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Weak discontinuity waves in materials with semi-empirical temperature scale

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IN THIS PAPER, weak discontinuity waves are investigated for rigid and elastic heat conductors. For rigid conductors, a weak wave is represented by a thermal wave which propagates with finite speed. A thermoelastic material admits two families of waves which transport both thermal and mechanical disturbances. In the case of a one-dimensional body, a differential equation is found to describe the evolution of the amplitude of the weak wave. It is shown that the amplitude in both rigid and elastic conductors can blow up in finite time which can lead to the formation of shocks.

1. Introduction

WE CONSIDER the problem of heat transported by conduction in rigid as well as in deformable bodies, in which heat pulses are transmitted by waves at finite speeds.

The first theory of heat conduction with finite wave speed was proposed by CATTANEO [1,2] and VERNOTTE [3]. However, it seems that it was MAXWELL [4] who was the first to modify Fourier's law. Hence we refer to

$$\tau \mathbf{q}_{,t} + \mathbf{q} = -k\nabla\vartheta \tag{1.1}$$

as the M.C.V. (Maxwell—Cattaneo—Vernotte) equation. Here $\tau > 0$ is a suitable relaxation time, ϑ denotes the absolute temperature, k is the conductivity and $\mathbf{q}$ the heat flux vector. When $\tau = 0$, Eq. (1.1) reduces to Fourier's law. However, if no terms are omitted from Eq. (1.1) and the internal energy ε is a function of ϑ, as for rigid solids, then the energy equation

$$\rho_0\varepsilon_{,t} + \operatorname{div}\mathbf{q} = 0 \tag{1.2}$$

combined with Eq. (1.1) leads to a telegraph equation

$$\tau\rho_0 c_V \vartheta_{,tt} - k\Delta\vartheta + \rho_0 c_V \vartheta_{,t} = 0\,, \tag{1.3}$$

where c_V represents the derivative of the energy with respect to ϑ, and is called the heat capacity, assumed to be constant in Eq. (1.3).

The last equation is hyperbolic if $k/(\tau\rho_0 c_V) = s^2$ is positive, and it transmits waves of temperature weak discontinuity at the speed s. (Here k has been assumed constant.)

Heat conduction with finite wave speed can be analyzed in the context of the theory of materials with memory (e.g. GURTIN and PIPKIN [5]). One can use a different approach that consists in employing internal state variables in modelling the behavior of heat conducting bodies. This model is obtained by enriching the set of the independent variables appearing in the constitutive equations by additional quantities called internal state variables. Suitable kinetic equations for the evolution of the internal state

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variables are then postulated. The additional equations are evolutionary first order differential equations (e.g. KOSIŃSKI, PERZYNA [6], KOSIŃSKI [7], KOSIŃSKI, SZMIT [8], MIHAILESCU, SULICIU [9], MORRO [10]). Differential approaches have been used by MORRO, RUGGERI [11] and MÜLLER [12]. To study the physical meaning of temperature waves and heat conduction, kinetic theory and the extended thermodynamics of LARECKI, PIEKARSKI [13] and PIEKARSKI [14], have been employed. The 'inertial' theory of heat conduction with modified Onsager's symmetry relations was developed by KALISKI [15].

In the present paper a different model is used; in the case of a rigid heat conductor, the heat flux vector is given by a Fourier-type law, in which the gradient of a semi-empirical temperature appears, instead of the absolute one. The concept of the semi-empirical temperature scale has been introduced by KOSIŃSKI [16], and the physical foundations of the new temperature scale have been given by CIMMELLI, KOSIŃSKI [17, 18].

The case of a deformable continuum considered in this paper is restricted to a thermoelastic body. In that case, the final system of governing equations is hyperbolic under mild assumptions concerning the partial derivatives of the internal (or free) energy functions and the heat conductivity.

One of the aims of this paper is to show that the propagation of the weak discontinuity waves in both rigid and deformable cases is governed by a Bernoulli equation in which the nonlinear term may lead to the finite time breakdown and formation of shock waves. On the other hand, if the heat capacity (the specific heat) is a suitable power of the absolute temperature, then the nonlinearity in the amplitude equation will disappear and no singularity will form in the propagation of weak discontinuity waves.

The organization of the paper is as follows: in Sect. 2 we shall derive the governing hyperbolic equations based on physical foundations. In Sect. 3, thermal waves are considered. Section 4 is devoted to the thermoelastic case, and in Sect. 5, concluding remarks are given.

The present paper can be regarded as a continuation of the investigation of the previous two papers, KOSIŃSKI, SAXTON [19], where the finite time blow up of the amplitude of one-dimensional temperature rate waves in a particular class of rigid and deformable conductors was investigated, and the paper CIMMELLI, KOSIŃSKI [20], where a local in time existence, uniqueness and continuous dependence result was obtained for solutions of a Cauchy problem of a rigid heat conductor with the semi-empirical temperature scale.

One of the open problems is the question of global existence. This problem will be discussed in further papers together with shock wave analysis.

2. Rigid heat conductor with thermal relaxation

The classical Fourier law relating the heat flux vector to the gradient of the absolute temperature does not allow for propagation of thermal waves. The first theory of heat conduction which admitted finite wave speeds was included in the Landau two-fluid model for helium II in 1940.

