

On anisotropic invariants of vectors and second order tensors

J. M. ZHANG (SINGAPORE)

IN THIS PAPER a different approach is presented for determination of the integrity bases of finite vectors and second order tensors relative to all types of crystal symmetry except for those of cubic system. The Lokhin's general representation theorem for integrity basis involving tensors of arbitrary order is proven. Several important theorems are proven by making use of the algebraic properties of the higher order structural tensors. These theorems make it possible for us to limit our search for the integrity and functional bases to a small number of given homogeneous polynomial invariants.

W pracy przedstawiono odmienne podejście do problemu określenia baz całkowitych (integrity bases) wektorów skończonych i tensorów drugiego rzędu w odniesieniu do wszystkich typów symetrii kryształów z wyjątkiem układu kubicznego. Udowodniono twierdzenia Łochina o reprezentacji dla baz zawierających tensory dowolnego rzędu. Udowodniono szereg ważnych twierdzeń, wykorzystując własności algebraiczne tensorów strukturalnych wyższych rzędów. Twierdzenia te pozwalają ograniczyć poszukiwanie baz całkowitych i funkcyjnych do małej liczby jednorodnych niezmienników wielomianowych.

В работе представлен другой подход к задаче определения полных базисов конечных векторов и тензоров второго порядка по отношению ко всем типам симметрии кристаллов, за исключением кубической системы. Доказана теорема Лохина о представлении для базисов, содержащих тензоры произвольного порядка. Доказан ряд важных теорем, используя алгебраические свойства структурных тензоров высших порядков. Эти теоремы позволяют ограничить поиск полных и функциональных базисов к малому количеству однородных многочленных инвариантов.

Notations

$$\mathbf{a}, \mathbf{b}, \mathbf{k}, \mathbf{e}: a_i, b_i, k_i, e_i, \quad i = 1, 2, 3,$$

$$\mathbf{A}, \mathbf{B}, \mathbf{C} : A_{i\dots j}, B_{i\dots j}, C_{i\dots j},$$

$$\mathbf{Q} \quad Q_{ij}, Q_{ik} Q_{kj} = \delta_{ij},$$

$$(\mathbf{AB})_{i\dots jk\dots l} = A_{i\dots jn} B_{nk\dots l},$$

$$(\mathbf{A} \otimes \mathbf{B})_{i\dots jk\dots l} = A_{i\dots j} B_{k\dots l},$$

$$(\mathbf{A} \# \mathbf{B})_{i\dots j} = A_{ik\dots jl} B_{k\dots l},$$

$$(\mathbf{Q} * \mathbf{A})_{i\dots j} = Q_{ik\dots} Q_{jl} A_{k\dots l},$$

$$\mathbf{A} \cdot \mathbf{B} = A_{i\dots j} B_{i\dots j},$$

$$\otimes^m \mathbf{A} = \mathbf{A} \otimes \dots \otimes \mathbf{A},$$

$\mathbf{A}^m = \mathbf{AA} \dots \mathbf{A}$ defined only for second-order tensors,

$\mathbf{e}^m = \mathbf{e} \otimes \mathbf{e} \otimes \dots \otimes \mathbf{e}$ defined only for vectors,

$\sigma[\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_n] = \mathbf{v}_i \otimes \mathbf{v}_j \otimes \dots \otimes \mathbf{v}_k \otimes \mathbf{v}_l$, for $\sigma = [ij\dots k]$ being a permutation of integer number sequence $[12\dots n]$.

Intoroduction

THE SYMMETRIES of crystal lattice of a solid are characterized phenomenologically by symmetries of the constitutive functions describing various physical phenomena which may happen in the solid. The material symmetries impose definite restrictions on the forms of the constitutive functions of the material. Therefore representations of anisotropic functions, which specifies all possible forms of the functions meeting the restrictions, are indispensable in obtaining constitutive equations for anisotropic materials.

The known results concerning integrity bases of vectors and tensors under the transformation groups corresponding to the crystal classes are mainly due to Smith and Rivlin. The only cases which have been dealt with are the invariants of a single symmetric second-order tensor [1, 2], of a vector and a symmetric second-order tensor [3], and of an arbitrary number of vectors [4]. Irreducible integrity bases have been obtained for these cases by making use of the theorems in the theory of symmetric polynomials which give integrity bases for polynomials in two and three sets of variables which are invariant under symmetric groups of transformations of the variables, and for polynomials in three sets of variables which are invariant under cyclic transformations. The algebraic manipulations leading to the results are usually very complicated.

An completely different approach was presented by BOEHLER [5] and developed by LIU [6], RYCHLEWSKI [7] who tackled the representation problems of anisotropic functions by using the well-known results for isotropic ones. ZHANG and RYCHLEWSKI [8] have found the structural tensor sequences, for all crystal and non-crystal symmetries, through which the representations of all anisotropic tensorial functions can be reduced to representations of isotropic tensorial functions, at least from a theoretical viewpoint. Unfortunately, the structural tensor sequences for most types of crystal symmetry contain tensors of rank higher than two. At present, no practicable approach is available for determination of the irreducible integrity or functional basis involving such higher order tensors.

This paper aims at presenting an approach for determination of the integrity basis of finite vectors and second order tensors, relative to all types of crystal symmetry except for those of cubic system. Several theorems will be established which make it possible to limit our search for the integrity basis to a small number of given homogeneous polynomial invariants. In the following section, the fundamentals of isotropic extension of anisotropic function will be briefly reviewed. Several theorems which will be used in discussion of the equivalence of two different extended arguments will be presented. A general theorem for integrity bases involving tensors of arbitrary

order is proven in Sect. 3. In Sect. 4, several important theorems are proven by making use of the algebraic properties of the higher order structural tensors. These theorems make it possible to largely reduce the number of elements of the integrity basis specified by the general representation theorem. In Sect. 5 the integrity bases of a single symmetric second order tensor for each of the crystal symmetry groups except for those in cubic system are obtained according to our theorems in Sect. 4.

2. Structural tensor sequence and isotropic extension

The constitutive equation describing a physical phenomenon in a solid usually takes the form as follows:

$$(2.1) \quad \mathbf{S} = \hat{\mathbf{F}}(\mathbf{E}_1, \dots, \mathbf{E}_p, \mathbf{a}_1, \dots, \mathbf{a}_q)$$

whose symmetry group consists of all the orthogonal transformations \mathbf{Q} satisfying

$$(2.2) \quad \mathbf{Q} * \hat{\mathbf{F}}(\mathbf{F}_1, \dots, \mathbf{E}_p, \mathbf{a}_1, \dots, \mathbf{a}_q) = \hat{\mathbf{F}}(\mathbf{Q} * \mathbf{E}_1, \dots, \mathbf{Q} * \mathbf{E}_p, \mathbf{Q} * \mathbf{a}_1, \dots, \mathbf{Q} * \mathbf{a}_q),$$

where $\mathbf{E}_1, \dots, \mathbf{E}_p$ are the second order tensors, symmetric or skew-symmetric, and $\mathbf{a}_1, \dots, \mathbf{a}_q$ are vectors. All the tensors and vectors are defined in the three-dimensional Euclidean space. $\hat{\mathbf{F}}$ may be a scalar- or tensor-valued function. Here, without loss of generality, it will be assumed that $\hat{\mathbf{F}}$ is scalar-valued.

