omega} \times \mathbf{n} = -\omega^\alpha e_\alpha{}^k{}_l n^l \mathbf{e}_k , \\ \dot{\mathbf{N}} &= \boldsymbol{\Omega} \times \mathbf{N} = -\Omega^\Theta E_\Theta{}^K{}_L N^L \mathbf{E}_K , \end{aligned}$$

where $\omega^\alpha = \dot{\phi}^\alpha$ and $\Omega^\Theta = \dot{\Phi}^\Theta$. The material derivative of the director $\mathbf{n}$ is also determined as

(B.2)
$$\dot{\mathbf{n}} = \dot{\mathbf{Q}}\mathbf{N} + \mathbf{Q}\dot{\mathbf{N}} ,$$

what can be rewritten in the form

(B.3)
$$\boldsymbol{\omega} \times \mathbf{QN} = \dot{\mathbf{Q}}\mathbf{N} + \mathbf{Q}\boldsymbol{\Omega} \times \mathbf{N}$$

so that

(B.4)
$$\dot{Q}^k{}_K = -\dot{\phi}^\alpha e_\alpha{}^k{}_l Q^l{}_K + \dot{\Phi}^\Theta E_\Theta{}^L{}_K Q^k{}_L .$$

Let us consider the following composite mapping:

(B.5)
$$\mathbf{Q} = \mathbf{Q}(\boldsymbol{\phi}(\mathbf{x}), \mathbf{\Phi}(\mathbf{X})) ,$$

its material derivative being determined by

(B.6)
$$\dot{Q}^k{}_K = Q^k{}_{K,\alpha}\dot{\phi}^\alpha + Q^k{}_{K,\Theta}\dot{\Phi}^\Theta .$$

Comparing (B.6) with (B.4) we identify the covariant derivatives as

(B.7)
$$Q^k{}_{K,\alpha} = -e_\alpha{}^k{}_l Q^l{}_K \quad \text{and} \quad Q^k{}_{K,\Theta} = -E_{\Theta K}{}^L Q^k{}_L .$$

On the other hand, the covariant derivatives of spatial-angular shifters vanish, thus

(B.8)
$$Q^\alpha{}_{\Theta,\beta} = -e_\beta{}^\alpha{}_\gamma Q^\gamma{}_\Theta \quad \text{and} \quad Q^\alpha{}_{\Theta,\Lambda} = -E_{\Lambda\Theta}{}^\Psi Q^\alpha{}_\Psi .$$

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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