Rigid heat conductors were discussed by CATTANEO [1, 2] and VERNOTTE [3]. In their derivation an extra term in the heat equation had appeared, which was proportional to the time derivative of the heat flux. The proportionality constant (or coefficient) was called by them a thermal relaxation time. It seems, however, that it was MAXWELL [4] who obtained, for the first time, such a term in his derivation of an energy balance equation based on statistical arguments. However, he dropped this term after his derivation.

The modifications of Cattaneo and Vernotte, as well as a particular case of that of Maxwell, can be summarized simply by the following so-called Maxwell—Cattaneo—Vernotte equation,

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -k\nabla\vartheta \,, \tag{2.1}$$

where the superposed dot denotes the time derivative, $\mathbf{q}$ is the heat flux vector, $\nabla\vartheta$ denotes the temperature gradient, while k and τ are, in general, material functions, called the heat conduction coefficient and thermal relaxation time, respectively. In most known physical models, the value of τ is very small, and varies in the range around $10^{-9} - 10^{-11} s^{-1}$.

In the subsequent parts of the paper, another model will be explored with a Fourier-type proportionality law. To this end, let us notice that the Fourier law, often written in the form

$$\mathbf{q} = -k\nabla\vartheta \,, \tag{2.2}$$

with the heat conductivity k depending in general on ϑ, is a consequence of an averaging procedure, performed at the macroscopic, or statistical level by CATTANEO [2]. The starting point of this procedure is a proportionality rule between the "protoplast" $\widetilde{\mathbf{q}}$ of the heat flux vector and the gradient of the kinetic energy of molecules G,

$$\widetilde{\mathbf{q}} = -A\nabla G \,, \tag{2.3}$$

where A is a constant. Since G, in that model, is proportional to the absolute temperature, i.e. $G = m\vartheta$, we can recover (2.2).

The proportionally rule (2.3) is the first approximation in which one has assumed a constant, in time, value of G. A time-dependent function G requires a second approximation in which an extra term is added to the right side of Eq. (2.3), that is proportional to the second mixed derivative of G, namely,

$$\widetilde{\mathbf{q}} = -A\nabla G + \sigma\nabla\dot{G} \,, \tag{2.4}$$

where $\sigma = \tau A$.

If we define, formally, a new semi-empirical temperature scale as a solution of the kinetic equation

$$\dot{\overline{m\beta}} = \frac{m}{\tau}(\vartheta - \beta) \,, \tag{2.5}$$

then the macroscopic counterpart of (2.4) will be a proportionality law (compare CIMMELLI and KOSIŃSKI [17]),

$$\mathbf{q} = -k\nabla\beta \,, \tag{2.6}$$

with β given as a solution of Eq. (2.5) under the initial condition $\beta(0) = \vartheta(0)$, provided that at the initial time thermodynamic equilibrium was maintained. In the general case, however, $\beta(0) = \beta_0$, may differ from $\vartheta(0) = \vartheta_0$.

The appearance of the coefficient m in the kinetic equation (2.5) can be explained in terms of statistical mechanics arguments; here the product $m\beta$ can be regarded as a non-equilibrium thermal energy(1) related linearly to the new semi-empirical temperature β. Moreover, the presence of m(2) in Eq. (2.5) will be of a special importance when a deformable continuum is discussed in Lagrangian and Eulerian setups.

For the purpose of this paper we assume a simplified version of Eq. (2.5) — m will be constant. First, we investigate the behavior of rigid conductors. The constitutive equations form a set of relations involving the following functions: the Helmholtz free energy ψ, specific entropy η, and heat flux $\mathbf{q}$, which are represented, in general, by functions of temperature ϑ, and gradient of the new temperature $\nabla\beta$,

$$\begin{aligned} \psi &= \hat{\psi}(\vartheta, \nabla\beta), \\ \eta &= \hat{\eta}(\vartheta, \nabla\beta) = -\psi_{,\vartheta}(\vartheta, \nabla\beta), \\ \mathbf{q} &= -k(\vartheta)\nabla\beta. \end{aligned} \tag{2.7}$$

We assume that the free energy is given by

$$\hat{\psi}(\vartheta, \nabla\beta) = \psi^1(\vartheta) + \frac{\tau_0}{2}\frac{k(\vartheta)}{\rho_0\vartheta}\nabla\beta\cdot\nabla\beta \tag{2.8}$$

and that the internal energy $\varepsilon = \psi + \eta\vartheta$ does not depend on $\nabla\beta$. Here $\tau = \tau_0$ is a constant relaxation time. As a consequence of the second assumption, we obtain

$$\varepsilon = \hat{\varepsilon}(\vartheta) = \psi^1(\vartheta) - \vartheta\psi^1_{,\vartheta}(\vartheta), \tag{2.9}$$

and for the heat conductivity

$$k(\vartheta) = \frac{k_0}{\vartheta_0^2}\vartheta^2, \tag{2.10}$$

where $k_0 = k(\vartheta_0)$.