If the symmetry group is the full orthogonal transformation group $\mathcal{O}(3)$ we say that $\hat{\mathbf{F}}$ is isotropic or the material, relative to the physical phenomenon, is isotropic. Otherwise the material and the constitutive function are said to be anisotropic.

To represent a function $\hat{\mathbf{F}}$ with a definite symmetry group is to find all the possible forms of the function which meet the requirement (2.2). Representations of isotropic functions with arguments consisting of tensors of order not higher than two have been extensively investigated [9, 10]. On the contrary, results for anisotropic constitutive function are available only for a few special symmetry groups and/or for very simple arguments.

In fact, material symmetry is a mathematical property of the phenomenological constitutive equation of a material with microscopic inhomogeneity and directionality at a given reference state. Then how can such mathematical property be represented by quantities which, at a macroscopic level, describe the microscopic inhomogeneity and directionality? Let us look at the following theorem.

ISOTROPIC EXTENSION THEOREM [7]: Let $\hat{F}(E_1, \dots, E_p, \mathbf{a}_1, \dots, \mathbf{a}_q)$ be any anisotropic function with symmetry group \mathcal{G} , and let $\mathbf{J} \equiv \{S_1, \dots, S_r\}$ be a constant tensor-valued parameter sequence, called structural tensor sequence, such that

$$\mathcal{G} = \{Q \in \mathcal{O}(3) \mid Q^* S_1 = S_1, \dots, Q^* S_r = Q^* S_r\};$$

then there exists at least one isotropic function \tilde{F} of $E_1, \dots, E_p, \mathbf{a}_1, \dots, \mathbf{a}_q$ and S_1, \dots, S_r , such that

$$(2.4) \quad \hat{F}(E_1, \dots, E_p, \mathbf{a}_1, \dots, \mathbf{a}_q) \equiv \tilde{F}(E_1, \dots, E_p, \mathbf{a}_1, \dots, \mathbf{a}_q; S_1, \dots, S_r)$$

for all $E_1, \dots, E_p, \mathbf{a}_1, \dots, \mathbf{a}_q$ in the domain of the constitutive function \hat{F} .

The special form of this theorem in the cases of transverse isotropy and orthotropy was obtained and applied to represent the constitutive equations with these kinds of symmetry by BOEHLER [5] before one decade. LIU [6] summed up this result into a theorem and found the tensor parameters \mathbf{J} for crystals of triclinic system, monoclinic system and orthorhombic system. Recently, ZHANG and RYCHLEWSKI [8] have found the structural tensors sequences \mathbf{J} for all orthogonal subgroups. Therefore the extension theorem implies that every anisotropic function with an orthogonal subgroup as its symmetry group can be obtained by fixing some variables as constant parameters in some isotropic function and, further, that the representation problem of any anisotropic function can be reduced to a representation problem of an isotropic function.

The readers who are interested in the proof of the extension theorem are referred to RYCHLEWSKI [7] or ZHANG and RYCHLEWSKI [8]. Here, let us only present, in Table 1, some results concerning the structural tensors of all thirty-two crystal classes; the data will be used later on in the paper.

In Table 1 we have made use of the following notations:

$$(2.5) \quad \begin{aligned} \mathbf{N}_1 &= \mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2, \quad \mathbf{N}_2 = \mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1, \\ \mathbf{N}_3 &= \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1, \end{aligned}$$

$$(2.6) \quad \mathbf{T}_{2,2} = 2\mathbf{e}_1 \otimes \mathbf{e}_1,$$

$$(2.7) \quad \mathbf{T}_{3,3} = 3/4(\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1),$$

$$(2.8) \quad \mathbf{T}_{4,4} = 2(\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2),$$

$$(2.9) \quad \begin{aligned} \mathbf{T}_{6,6} &= 33/16(\mathbf{e}_1^6 - \mathbf{e}_1^4 \otimes \mathbf{e}_2^2 - \mathbf{e}_1^3 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_1^3 \otimes \mathbf{e}_2^2 \otimes \mathbf{e}_1 \\ &\quad - \mathbf{e}_1 \otimes \mathbf{e}_2^2 \otimes \mathbf{e}_1^3 + \mathbf{e}_1 \otimes \mathbf{e}_2^2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2^2 + \mathbf{e}_1 \otimes \mathbf{e}_2^3 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes \mathbf{e}_2^4 \otimes \mathbf{e}_1 \\ &\quad - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1^3 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2^2 \\ &\quad + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2^2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2^3 \otimes \mathbf{e}_1 \\ &\quad - \mathbf{e}_2^2 \otimes \mathbf{e}_1^4 + \mathbf{e}_2^2 \otimes \mathbf{e}_1^2 \otimes \mathbf{e}_2^2 + \mathbf{e}_2^2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2^2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2^2 \otimes \mathbf{e}_1), \end{aligned}$$

$$(2.10) \quad \begin{aligned} \mathbf{Z} &= \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_3 \otimes \mathbf{e}_1 - \mathbf{e}_3 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \\ &\quad + \mathbf{e}_3 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_3 \otimes \mathbf{e}_2, \end{aligned}$$

Table 1. Structural tensors for 32 point groups.

CRYSTAL SYSTEM	NO	Schonflies notations	STRUCTURAL SETS
Triclinic	1	C_1	e_1, e_2, e_3
	2	C_i	N_1, N_2, N_3
Monoclinic	3	C_s	$e_1, e_2, e_3 \otimes e_3$
	4	C_2	$T_{2,2}, e_3, Z$
	5	C_{2h}	$T_{2,2}, N_3, e_3 \otimes e_3$
Orthorhombic	6	C_{2v}	$T_{2,2}, e_3$
	7	D_2	$T_{2,2}, Z, e_3 \otimes e_3$
	8	D_{2h}	$T_{2,2}, e_3 \otimes e_3$
Tetragonal	9	C_4	$T_{4,4}, e_3, Z$
	10	S_4	S_4, N_3
	11	C_{4h}	$T_{4,4}, N_3$
	12	C_{4v}	$T_{4,4}, e_3$
	13	D_{2d}	$T_d, e_3 \otimes e_3$
	14	D_4	$T_{4,4}, Z$
	15	D_{4h}	$T_{4,4}$
Rhombohedral	16	C_3	$T_{3,3}, e_3, Z$
	17	C_{3i}	$T_{3,3} \otimes e_3, N_3$
	18	C_{3v}	$T_{3,3}, e_3$
	19	D_3	$T_{3,3}, Z$
	20	D_{3d}	$T_{3,3} \otimes e_3$
Hexagonal	21	C_{3h}	$T_{3,3}, N_3$
	22	C_6	$T_{6,6}, e_3, Z$
	23	C_{6h}	$T_{6,6}, N_3$
	24	D_{3h}	$T_{3,3}$
	25	C_{6v}	$T_{6,6}, e_3$
	26	D_6	$T_{6,6}, Z$
	27	D_{6h}	$T_{6,6}$
Cubic	28	T	T_h, Z
	29	T_h	T_h
	30	T_d	T_d
	31	O	O_h, Z
	32	O_h	O_h

$$(2.11) \quad S_4 = e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_1 \otimes e_3,$$

$$(2.12) \quad O_h = e_1^4 + e_2^4 + e_3^4,$$

$$(2.13) \quad T_h = e_1^2 \otimes e_2^2 + e_2^2 \otimes e_3^2 + e_3^2 \otimes e_1^2,$$

$$(2.14) \quad T_d = e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_1 \otimes e_3 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_2 \otimes e_1 \\ + e_3 \otimes e_1 \otimes e_2 + e_1 \otimes e_3 \otimes e_2.$$

The structural tensor $T_{n,n}$ can also be expressed generally as

$$(2.15) \quad T_{n,n} = n(k_1^n + k_2^n) \quad \text{if } n \text{ is odd,}$$

$$(2.16) \quad T_{n,n} = n\left(k_1^n + k_2^n + \frac{1}{2n} T_{2n,n}\right) \quad \text{if } n \text{ is even,}$$

where

$$(2.17) \quad k_1 = \frac{1}{2}(e_1 - i e_2),$$

$$(2.18) \quad k_2 = \frac{1}{2}(e_1 + i e_2),$$

$$(2.19) \quad T_{2n,n} = 2n [k_1^{n/2} \otimes k_1^{n/2}],$$

$[k_1^{n/2} \otimes k_2^{n/2}]$ denotes the sum of all the possible permutations of the tensor $k_1 \otimes \dots \otimes k_1 \otimes k_2 \otimes \dots \otimes k_2$ and i is the unit imaginary number.

We call the argument $(E_1, \dots, E_p, a_1, \dots, a_q; S_1, \dots, S_r)$ the extended argument of the tensors $(E_1, \dots, E_p, a_1, \dots, a_q)$, relative to group \mathcal{G} . Two extended arguments are said to be equivalent if they have same isotropic integrity basis.

THEOREM 1. *Two tensor sequences (S_1, \dots, S_p, M_1) and (S_1, \dots, S_p, M_2) are structural tensor sequences for the same group \mathcal{G} if M_1 can be represented as an isotropic function of (S_1, \dots, S_p, M_2) and M_2 can be represented as an isotropic function of (S_1, \dots, S_p, M_1) .*

The proof of this theorem is obvious.

The structural tensors for some crystal groups contain the third-order alternating tensor Z , the entry of which into the argument makes it a little more complicated to find integrity basis or functional basis for the extended argument. The following two theorems show that, in fact, we can avoid this difficulty in most cases.

THEOREM 2 [11]. *When only tensors of even order occur as arguments and values, there is no distinction between isotropic and hemitropic functions.*

The proof of this theorem is obvious because the transformation $Q * A$ of a tensor A of even order remains unchanged when an improper orthogonal tensor Q is replaced by the proper one $-Q$.

Let Y be a tensor of odd order, define

$$(2.20) \quad \langle Y \rangle \equiv YZ = Y_{i\dots jk} G_{kmn} e_i \otimes \dots \otimes e_j \otimes e_m \otimes e_n,$$

where Z is defined by Eq. (2.10). It is easy to check that for *proper* orthogonal tensor Q ,

$$(2.21) \quad Q * \langle Y \rangle = (Q * Y) = (Q * Y) Z = \langle Q * Y \rangle.$$

THEOREM 3. For a tensorial function $F(Y, A)$ of even order where A is a tensor sequence of even order, define

$$(2.22) \quad \hat{F}(\langle Y \rangle, A) = F(Y, A).$$

Then $F(Y, A)$ is a hemitropic function if and only if $\hat{F}(\langle Y \rangle, A)$ is an isotropic one.

Proof. If $F(Y, A)$ is hemitropic, then for any orthogonal transformation Q

$$(2.23) \quad Q * \hat{F}(\langle Y \rangle, A) = Q * F(Y, A) = \|Q\| * F(Y, A) = F(\|Q\| * Y, \|Q\| * A) \\ = \hat{F}(\langle \|Q\| * Y \rangle, \|Q\| * A) = \hat{F}(\|Q\| * \langle Y \rangle, \|Q\| * A) = F(Q * \langle Y \rangle, Q * A),$$

that is, $\hat{F}(\langle Y \rangle, A)$ is isotropic, where

$$(2.24) \quad \|Q\| \equiv \begin{cases} Q & \text{if } Q \text{ is proper,} \\ -Q & \text{if } Q \text{ is improper.} \end{cases}$$

The inverse is obviously by definition. *QED*

When the argument of the function \hat{F} contains more than one tensor of odd order, say Y_1, \dots, Y_s , we define

$$(2.25) \quad Y \equiv (Y_1, \dots, Y_s), \quad \langle Y \rangle \equiv (\langle Y_1 \rangle, \dots, \langle Y_s \rangle).$$

The Theorem 3 still holds.

3. Representations theorem for isotropic functions

Material symmetry imposes certain restrictions on the forms of the response functions or functionals. If a response function or functional satisfies the material symmetry requirements, then it must be representable in certain canonical form. Our theorem for isotropic extension of anisotropic functions stated in the previous section shows that the representations of the isotropic functions, depending on an arbitrary number of tensors of arbitrary ranks, are fundamental for descriptions of the anisotropic behavior of a wide range of engineering materials. In this section, we will prove the following *main representation theorem*, which was first presented by LOKHIN [12] without proof.

Let $\mathcal{L}(\mathcal{V}, n_i)$ be the space of n_i -th order tensors on the 3-dimensional space \mathcal{V} . Denote the product space $\mathcal{L}(\mathcal{V}, n_1) \times \mathcal{L}(\mathcal{V}, n_2) \times \dots \times \mathcal{L}(\mathcal{V}, n_p)$ by $\mathcal{L}(\mathcal{V}, n_1, n_2, \dots, n_p)$.

THEOREM 4. *A function $\mathcal{F}: \mathcal{L}(\mathcal{V}; n_1, n_2, \dots, n_p) \rightarrow \mathbf{R}$ is an isotropic function if and only if it can be expressed as a function of finite invariants*

$$(3.1) \quad \{I_1, I_2, \dots, I_k\},$$

where I_i are homogeneous polynomial invariants formed only through the operations of tensor multiplication, permutation and contraction of the argument $\mathbf{A}_1, \dots, \mathbf{A}_p$ and the second order unit tensor \mathbf{I} .