The system of equations for a rigid conductor consists of Eq. $(2.7)_1$, the evolution equation for β (2.5) (with m = const), and the energy balance law:

$$\begin{aligned} \mathbf{q} &= -k(\vartheta)\nabla\beta, \\ \dot{\beta} &= \frac{1}{\tau_0}(\vartheta - \beta), \\ \rho_0\dot{\varepsilon} &+ \operatorname{div}\mathbf{q} = 0. \end{aligned} \tag{2.11}$$

Here ρ_0 is a reference density, and we assume there to be no heat supply. Using Eqs. (2.8), (2.9) and (2.10), we obtain a second order partial differential equation for β of the form

$$-\rho_0\tau_0\vartheta_0^2c_V(\vartheta)\beta_{,tt} + 2k_0\tau_0\vartheta\nabla\beta\cdot\nabla\beta_{,t} + k_0\vartheta^2\Delta\beta - \rho_0\vartheta_0^2c_V(\vartheta)\beta_{,t} + 2k_0\vartheta\nabla\beta\cdot\nabla\beta = 0, \tag{2.12}$$

where by Eq. (2.11) $\vartheta = \vartheta(\beta, \beta_{,t}) = \tau_0\beta_{,t} + \beta$, and $c_V(\vartheta) = \vartheta\eta_{,\vartheta} > 0$ is the specific heat at constant volume.

By introducing new variables w and $\mathbf{p}$ through the equations

$$\beta_{,t} = w, \quad \mathbf{p} = \nabla\beta, \tag{2.13}$$

(1) Formally, the relationships (2.4)–(2.6) are compatible with an extra term in Eq. (2.5), $m\tau\dot{\vartheta}$, which is negligible.

(2) Notice that m has the dimension of the heat capacity.

Eq. (2.12) can be written as a quasi-linear system,

$$(2.14)\qquad \begin{aligned} -C(\beta,w)w_{,t} + \mathbf{b}(\beta,w,\mathbf{p})\cdot\nabla w + a(\beta,w)\nabla\cdot\mathbf{p} + H(\beta,w,\mathbf{p}) &= 0\,,\\ \beta_{,t} - w &= 0\,,\\ \mathbf{p}_{,t} - \nabla w &= 0\,. \end{aligned}$$

The coefficients in the above system (2.14) are

$$(2.15)\qquad \begin{aligned} C(\beta,w) &= \rho_0\tau_0\vartheta_0^2 c_V(\vartheta)\,,\\ \mathbf{b}(\beta,w,\mathbf{p}) &= 2k_0\tau_0\vartheta\mathbf{p}\,,\\ a(\beta,w) &= k_0\vartheta^2\,,\\ H(\beta,w,\mathbf{p}) &= 2k_0\vartheta\mathbf{p}\cdot\mathbf{p} - \rho_0\vartheta_0^2 c_V(\vartheta)w\,, \end{aligned}$$

where

$$(2.16)\qquad \vartheta = \vartheta(\beta,w) = \tau_0 w + \beta\,,$$

and we used the heat condictivity $k(\vartheta)$ given by Eq. (2.10).

Nonzero eigenvalues λ for the system (2.14) satisfy a quadratic equation,

$$C\lambda^2 + \lambda\mathbf{b}\cdot\mathbf{n} - a = 0\,.$$

This system is hyperbolic; for $k_0 > 0, \tau_0 > 0, c_V > 0$, all eigenvalues are real, and the corresponding set of five eigenvectors are linearly independent for all orientations of the wave normal $\mathbf{n}$. This can be shown by following the same calculation as in the paper by KOSIŃSKI, SAXTON [19], although here some of the coefficients in Eqs. (2.14) are not constants but functions of β, w and $\mathbf{p}$.

In the next section thermal waves governed by the system (2.14) will be discussed. The question of how the present model is related to the classical one is, however, still open. To answer this question, let us consider the consequence of a vanishing thermal relaxation time τ_0.

The first consequence, given by Eq. (2.16), is that

$$(2.17)\qquad \vartheta = \beta\,,$$

i.e. both scales coincide. Then, instead of Eq. (2.12), the equation in terms of ϑ can be written

$$(2.18)\qquad \rho_0\vartheta_0^2 c_V(\vartheta)\vartheta_{,t} - k_0\vartheta^2\Delta\vartheta - 2k_0\vartheta\nabla\vartheta\cdot\nabla\vartheta = 0\,,$$

which is the only equation remaining as a consequence of the balance energy equation

$$(2.19)\qquad \rho_0\vartheta_0^2 c_V(\vartheta)\vartheta_{,t} - \nabla\cdot(k_0\vartheta^2\nabla\vartheta) = 0\,.$$

Since the coefficient k_0 is constant and Eq. (2.9) holds, we can rewrite the last equation as

$$(2.20)\qquad \rho_0\vartheta_0^2(\varepsilon(\vartheta))_{,t} - \frac{k_0}{3}\Delta(\vartheta^3) = 0\,.$$

The cases of the free energy function ψ, and heat capacity c_V, discussed later (cf. Eqs. (3.14), (3.15)) lead to the following form of the internal energy function ε(3),

$$(2.21)\qquad \varepsilon = \varepsilon_0 + \varepsilon_1\left(\frac{\vartheta}{\vartheta_0}\right)^5, \quad \varepsilon_1 = \frac{1}{5}c_0\vartheta_0\,,$$

(3) Note that, by Eq. (2.9), $\varepsilon_{,\vartheta} = c_V$.