Before we begin to prove this theorem, we should restate a number of well-known and essential general results concerning representations of scalar-valued functions.

HILBERT'S THEOREM. *For any finite system of vectors and tensors there exists an integrity basis which consists of a finite number of invariants.*

The proof of this theorem is rather lengthy and will not be given here. It is given, for example, in [13] and [14]. This theorem is of great importance in that it asserts the existence of finite integrity basis and so justifies the search for such basis. Therefore, to represent a scalar-valued function, we need only to find an irreducible integrity basis.

Another important and well-established result to be employed in the proof of Theorem 4 is about the general forms of isotropic constant tensors.

LEMMA 1 [10]. *Any even order isotropic tensor has the expression*

$$(3.2) \quad \mathbf{I}^{(2p)} = \sum_{\sigma} a_{\sigma} \sigma [\mathbf{I} \otimes \mathbf{I} \otimes \dots \otimes \mathbf{I}],$$

where $\sigma = [\alpha\beta\dots\gamma]$ are arbitrary permutations of $[12\dots 2p]$, a_{σ} constant parameters, the summation is taken over all possible permutations.

Isotropic tensor of odd order must be the zero tensor of the order.

PROOF of Theorem 4. Now we begin to prove our main theorem which can be completed by showing that any polynomial invariant can be expressed as a sum of finite homogeneous polynomial invariants, and that the homogeneous invariants can be obtained through the operations of tensor multiplication, permutation and contraction.

Let $I(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p)$ be a polynomial invariant and t_1, t_2, \dots, t_p be p arbitrary real numbers. Then $I(t_1 \mathbf{A}_1, t_2 \mathbf{A}_2, \dots, t_p \mathbf{A}_p)$ can be rewritten as a sum of the homogeneous terms as follows:

$$(3.3) \quad I(t_1 \mathbf{A}_1, t_2 \mathbf{A}_2, \dots, t_p \mathbf{A}_p) = I_0 + \Sigma t_1^{\alpha_1} t_2^{\alpha_2} \dots t_p^{\alpha_p} I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p),$$

where $I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \dots, \mathbf{A}_p)$ is the sum of all the terms which are homogeneous of order α_1 in the components of tensor \mathbf{A}_1 , of order α_2 in the components of tensor \mathbf{A}_2 and so on: I_0 is a real constant.

After an orthogonal symmetry transformation of the argument involved, we have

$$(3.4) \quad \begin{aligned} I(t_1 \mathbf{Q} * \mathbf{A}_1, t_2 \mathbf{Q} * \mathbf{A}_2, \dots, t_p \mathbf{Q} * \mathbf{A}_p) \\ = I_0 + \Sigma t_1^{\alpha_1} t_2^{\alpha_2} \dots t_p^{\alpha_p} I_{\alpha_1 \dots \alpha_p}(\mathbf{Q} * \mathbf{A}_1, \mathbf{Q} * \mathbf{A}_2, \dots, \mathbf{Q} * \mathbf{A}_p) \\ = I_0 + \Sigma t_1^{\alpha_1} t_2^{\alpha_2} \dots t_p^{\alpha_p} I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p). \end{aligned}$$

Because the second equality holds for arbitrary real number (t_1, t_2, \dots, t_p) , it follows that

$$(3.5) \quad I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p) = I_{\alpha_1 \dots \alpha_p}(\mathbf{Q} * \mathbf{A}_1, \mathbf{Q} * \mathbf{A}_2, \dots, \mathbf{Q} * \mathbf{A}_p).$$

That is to say, $I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p)$ are invariant under any symmetry transformations.

Now, the homogeneous invariants $I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p)$ can be rewritten as

$$(3.6) \quad \begin{aligned} I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p) &= ((\otimes^{\alpha_1} \mathbf{A}_1) \otimes \dots \otimes (\otimes^{\alpha_p} \mathbf{A}_p)) \cdot \mathbf{H} \\ &= ((\otimes^{\alpha_1} (\mathbf{Q} * \mathbf{A}_1)) \otimes \dots \otimes (\otimes^{\alpha_p} (\mathbf{Q} * \mathbf{A}_p))) \cdot (\mathbf{Q} * \mathbf{H}), \end{aligned}$$

where \mathbf{Q} is an arbitrary orthogonal transformation and \mathbf{H} is a real constant tensor of order $(\alpha_1 n_1 + \alpha_2 n_2 + \dots + \alpha_p n_p)$. The constant tensor \mathbf{H} may not be unique, because tensor $(\otimes^{\alpha_1} \mathbf{A}_1) \otimes \dots \otimes (\otimes^{\alpha_p} \mathbf{A}_p)$ must possess some kind of subscript symmetries when tensors $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p$ themselves possess index symmetries or when some integers among $\alpha_1, \alpha_2, \dots, \alpha_p$ are larger than 1. But the uniqueness can be ensured by requiring that \mathbf{H} possesses the same subscript symmetry properties as $(\otimes^{\alpha_1} \mathbf{A}_1) \otimes \dots \otimes (\otimes^{\alpha_p} \mathbf{A}_p)$.

If \mathbf{Q}^T is a symmetry transformation, then

$$(3.8) \quad \begin{aligned} I_{\alpha_1 \dots \alpha_p}(\mathbf{Q}^T * \mathbf{A}_1, \mathbf{Q}^T * \mathbf{A}_2, \dots, \mathbf{Q}^T * \mathbf{A}_p) &= ((\otimes^{\alpha_1} \mathbf{A}_1) \otimes \dots \otimes (\otimes^{\alpha_p} \mathbf{A}_p)) \cdot (\mathbf{Q} * \mathbf{H}) \\ &= I_{\alpha_1 \dots \alpha_p}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_p) = ((\otimes^{\alpha_1} \mathbf{A}_1) \otimes \dots \otimes (\otimes^{\alpha_p} \mathbf{A}_p)) \cdot \mathbf{H}, \end{aligned}$$

which in fact means that

$$(3.8) \quad ((\otimes^{\alpha_1} \mathbf{A}_1) \otimes \dots \otimes (\otimes^{\alpha_p} \mathbf{A}_p)) \cdot (\mathbf{Q} * \mathbf{H} - \mathbf{H}) = 0.$$

The tensor in the second bracket must be zero because the above equality is valid for arbitrary (A_1, A_2, \dots, A_p) . This leads to the conclusion that H is a constant tensor with the same symmetry group as the homogeneous polynomial invariant considered. For isotropic invariant, H should be an isotropic tensor of order $(\alpha_1 n_1 + \alpha_2 n_2 + \alpha_p n_p)$. Taking into account Lemma 1, we have proven the main theorem. *QED*

Theorem 4 can be easily generalized to tensor-valued functions:

THEOREM 5 [12]. *A tensor-valued function $\hat{F}: \mathcal{L}(\mathcal{V}; n_1, n_2, \dots, n_p) \rightarrow \mathcal{L}(\mathcal{V}, n_q)$ is isotropic if and only if there exist a finite integrity basis $\{I_1, \dots, I_r\}$ and finite tensors (H_1, \dots, H_s) of order n_q such that*

$$(3.9) \quad \hat{F}(A_1, \dots, A_p) = \sum_{i=1}^s f_i(I_1, \dots, I_r) H_i,$$

where H_i are also formed through the operations of tensor multiplication, permutation and contraction and are usually called the generating set, f_i are arbitrary scalar-valued functions of the invariants I_1, \dots, I_r .

4. General results for integrity bases of vectors and second-order tensors

According to our results in the previous sections, it is possible to transform the representation problems of the constitutive functions of anisotropic solids into those of isotropic functions. However, up to now, almost all the results concerning representations of isotropic functions are confined to functions of vectors and second-order tensors [9, 10]. Relatively little is known about the representations of isotropic functions whose arguments contain tensors of rank higher than two (see [15., 16., 17]). Our representation Theorems 4 and 5 in the previous section have already established a theoretical framework for this purpose and furnished an approach to find the integrity bases and generators involving higher-order tensors. However, it still is an open problem how to seek finite and even irreducible integrity basis and generating set from the infinite homogeneous polynomial invariants.

This section is devoted to the representation problems of scalar-valued function of vectors and second-order tensors. In this case, the only tensor of order higher than 2 stems from the structural tensor sequence. In order to establish a general procedure of reducing as many elements as possible of the integrity basis specified in the general representation theorem, it seems essential to investigate the algebraic properties of the homogeneous polynomial invariants involving higher-order structural tensors. By this process, we hope to arrive at minimal bases; the procedure in itself, however, gives no assurance that the bases finally obtained are minimal, except in some of the simpler cases where the irreducibility of the basis elements is evident by inspection.

Let (E_1, E_2, \dots, E_p) consist of vectors and second-order tensors whose integrity bases relative to some orthogonal subgroup \mathcal{G} are to be investigated. Assume that $(S_1, \dots, S_q, T_{n,n})$ is a structural tensor sequence for group \mathcal{G} where $T_{n,n}$ is the n -th order tensor characterizing the n -fold rotation symmetry operation. S_1, \dots, S_q are all assumed to be of order not higher than 2. This is the case for most crystal symmetry groups. Therefore, the extended argument

$$(4.1) \quad (E_1, E_2, \dots, E_p; T_{n,n}, S_1, \dots, S_q)$$

contains only one tensor whose order n is probably higher than 2 (for crystals n may take values 2, 3, 4 or 6).

The following properties of tensor $T_{n,n}$ make it possible for us to represent (4.1) without knowing the general results concerning the integrity bases for tensors of order higher than 2:

$T_{n,n}$ is invariant under any permutation σ , i.e.,

$$(4.2) \quad \sigma[T_{n,n}] = T_{n,n}$$

Any $m \leq n$ times contraction of tensor $(T_{n,n} \otimes T_{n,n})$ can be written as dot multiplication $\sigma[T_{n,n} \otimes T_{n,n}] \cdot (\otimes^m \mathbf{I})$. Hence, the inner product of the contracted tensor of $(T_{n,n} \otimes T_{n,n})$ with a $2(n-m)$ -order tensor \mathbf{A} is equal to $\sigma[T_{n,n} \otimes T_{n,n}] \cdot ((\otimes^m \mathbf{I}) \otimes \mathbf{A})$, where σ is a permutation of integer sequence $[12 \dots (2n)]$.

THEOREM 6. *The integrity basis of argument (4.1) consists of the integrity basis of the argument $(E_1, E_2, \dots, E_p; S_1, \dots, S_q, \mathbf{I}_\alpha)$ and the invariants*

$$(4.3) \quad T_{n,n} \cdot G_1, \dots, T_{n,n} \cdot G_t,$$

and, further, if n is odd number

$$(4.4) \quad T_{2n,2n} \cdot M_1, \dots, T_{2n,2n} \cdot M_s,$$

where

$$(4.5) \quad \mathbf{I}_\alpha = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2,$$

G_1, \dots, G_t are the n -th order generators of the argument $(E_1, E_2, \dots, E_p; S_1, \dots, S_q, \mathbf{I}_\alpha)$, to within a permutation, and M_1, \dots, M_s are the $2n$ -th order generators of the argument $(E_1, E_2, \dots, E_p; S_1, \dots, S_q, \mathbf{I}_\alpha)$, also to within a permutation.

Proof. According to the main representation Theorem 4, the integrity basis of tensors (4.1) consists of the homogeneous polynomial invariants

formed only through the operations of tensor multiplication, permutation and contraction of the argument (4.1), and the second order unit tensor \mathbf{I} . The algebraic properties of $\mathbf{T}_{n,n}$ described above show, that the general form of the homogeneous polynomial invariants involving $\mathbf{T}_{n,n}$ is

$$(4.6) \quad (\otimes^m \mathbf{T}_{n,n}) \cdot \mathbf{H},$$

where \mathbf{H} is any tensor formed by tensor multiplications, permutations and contractions among the tensors $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q)$ and \mathbf{I} , up to the order of mn . First, we assume that the mn -th order tensor \mathbf{H} can be simply decomposed into

$$(4.7) \quad \mathbf{H} = I(\mathbf{E}_i; \mathbf{S}_i) (\mathbf{A}_1 \otimes \mathbf{A}_2 \dots \otimes \mathbf{A}_a) \otimes \dots \otimes (\mathbf{B}_1 \otimes \mathbf{B}_2 \otimes \dots \otimes \mathbf{B}_b),$$

where $I(\mathbf{E}_i; \mathbf{S}_i)$ is a homogeneous polynomial invariant concerning the tensors involved, $\mathbf{A}_i, \mathbf{B}_i$ are vectors or second-order tensors formed by tensor multiplications, permutations and contractions among the tensors $(\mathbf{E}_i; \mathbf{S}_i, \mathbf{I})$, and all the bracketed tensors $(\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_a), \dots, (\mathbf{B}_1 \otimes \mathbf{B}_2 \otimes \dots \otimes \mathbf{B}_b)$ are tensors of order n or of order $2n$ ⁽¹⁾. Therefore, the scalar invariant (4.6) can be rewritten as

$$(4.8) \quad I(\mathbf{E}_i; \mathbf{S}_i) \mathbf{T}_{n,n} \cdot (\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_a) \times \dots \times \mathbf{T}_{n,n} \cdot (\mathbf{B}_1 \otimes \mathbf{B}_2 \otimes \dots \otimes \mathbf{B}_b)$$

for n being even, and as

$$(4.9) \quad I(\mathbf{E}_i; \mathbf{S}_i) \mathbf{T}_{n,n} \cdot (\mathbf{A}_1 \otimes \dots \otimes \mathbf{A}_a) \times \dots \times \mathbf{T}_{n,n} \cdot (\mathbf{B}_1 \otimes \dots \otimes \mathbf{B}_b) \\ \times (\mathbf{T}_{n,n} \otimes \mathbf{T}_{n,n}) \cdot (\mathbf{C}_1 \otimes \dots \otimes \mathbf{C}_u) \times \dots \times (\mathbf{T}_{n,n} \otimes \mathbf{T}_{n,n}) \cdot (\mathbf{D}_1 \otimes \dots \otimes \mathbf{D}_v)$$

for n being odd.