and consequently to the following form of (2.20)

$$\varepsilon_1 \rho_0 \vartheta_0^{-3} (\vartheta^5)_{,t} - \frac{k_0}{3} \Delta(\vartheta^3) = 0 .$$

3. One-dimensional thermal waves

In this section we will investigate the conditions which lead to the blow-up, in finite time, of the amplitude of thermal waves for one-dimensional rigid conductors. The system (2.14) reduces to 3×3, in terms of β, w and p:

$$\begin{aligned} -C(\beta, w) w_{,t} + b(\beta, w, p) w_{,x} + a(\beta, w) p_{,x} + H(\beta, w, p) &= 0 , \\ \beta_{,t} - w &= 0 , \\ p_{,t} - w_{,x} &= 0 , \end{aligned} \tag{3.1}$$

where coefficients are given by Eq. (2.15) with **p** now scalar-valued.

We will call a one-dimensional smooth curve an acceleration wave having speed of propagation s, if across the curve β, w, and p are continuous, but certain first derivatives are not. Namely,

$$[\![w_{,t}]\!] \neq 0, \qquad [\![p_{,t}]\!] \neq 0 \tag{3.2}$$

and since $\beta_{,t} = w$,

$$[\![\beta_{,t}]\!] = 0 . \tag{3.3}$$

As a result of Eqs. (3.2), (3.3), and the definition of the directional derivative along the curve:

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + s \frac{\partial}{\partial x} =: \frac{d}{dt} \tag{3.4}$$

there follows:

$$[\![w_{,x}]\!] \neq 0 , \qquad [\![p_{,x}]\!] \neq 0, \qquad [\![\beta_{,x}]\!] = 0 ,$$

also by Eq. $(3.1)_3$,

$$[\![p_{,x}]\!] = -\frac{1}{s} [\![p_{,t}]\!] = -\frac{1}{s} [\![w_{,x}]\!] = \frac{1}{s^2} [\![w_{,t}]\!] . \tag{3.5}$$

Evaluating the system (3.1) across the wave, and using Eqs. (3.4), (3.5), we obtain the quadratic equation for nonzero s:

$$C^+ s^2 + b^+ s - a^+ = 0 . \tag{3.6}$$

Here we defined $C^+ = C(\beta^+, w^+)$, $b^+ = b(\beta^+, w^+, p^+)$, $a^+ = a(\beta^+, w^+)$, and β^+ and w^+ are the values of β and w, respectively, at the wave.

If we assume additionally that the wave is propagating into a material which is in a state of rest, such that

$$\beta(x, t) = \beta^+ = \text{const} \neq 0 , \qquad p(x, t) = p^+ = 0, \qquad w(x, t) = w^+ = 0 \tag{3.7}$$

in the region in front of the wave, then

$$s^2 = \frac{a^+}{C^+} = \frac{a(\beta^+, 0)}{C(\beta^+, 0)} = \frac{k_0 (\beta^+)^2}{\rho_0 \tau_0 \vartheta_0^2 c_V(\beta^+)} , \tag{3.8}$$

because, by Eq. (2.16), $\vartheta^+ = \vartheta(\beta^+, 0) = \beta^+$.

We define the amplitude of the acceleration wave by

$$\alpha(t) = [\![w_{,t}]\!](t). \tag{3.9}$$

Note that by Eq. $(2.11)_2$, $\alpha = [\![w_{,t}]\!] = \dfrac{1}{\tau_0}[\![\vartheta_{,t}]\!]$.

The amplitude $\alpha(t)$ evolves in time along the characteristic accordingly to a first order ordinary differential equation. To derive this equation (cf. KOSIŃSKI, SAXTON [19]), we differentiate the system (3.1) with respect to t, evaluate it at the wave by taking the values of the corresponding continuous coefficients at $(\beta^+, 0, 0)$ (cf. Eqs. (3.7)), and use the following relation for the constant speed s given (Eq. (3.8)):

$$2\frac{d}{dt}[\![w_{,t}]\!] = [\![w_{,tt}]\!] - s^2[\![w_{,xx}]\!]. \tag{3.10}$$

The equation obtained for α is of Bernoulli type,

$$\frac{d}{dt}\alpha - n\tau_0\alpha^2 + \frac{1}{2\tau_0}\alpha = 0, \tag{3.11}$$

where

$$n = \frac{2}{\beta^+} - \frac{c^+_{V,\vartheta}}{2c^+_V}. \tag{3.12}$$

Since $(1/(2\tau_0)) > 0$, the solution $\alpha(t)$ of Eq. (3.11) has the property; $\alpha(t) \to \pm\infty$ as $t \to t_1$, t_1 finite time, if the initial condition $\alpha_0 = \alpha(0)$ satisfies the inequality,

$$2n\alpha_0\tau_0^2 - 1 > 0, \tag{3.13}$$

and the blow-up time t_1 can be calculated to be

$$t_1 = 2\tau_0 \log\frac{2n\alpha_0\tau_0^2}{2n\alpha_0\tau_0^2 - 1}.$$

It can be easily verified that t_1 is a decreasing function of τ_0 for fixed α_0.