All the invariants appearing in (4.8) are expressible through invariants (4.3), because the n -th order tensor $(\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_a)$ can be algebraically represented by the n -th order generators $(\mathbf{G}_{11}, \dots, \mathbf{G}_{1p}, \dots, \mathbf{G}_{t1}, \dots, \mathbf{G}_{tq})$:

$$(4.10) \quad \mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_a = \sum_i \alpha_i \mathbf{G}_{1i} + \dots \sum_j \beta_j \mathbf{G}_{tj},$$

and

$$(4.11) \quad (\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_a) \cdot \mathbf{T}_{n,n} = (\sum_i \alpha_i) \mathbf{G}_{11} \cdot \mathbf{T}_{n,n} + \dots + (\sum_j \beta_j) \mathbf{G}_{t1} \cdot \mathbf{T}_{n,n},$$

⁽¹⁾ This may happen when n is odd. In this case, if all of $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_a$ are of order two, the tensor product $\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \dots \otimes \mathbf{A}_a$ may not be decomposed into product of two tensors of order n .

where $G_{i_1, \dots}, G_{i_p}$ are generators which differ from one another only by a permutation.

Similarly, we can easily prove that for any odd number n , all the invariants appearing in (4.9) are expressible algebraically by the invariants in (4.3) and (4.4).

To complete the proof of our theorem, we are required only to prove that for any integer m , any permutation σ of $[12\dots(mn)]$ and any tensor H of form (4.7), the invariant $(\otimes T_{n,n}) \cdot \sigma(H)$ may be expressed algebraically by the invariants in (4.3) for even n and by the invariants in (4.3) and (4.4) for odd n .

We will prove this by using the method of mathematical induction. Assume that n is odd.

For $m = 2$, $T_{n,n} \otimes T_{n,n}$ can be expressed as

$$(4.12) \quad T_{n,n} \otimes T_{n,n} = k_1^{2n} + k_2^{2n} + k_1^n \otimes k_2^n + k_2^n \otimes k_1^n,$$

and hence for any $2n$ -th order tensor H_2 of the form (4.7), we have

$$(4.13) \quad (T_{n,n} \otimes T_{n,n}) \cdot \sigma(H_2) = (k_1^{2n} + k_2^{2n}) \cdot \sigma(H_2) + (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) \cdot \sigma(H_2) \\ = (T_{n,n} \otimes T_{n,n}) \cdot H_2 + (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) \cdot (\sigma(H_2) - H_2),$$

where we have made use of the fact that $(k_1^{2n} + k_2^{2n}) \cdot \sigma(H_2) = (k_1^{2n} + k_2^{2n}) \cdot H_2$, and the fact that the first equality also holds for identity permutation $\sigma = [12\dots(2n)]$. It has been previously shown that the tensor $(k_1^n \otimes k_2^n + k_2^n \otimes k_1^n)$ is a transversely isotropic tensor. So the last term in the above equation is an transversely isotropic invariant of the argument $(E_1, \dots, E_p; S_1, \dots, S_q; I_\alpha)$.

For $m = 3$, $T_{n,n} \otimes T_{n,n} \otimes T_{n,n}$ can be expressed as

$$(4.14) \quad 2T_{n,n} \otimes T_{n,n} \otimes T_{n,n} = T_{n,n} \otimes (T_{n,n} \otimes T_{n,n}) + (T_{n,n} \otimes T_{n,n}) \otimes T_{n,n} \\ = 2k_1^{3n} + 2k_2^{3n} + k_1^n \otimes (k_1^n + k_2^n) \otimes k_2^n + k_2^n \otimes (k_1^n + k_2^n) \otimes k_1^n \\ + (k_1^n + k_2^n) \otimes (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) + (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) \otimes (k_1^n + k_2^n).$$

Hence, for any $3n$ -th order tensor H_3 of the form (4.7), we have

$$(4.15) \quad (2T_{n,n} \otimes T_{n,n} \otimes T_{n,n}) \cdot \sigma(H_3) \\ = 2(k_1^{3n} + k_2^{3n}) \cdot \sigma(H_3) + (k_1^n \otimes (k_1^n + k_2^n) \otimes k_2^n) \cdot \sigma(H_3) + (k_2^n \otimes (k_1^n + k_2^n) \otimes k_1^n) \cdot \sigma(H_3) \\ + [(k_1^n + k_2^n) \otimes (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) + (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) \otimes (k_1^n + k_2^n)] \cdot \sigma(H_3) \\ = (2T_{n,n} \otimes T_{n,n} \otimes T_{n,n}) \cdot H_3 + (k_1^n \otimes T_{n,n} \otimes k_2^n) \cdot (\sigma(H_3) - H_3) + (k_2^n \otimes T_{n,n} \otimes k_1^n) \cdot (\sigma(H_3) - H_3) \\ + [T_{n,n} \otimes (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) + (k_1^n \otimes k_2^n + k_2^n \otimes k_1^n) \otimes T_{n,n}] \cdot (\sigma(H_3) - H_3).$$

The first term at the right-hand side of the last equation has already been proven to be expressible by invariants in (4.3) and (4.4) because H_3 meets the

decomposition property (4.7). And, further, as $\mathbf{k}_1^n \otimes \mathbf{k}_2^n$ is a transversely-isotropic complex tensor, the second term can be rewritten as

$$(4.16) \quad (\mathbf{T}_n, \otimes \mathbf{k}_1^n \otimes \mathbf{k}_2^n) \cdot \lambda [\sigma(\mathbf{H}_3) - \mathbf{H}_{-3}] = \sum_i I_i(\sigma, \mathbf{H}_3, \mathbf{I}_\alpha) \mathbf{T}_{n,n} \cdot L_i(\sigma, \mathbf{H}_3, \mathbf{I}_\alpha),$$

where $\lambda \equiv [(n+1)\dots(2n)12\dots n(2n+1)\dots(3n)]$ is a permutation of $[1\dots(3n)]$, $I_i(\sigma, \mathbf{H}_3, \mathbf{I}_\alpha)$ are invariants of the tensors $(\mathbf{E}_1, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q; \mathbf{I}_\alpha)$ and $L_i(\sigma, \mathbf{H}_3, \mathbf{I}_\alpha)$ are some n -th order tensor-valued generators of the argument. As a result, the second term, as well as all other terms in Eq. (4.15), can be algebraically expressed by the invariants in (4.3) and (4.4).