In the model presented, the heat conductivity can result in a mixture of stabilizing (due to diffusion) and destabilizing effects, for $n \neq 0$, on the behavior of the amplitude α. This fact can lead to the formation of shocks. Although destabilization exists, it is not very strong, and depends on the order of the coefficient $n\tau_0$. If τ_0 is very small, then the stabilizing effect is dominating. Consequently, if $\tau_0 \to 0$, then $s^2 \to \infty$, $t_1 \to \infty$, and $\alpha(t) \to 0$ as $t \to \infty$, which is the case corresponding to the classical Fourier law (2.2).

For the special case of the specific heat c_V, i.e. where

$$c_V = c_0\left(\frac{\vartheta}{\vartheta_0}\right)^4 \tag{3.14}$$

then the amplitude decays along the wave, and blow-up does not occur,

$$\alpha(t) = \alpha_0 e^{-\frac{t}{2\tau_0}}.$$

The heat capacity represented by Eq. (3.14) is possible if the function $\psi^1(\vartheta)$ in the free energy function (2.8) together with (2.10), has a form

$$\psi^1(\vartheta) = e_0 + e_1\vartheta + e_2\vartheta^5, \tag{3.15}$$

where e_0, e_1, e_2 are constants, and $c_0 = -20e_2\vartheta_0^4$.

4. Acceleration waves in thermoelastic materials

We will consider constitutive equations for thermoelastic materials as represented by the following quantities: free energy ψ, Piola—Kirchhoff stress tensor $\mathbf{T}_\kappa$, entropy η, heat flux $\mathbf{q}$,

$$(4.1)\qquad \begin{aligned} \psi &= \hat{\psi}(\mathbf{F}, \vartheta, \nabla\beta) = \psi^1(\mathbf{F}, \vartheta) + \frac{\tau_0}{2}\frac{k(\vartheta)}{\rho_0\vartheta}\nabla\beta\cdot\nabla\beta\,, \\ \mathbf{T}_\kappa &= \hat{\mathbf{T}}_\kappa(\mathbf{F}, \vartheta) = \rho_0\hat{\psi}_{,\mathbf{F}}(\mathbf{F}, \vartheta, \nabla\beta) = \rho_0\psi^1_{,\mathbf{F}}(\mathbf{F}, \vartheta)\,, \\ \eta &= \hat{\eta}(\mathbf{F}, \vartheta, \nabla\beta) = -\hat{\psi}_{,\vartheta}(\mathbf{F}, \vartheta, \nabla\beta)\,, \\ \mathbf{q} &= -k(\vartheta)\nabla\beta\,, \end{aligned}$$

where $\mathbf{F}$ is the deformation gradient, and $k(\vartheta)$ is given by Eq. (2.10).

The system of equations contains: the law of motion, compatibility condition, energy balance law, and the evolution equation for β,

$$(4.2)\qquad \begin{gathered} \rho_0\mathbf{v}_{,t} - \operatorname{div}\mathbf{T}_\kappa = 0\,, \\ \mathbf{F}_{,t} - \operatorname{grad}\mathbf{v} = 0\,, \\ \rho_0\vartheta\hat{\eta}_{,\vartheta}\vartheta_{,t} + \rho_0\vartheta\hat{\eta}_{,\mathbf{F}}\cdot\mathbf{F}_{,t} + \operatorname{div}\mathbf{q} = 0\,, \\ \beta_{,t} - \frac{1}{\tau_0}(\vartheta - \beta) = 0\,, \end{gathered}$$

where $\mathbf{v}_{,t}$ is the particle acceleration.

Next, we write the system (4.2) in terms of new unknowns $\beta_{,t} = w$ and $\nabla\beta = \mathbf{p}$.

$$(4.3)\qquad \begin{gathered} v^l_{,t} - \rho_0^{-1}A^{lKM}_n F^n_{M,K} - \tau_0\rho_0^{-1}P^{lK}w_{,K} - G^l = 0\,, \\ F^l_{L,t} - v^l_{,L} = 0\,, \\ \beta_{,t} - w = 0\,, \\ w_{,t} - (\tau_0 c_V)^{-1}\vartheta P^K_l v^l_{,K} - (\rho_0 c_V)^{-1}k_{,\vartheta}p^K w_{,K} - (\rho_0\tau_0 c_V)^{-1}k\delta^L_K p^K_{,L} + H = 0\,, \\ p_{K,t} - w_{,K} = 0\,, \end{gathered}$$

where we introduced the notation:

$$(4.4)\qquad \begin{gathered} A^{lKM}_n = \frac{\partial T^{lK}_\kappa}{\partial F^n_M}\,, \qquad P^{lK} = \frac{\partial T^{lK}_\kappa}{\partial\vartheta}\,, \\ H(F^n_M, \beta, w, p_K) = \tau_0^{-1}w - (\rho_0\tau_0 c_V)^{-1}k_{,\vartheta}p_K p^K\,, \\ G^l(F^n_M, \beta, w, p_K) = \rho_0^{-1}P^{lK}p_K \end{gathered}$$

and $\vartheta = \tau_0 w + \beta$.

The system of differential equations (4.3) is quasilinear, of the form

$$(4.5)\qquad \mathbf{U}_{,t} + \Omega_1(\mathbf{U})\mathbf{U}_{,1} + \Omega_2(\mathbf{U})\mathbf{U}_{,2} + \Omega_3(\mathbf{U})\mathbf{U}_{,3} + \mathbf{B}(\mathbf{U}) = \mathbf{0}$$

where we write $\mathbf{U}_{,i} = \mathbf{U}_{,x_i}, i = 1, 2, 3,$

$$\mathbf{U} = \begin{bmatrix} \mathbf{v} \\ \mathbf{F} \\ \beta \\ w \\ \mathbf{p} \end{bmatrix},$$

and Ω_i are square 17×17 matrices.