Assume that for any $m = (p-1)$ and any nm -th order tensor \mathbf{H}_m of form (4.7), the invariant $(\otimes^m \mathbf{T}_{n,n}) \cdot \sigma(\mathbf{H}_m)$ can be algebraically expressed by the invariants in (4.3) and (4.4). The expression for $\otimes^m \mathbf{T}_{n,n}$, $m \leq p-1$, is assumed to be in the form of

$$(4.17) \quad \otimes^m \mathbf{T}_{n,n} = \mathbf{k}_1^{mn} + \mathbf{k}_2^{mn} + \sum_i a_i \lambda_i [(\otimes^{(m-2)} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^n \otimes \mathbf{k}_2^n] \\ + \dots + \sum_i b_i \gamma_i [\mathbf{T}_{n,n} \otimes \mathbf{k}_1^{(m-1)n/2} \otimes \mathbf{k}_2^{(m-1)n/2}] \quad \text{if } m \text{ is odd,}$$

and

$$(4.18) \quad \otimes^m \mathbf{T}_{n,n} = \mathbf{k}_1^{mn} + \mathbf{k}_2^{mn} + \sum_i a_i \lambda_i [(\otimes^{m-2} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^n \otimes \mathbf{k}_2^n] \\ + \dots + \sum_i c_i \pi_i [\mathbf{k}_1^{mn/2} \otimes \mathbf{k}_2^{mn/2}] \quad \text{if } m \text{ is even,}$$

where a_i, b_i, c_i are constants, $\lambda_i, \gamma_i, \pi_i$ are some permutations of the integer sequence $[1\dots (nm)]$. Then the general expression for $\otimes^p \mathbf{T}_{n,n}$ is as follows:

$$(4.19) \quad \otimes^p \mathbf{T}_{n,n} = 1/2(\mathbf{T}_{n,n} \otimes (\otimes^{(p-1)} \mathbf{T}_{n,n}) + (\otimes^{(p-1)} \mathbf{T}_{n,n}) \otimes \mathbf{T}_{n,n}) \\ = \mathbf{k}_1^{pn} + \mathbf{k}_2^{pn} + \frac{1}{2} \{ \mathbf{k}_1^n \otimes (\mathbf{k}_1^{(p-2)n} + \mathbf{k}_2^{(p-2)n}) \otimes \mathbf{k}_2^n + \mathbf{k}_2^n \otimes (\mathbf{k}_1^{(p-2)n} + \mathbf{k}_2^{(p-2)n}) \otimes \mathbf{k}_1^n \} \\ + \frac{1}{2} \sum_i a_i \{ \mathbf{T}_{n,n} \otimes \lambda_i [(\otimes^{(p-3)} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^n \otimes \mathbf{k}_2^n] + \lambda_i [(\otimes^{(p-3)} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^n \otimes \mathbf{k}_2^n] \otimes \mathbf{T}_{n,n} \} \\ + \dots + \frac{1}{2} \sum_i b_i \{ \mathbf{T}_{n,n} \otimes \gamma_i [\mathbf{T}_{n,n} \otimes \mathbf{k}_1^{(p-2)n/2} \otimes \mathbf{k}_2^{(p-2)n/2}] \\ + \gamma_i [\mathbf{T}_{n,n} \otimes \mathbf{k}_1^{(p-2)n/2} \otimes \mathbf{k}_2^{(p-2)n/2}] \otimes \mathbf{T}_{n,n} \} \quad \text{if } p \text{ is even}$$

and

$$\begin{aligned}
 (4.20) \quad \otimes^p \mathbf{T}_{n,n} &= \frac{1}{2} (\mathbf{T}_{n,n} \otimes (\otimes^{(p-1)} \mathbf{T}_{n,n}) + (\otimes^{(p-1)} \mathbf{T}_{n,n}) \otimes \mathbf{T}_{n,n}) \\
 &= \mathbf{k}_1^{pn} + \mathbf{k}_2^{pn} + \frac{1}{2} \{ \mathbf{k}_1^n \otimes (\mathbf{k}_1^{(p-2)n} + \mathbf{k}_2^{(p-2)n}) \otimes \mathbf{k}_2^n + \mathbf{k}_2^n \otimes (\mathbf{k}_1^{(p-2)n} + \mathbf{k}_2^{(p-2)n}) \otimes \mathbf{k}_1^n \} \\
 &+ \frac{1}{2} \sum_i a_i \{ \mathbf{T}_{n,n} \otimes \lambda_i [(\otimes^{(p-3)} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^n \otimes \mathbf{k}_2^n] + \lambda_i [(\otimes^{(p-3)} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^n \otimes \mathbf{k}_2^n] \otimes \mathbf{T}_{n,n} \} \\
 &\quad + \dots + \frac{1}{2} \sum_i c_i \{ \mathbf{T}_{n,n} \otimes \pi_i [(\mathbf{k}_1^{(p-1)n/2} \otimes \mathbf{k}_2^{(p-1)n/2}) \\
 &\quad + \pi_i [(\mathbf{k}_1^{(p-1)n/2} \otimes \mathbf{k}_2^{(p-1)n/2}) \otimes \mathbf{T}_{n,n} \} \quad \text{if } p \text{ is odd.}
 \end{aligned}$$

Finding $(\mathbf{k}_1^{(p-2)n} + \mathbf{k}_2^{(p-2)n})$ from Eq. (4.17) in the case of p being odd or from Eq.(4.18) in the case of p being even and substituting the result into Eq. (4.20), respectively, we can easily show that (4.17) and (4.18) are also valid for $m = p$.

Now, let us analyze a typical term in the invariant $(\otimes^p \mathbf{T}_{n,n}) \cdot (\mathbf{H}_p - \sigma(\mathbf{H}_p))$ for any np -th order tensor \mathbf{H}_p of form (4.7) and any permutation σ of integer sequence $[12\dots(np)]$, i.e.,

$$\begin{aligned}
 (4.21) \quad \nu_i [(\otimes^{(p-2t)} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^{tn} \otimes \mathbf{k}_2^{tn}] \cdot (\mathbf{H}_p - \sigma(\mathbf{H}_p)) \\
 = [(\otimes^{(p-2t)} \mathbf{T}_{n,n}) \otimes \mathbf{k}_1^{tn} \otimes \mathbf{k}_2^{tn}] \cdot \nu_i^{-1} [\mathbf{H}_p - \sigma(\mathbf{H}_p)] \\
 = (\otimes^{(p-2t)} \mathbf{T}_{n,n}) \cdot \sum_a L_a(\mathbf{E}_i, \mathbf{S}_i, \mathbf{I}_a) \mu_a[\mathbf{X}_a],
 \end{aligned}$$

which we have assumed to be expressible algebraically by invariants in Eqs. (4.3) and (4.4). Here, $L_a(\mathbf{E}_i, \mathbf{S}_i, \mathbf{I}_a)$ are invariants of the tensors involved, μ_a permutations of the integer sequence $[12\dots((p-2t)n)]$, \mathbf{X}_a the $(p-2t)$ n -th order tensors of form (4.7).