Let Σ be a characteristic surface for Eqs. (4.3) (or Eqs. (4.5)), $\Sigma : f(\mathbf{x}, t) = 0$, λ the normal speed of propagation, and $\mathbf{n}$ the unit normal vector to Σ:

$$\lambda = -\frac{f_{,t}}{|\nabla f|}\,. \tag{4.6}$$

The speed λ is a solution of

$$\det|\Omega_1 n^1 + \Omega_2 n^2 + \Omega_3 n^3 - \lambda \mathbf{I}| = 0\,. \tag{4.7}$$

Across the surface Σ, $\mathbf{U}$ is continuous, but its first derivatives are discontinuous, i.e.

$$[\![\mathbf{U}]\!] = \mathbf{0}\,, \qquad [\![\mathbf{U}_{,t}]\!] \neq \mathbf{0}\,, \qquad [\![\mathbf{U}_{,i}]\!] \neq \mathbf{0}, \qquad i = 1,2,3\,. \tag{4.8}$$

We have the compatibility relation

$$[\![\mathbf{U}_{,t}]\!] = -\lambda n^K [\![\mathbf{U}_{,K}]\!]\,. \tag{4.9}$$

Equation (4.5) must hold on either side of Σ, thus after using Eq. (4.9), we get

$$(\Omega_K n^K - \lambda \mathbf{I})\phi = \mathbf{0}\,, \tag{4.10}$$

where $\phi = n^K [\![\mathbf{U}_{,K}]\!]$ is the amplitude of the jump in the field $\mathbf{U}$.

Equation (4.7) for the system (4.3) has nine vanishing λ, the remaining λ different from zero can be obtained whenever the following new amplitudes are introduced,

$$\omega^l = n^K n^L [\![F^l_{K,L}]\!] \tag{4.11}$$

and

$$\gamma = n^K [\![w_{,K}]\!]\,. \tag{4.12}$$

As a result, Eq. (4.10) corresponding to system (4.3) evaluated at Σ, reduces to

$$(\mathbf{Q} - \rho_0\lambda^2\mathbf{I})\boldsymbol{\omega} + \tau_0\mathbf{z}\gamma = \mathbf{0}, \quad \lambda^2\rho_0\vartheta\mathbf{z}\cdot\boldsymbol{\omega} + (k - \lambda^2\rho_0\tau_0 c_V - \lambda\tau_0 r)\gamma = 0\,. \tag{4.13}$$

Here $\mathbf{Q}$ is a symmetric acoustic tensor

$$Q^l_n = A^{lKM}_n n_K n_M\,, \tag{4.14}$$

and for simplicity we introduced the following notation

$$r = k_{,\vartheta}\mathbf{p}\cdot\mathbf{n}, \qquad \mathbf{z} = \mathbf{P}\mathbf{n}\,. \tag{4.15}$$

Amplitudes $\boldsymbol{\omega}$ and γ are not zero if the corresponding determinant of Eqs. (4.13) is zero. This condition gives an 8th order polynomial in terms of λ:

$$\begin{aligned}(\rho_0\lambda^2)^4\tau_0 c_V &- (\rho_0\lambda^2)^3\{\tau_0 c_V \operatorname{tr}\mathbf{Q} + \tau_0\vartheta\mathbf{z}\cdot\mathbf{z} + k)\} \\ &+(\rho_0\lambda^2)^2\{\tau_0\vartheta(\mathbf{z}\cdot\mathbf{z}\operatorname{tr}\mathbf{Q} - \mathbf{z}\cdot\mathbf{Q}\mathbf{z}) + k\operatorname{tr}\mathbf{Q} - \tau_0 c_V \Pi_Q\} \\ &-\rho_0\lambda^2\{\tau_0\vartheta(\mathbf{Q}\mathbf{z}\cdot\mathbf{Q}\mathbf{z} - \operatorname{tr}\mathbf{Q}(\mathbf{z}\otimes\mathbf{z})\cdot\mathbf{Q} - (\mathbf{z}\cdot\mathbf{z})\Pi_Q\} + k\det\mathbf{Q} \\ &-\lambda\tau_0 r\{-(\rho_0\lambda)^3 + (\rho_0\lambda)^2\operatorname{tr}\mathbf{Q} + \rho_0\lambda\Pi_Q + \det\mathbf{Q}\} = 0\,,\end{aligned} \tag{4.16}$$

where

$$\Pi_Q = \frac{1}{2}(\operatorname{tr}\mathbf{Q}^2 - (\operatorname{tr}\mathbf{Q})^2)\,.$$

The coefficients in Eqs. (4.16) are evaluated at Σ, and since $\mathbf{v}, \mathbf{F}, \beta, w$ and $\mathbf{p}$ are continuous across Σ, this implies that they are evaluated at $\mathbf{v} = \mathbf{v}^+$, $\mathbf{F} = \mathbf{F}^+$, $w = w^+$, $\mathbf{p} = \mathbf{p}^+$. If we assume (cf. rigid conductor in Sect. 3) that in front of the wave the state "+" is defined as

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{F}^+ = (\text{constant matrix})\,, \qquad \beta(\mathbf{x}, t) = \beta^+ = \text{const}\,, \tag{4.17}$$

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}^+ = \mathbf{0}, \quad \mathbf{p}(\mathbf{x}, t) = \mathbf{p}^+ = \mathbf{0}, \tag{4.17 cont.}$$

$$w(\mathbf{x}, t) = w^+ = 0$$

for points $(\mathbf{x}, t)$ in front of Σ, then this gives us a simplified version of Eq. (4.16), as by Eqs. (4.15)

$$r = r^+ = 0,$$

and Eq. (4.16) can be written in the shorter form,

$$d_4(\rho_0\lambda^2)^4 - d_3(\rho_0\lambda^2)^3 + d_2(\rho_0\lambda^2)^2 - d_1\rho_0\lambda^2 + d_0 = 0. \tag{4.18}$$

The coefficients d_0, d_1, d_2, d_3, d_4, which come from Eq. (4.16), are all positive under the assumption that the acoustic tensor $\mathbf{Q}$ is positive definite, and $c_V > 0$. Note that then $\Pi_Q < 0$. Although these are not sufficient conditions for the existence of real roots of Eq. (4.18), sufficient condition comes directly from single polynomial theory, as was shown by KOSIŃSKI, SZMIT [8].

In the last part of this section we concentrate on one-dimensional thermoelastic material, whose behavior is described by Eq. (4.1). There we have five unknowns (v, F, β, w, p), and the basic system of equations can be written in the following form,

$$\begin{gathered}
v_{,t} - T_F(F, \beta, w)F_{,x} - B(F, \beta, w)w_{,x} - G(F, \beta, w, p) = 0,\\
F_{,t} - v_{,x} = 0,\\
\beta_{,t} - w = 0,\\
-C(F, \beta, w)w_{,t} + D(F, \beta, w)v_{,x} + b(\beta, w, p)w_{,x} + a(\beta, w)p_{,x} + H(F, \beta, p) = 0,\\
p_{,t} - w_{,x} = 0,
\end{gathered} \tag{4.19}$$

where the following new notation, more convenient for the one-dimensional case, was introduced,

$$\begin{aligned}
T_F(F, \beta, w) &= T_F(F, \vartheta) = \psi^1_{,FF}(F, \vartheta),\\
B(F, \beta, w) &= \tau_0 T_\vartheta(F, \vartheta) = \tau_0\psi^1_{,F\vartheta}(F, \vartheta),\\
G(F, \beta, w, p) &= T_\vartheta(F, \vartheta)p,\\
C(F, \beta, w) &= \rho_0\tau_0\vartheta_0^2 c_V(F, \vartheta),\\
D(F, \beta, w) &= \rho_0\vartheta_0^2\vartheta T_\vartheta(F, \vartheta),\\
a(\beta, w) &= k_0\vartheta^2,\\
b(\beta, w, p) &= 2k_0\tau_0\vartheta p,\\
H(F, \beta, w, p) &= 2k_0\vartheta p^2 - \rho_0\vartheta_0^2 c_V(F, \vartheta)w,\\
\vartheta &= \vartheta(\beta, w) = \tau_0 w + \beta.
\end{aligned} \tag{4.20}$$

As in the rigid conductor, we investigate an acceleration wave as a curve across which all unknown functions are continuous, but

$$[\![v_{,t}]\!] \neq 0, \quad [\![F_{,t}]\!] \neq 0, \quad [\![p_{,t}]\!] \neq 0, \quad [\![\beta_{,t}]\!] = 0. \tag{4.22}$$

We have two amplitudes:

$$\alpha(t) = [\![w_{,t}]\!](t) \tag{4.22}$$

and

$$\delta(t) = [\![v_{,t}]\!](t). \tag{4.23}$$

For s being a speed of propagation of the acceleration wave, next follows,

$$(4.24)\quad [\![v_{,x}]\!] = [\![F_{,t}]\!] = -\frac{1}{s}\delta, \quad [\![F_{,x}]\!] = \frac{1}{s^2}\delta, \quad [\![w_{,x}]\!] = [\![p_{,t}]\!] = \frac{-1}{s}\alpha, \quad [\![p_{,x}]\!] = \frac{1}{s^2}\alpha.$$

Evaluating the system across the wave, we find

$$(4.25)\qquad (s^2 - T_F^+)\delta + B^+ s\alpha = 0, \quad -D^+ s\delta + (a^+ - b^+ s - C^+ s^2)\alpha = 0,$$

and the nonzero speeds satisfy the equation

$$(4.26)\qquad -C^+ s^4 - b^+ s^3 + (a^+ + T_F^+ C^+ + B^+ D^+)s^2 + T_F^+ b^+ s - T_F^+ a^+ = 0.$$

The symbol "+", as previously, means the value of a function at the wave where $w = w^+$, $F = F^+$, $\beta = \beta^+$, $p = p^+$, and if the wave is propagating into an undisturbed material such that Eq. (4.17) holds in the case of one dimension, then the fastest wave $s = s_1$ will propagate into such a state. This means that s_1 satisfies Eq. (4.26) with $b^+ = 0$,

$$(4.27)\qquad s_1^2 = \frac{1}{2}\left[\left\{\frac{a^+}{C^+} + T_F^+ + \frac{B^+D^+}{C^+}\right\} + \sqrt{\left(\frac{a^+}{C^+} + T_F^+ + \frac{B^+D^+}{C^+}\right)^2 - 4\frac{a^+T_F^+}{C^+}}\right].$$

We are making the assumption $T_F^+ > 0$, $c_V^+ > 0$. Equations (4.25) gives the relation between the amplitude α and δ at the wave which is propagating with speed s_1,

$$(4.28)\qquad \delta = -\frac{B^+ s_1}{s_1^2 - T_F^+}\alpha = \frac{a^+ - C^+ s_1^2}{D^+ s_1}\alpha.$$

The amplitude α or δ evolves along the wave according to a first order differential equation, which can be derived by differentiation of Eq. (4.19) with respect to t, using the relations (4.24), and

$$(4.29)\qquad \begin{aligned}
[\![v_{,tt}]\!] &= 2\frac{d}{dt}\delta + s_1^2[\![F_{,tx}]\!],\\
[\![w_{,tt}]\!] &= 2\frac{d}{dt}\alpha + s_1^2[\![p_{,tx}]\!],\\
[\![p_{,tt}]\!] &= -\frac{1}{s_1}\frac{d}{dt}\alpha - s_1[\![p_{,tx}]\!],\\
[\![F_{,tt}]\!] &= -\frac{1}{s_1}\frac{d}{dt}\delta - s_1[\![F_{,tx}]\!].
\end{aligned}$$

As a result we obtain coupled equations for δ and α.

$$(4.30)\qquad \begin{aligned}
\varphi_1 + (s_1^2 - T_F^+)[\![F_{,xt}]\!] + B^+ s_1[\![p_{,tx}]\!] &= 0,\\
\varphi_2 - s_1 D^+[\![F_{,xt}]\!] + (a^+ - C^+ s_1^2)[\![p_{,tx}]\!] &= 0,
\end{aligned}$$

where φ_1 and φ_2 are functions of α, δ and their derivatives. Since the coefficients in terms containing $[\![p_{,tx}]\!]$ and $[\![F_{,tx}]\!]$ satisfy Eqs. (4.25) (with $b^+ = 0$, and $s = s_1$), as a consequence we have a linear relation between φ_1 and φ_2:

$$(4.31)\qquad \varphi_1 = \zeta\varphi_1,$$

where

$$\zeta = -\frac{s_1^2 - T_F^+}{s_1 D^+} = \frac{B^+ s_1}{a^+ - c^+ s_1^2}.$$

Combining relations (4.28), (4.30) and (4.31), we obtain the desired equation for δ (or α):

$$\frac{d}{dt}\delta + M_1\delta^2 + M_2\delta = 0 \tag{4.32}$$

and

$$M_1 = \frac{1}{2(a^+T_F^+ - C^+s_1^4)}\left\{\frac{a^+ - C^+s_1^2}{s_1^2D^+}T_{F,F} + s_1^2B^+(2D^+_{,w} - \rho_0\vartheta_0^2B^+ - C^+_{,F})\right.$$
$$\left. - 3(a^+ - C^+s_1^2)B^+_{,F} + (s_1^2 - T_F^+)(4k_0\tau_0\vartheta^+ - s_1^2C^+_{,w})\right\}, \tag{4.33}$$
$$M_2 = \frac{a^+(s_1^2 - T_F^+)}{2\tau_0(C^+s_1^4 - a^+T_F^+)} > 0\,.$$

Equation (4.32) has the same property as Eq. (3.11) for the rigid conductor, that is, the solution blows up in finite time t_1, if the initial condition satisfies

$$M_1\delta(0) + M_2 < 0\,. \tag{4.34}$$

In contrast to the classical thermoelastic material of the Fourier law, there are two acceleration waves, propagating in the positive direction. For the case of $\tau_0 \to 0, s_1 \to \infty$, and the second wave $s_1 \to s_0$, where $s_0^2 = T_F^+$ is the speed of propagation of acceleration wave in a classical thermoelastic material.

For $s_2 = s_0$, the corresponding equation for the amplitude δ takes the form derived by DAFERMOS [21],

$$\frac{d}{dt}\delta + \frac{T^+_{F,F}}{2s_0^3}\delta^2 + \frac{\rho_0\vartheta^+(T^+_\vartheta)^2}{2k^+}\delta = 0\,.$$

5. Concluding remarks

In the last section, the constitutive model of an elastic heat conductor allowed discussion of coupled thermo-mechanical waves of weak discontinuity. These may grow without bound in finite time, provided their initial amplitude is sufficiently large, and possess appropriate sign. This observation is rather obvious for nonlinear acoustic waves, which can transform into strong discontinuity waves. The influence of the thermal properties, however, may have a stabilizing effect on the acceleration wave propagation. This will be discussed in more detail in future work. Here, one should notice that the appropriate form of the specific heat function (cf. Eq. (3.14)) can make the amplitude equation linear.

At the present state of the theory, the available experimental data can support a limited number of situations of physical interest, since the measurement of τ at higher temperatures is restrictive.

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