According to the principle of mathematical induction, we have proven the theorem for the case of n being odd.

When n is even, define

$$(4.22) \quad \hat{\mathbf{T}}_{n,n} = \mathbf{T}_{n,n} - \frac{1}{2} \mathbf{T}_{2n,n} = \mathbf{k}_1^n + \mathbf{k}_2^n.$$

$\hat{\mathbf{T}}_{n,n}$ possesses the same symmetry properties as $\mathbf{T}_{n,n}$ and hence they determine the same group, since $\mathbf{T}_{2n,n}$ is a transversely isotropic tensor. When we replace $\mathbf{T}_{n,n}$ by $\hat{\mathbf{T}}_{n,n}$, the above deduction for odd n is still valid because $\hat{\mathbf{T}}_{n,n}$ satisfies the essential requirements (4.2). As a result, all the invariants of $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_p; \mathbf{T}_{n,n}, \mathbf{S}_1, \dots, \mathbf{S}_q)$ can be represented by the invariants

$$(4.23) \quad (\hat{\mathbf{T}}_{n,n} \cdot \mathbf{G}_1, \dots, \hat{\mathbf{T}}_{n,n} \cdot \mathbf{G}_t),$$

$$(4.24) \quad \hat{T}_{2n,2n} \cdot \mathbf{M}_1, \dots, \hat{T}_{2n,2n} \cdot \mathbf{M}_s$$

The fact that the invariants in (4.24) can again be expressed by those in (4.23), which are equivalent to those in (4.3), furnishes our proof. *QED*

The isotropic integrity bases involving vectors and second order tensors have been extensively investigated. So, according to Theorem 6 we need to focus our attention only on the small number of invariants in (4.3) and (4.4) in order to investigate the minimality of the integrity basis. Based on the previous results on representations of vector-valued and symmetric second order tensor-valued functions and on the subscript symmetry property (4.2), the following conclusions can be easily drawn;

For even n , all the tensors $\mathbf{A}_i, \mathbf{B}_i$ in Eq. (4.7) can be assumed to be symmetric second order tensors and take values among the symmetric second order generators of the argument $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{I}_\alpha, \mathbf{I})$.

For odd n , $\mathbf{A}_i, \mathbf{B}_i$ in the above theorem can be either vector or symmetric second-order tensor. It is enough for them to take values from among the vector- and symmetric second-order tensor-generators of the argument $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{I}_\alpha, \mathbf{I})$.

None of \mathbf{A}_i and \mathbf{B}_i is required to take value \mathbf{I}_α or \mathbf{I} , because the dot multiplications of $\mathbf{T}_{n,n}$ and $\mathbf{T}_{2n,2n}$ (for n being odd) with \mathbf{I}_α or \mathbf{I} are either zero or a transversely isotropic tensor.

Theorem 6 is also valid if we replace "integrity basis" by "functional basis".

The following theorem describes another property of the structural tensor $\mathbf{T}_{n,n}$ which can help us to reduce the redundant elements of the integrity basis in most cases of symmetry.

THEOREM 7. For any integer sequence (a, b, \dots, c) , we have

$$(4.25) \quad (\mathbf{N}_3^a \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \blacksquare \mathbf{T}_{n,n} = (\mathbf{N}_3^{a+b+\dots+c} \otimes (\otimes^{n-1} \mathbf{I})) \blacksquare \mathbf{T}_{n,n}, \quad n \text{ is odd,}$$

$$(4.26) \quad (\mathbf{N}_3^a \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \blacksquare \hat{\mathbf{T}}_{n,n} = (\mathbf{N}_3^{a+b+\dots+c} \otimes (\otimes^{n-1} \mathbf{I})) \blacksquare \hat{\mathbf{T}}_{n,n} \quad n \text{ is even,}$$

where \mathbf{N}_3 is given by Eq. (2.5).

Proof. Here, we prove only Eq. (4.25). The proof of equation (4.26) is similar. Substituting Eq. (2.15) into the left-hand side of Eq. (4.25) and taking into account the easily proven facts

$$(4.27) \quad \begin{aligned} \mathbf{N}_3 \mathbf{k}_1 &= -i \mathbf{k}_1, & \mathbf{N}_3^a \mathbf{k}_1 &= (-i)^a \mathbf{k}_1, \\ \mathbf{N}_3 \mathbf{k}_2 &= i \mathbf{k}_2, & \mathbf{N}_3^a \mathbf{k}_2 &= i^a \mathbf{k}_2, \end{aligned}$$

we get

$$(4.28) \quad \begin{aligned} & (\mathbf{N}_3^a \mathbf{k}_1 \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \blacksquare \mathbf{T}_{n,n} \\ &= n (\mathbf{N}_3^a \mathbf{k}_1 \otimes \mathbf{N}_3^b \mathbf{k}_1 \otimes \dots \otimes \mathbf{N}_3^c \mathbf{k}_1 + \mathbf{N}_3^a \mathbf{k}_2 \otimes \mathbf{N}_3^b \mathbf{k}_2 \otimes \dots \otimes \mathbf{N}_3^c \mathbf{k}_2) \\ &= n i^{a+b+\dots+c} ((-1)^{a+b+\dots+c} \mathbf{k}_1^n + \mathbf{k}_2^n). \end{aligned}$$

If $(a + b + \dots + c)$ is even, then the above quantity is equal to $\pm T_{n,n}$; and if $(a + b + \dots + c)$ is odd, then

$$(4.29) \quad (\mathbf{N}_3^a \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \blacksquare T_{n,n} = \pm ni(\mathbf{k}_2^a - \mathbf{k}_1^a)$$

is $2n$ times the imaginary part of the complex tensor \mathbf{k}_2^a . Therefore, the transformed tensor $(\mathbf{N}_3^a \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \blacksquare T_{n,n}$ is dependent on the power sequence (a, b, \dots, c) only through its sum $(a + b + \dots + c)$. *QED*

THEOREM 8. *Let $(S_1, \dots, S_q, N_3, T_{n,n})$ be a structural tensor suquence for some group \mathcal{G} . Let (E_1, \dots, E_p) be a sequence containing only vectors and the second-order symmetric or/and skew-symmetric tensors. If n is even, then the functional basis of $(E_1, \dots, E_p; S_1, \dots, S_q, N_3, T_{n,n})$ consists of the functional basis of tensors $(E_1, \dots, E_p; S_1, \dots, S_q, N_3)$ and the invariants*

$$(4.30) \quad \begin{aligned} & T_{n,n} \cdot X_1, \dots, T_{n,n} \cdot X_t, \\ & (\mathbf{N}_3 T_{n,n}) \cdot X_1, \dots, (\mathbf{N}_3 T_{n,n}) \cdot X_t, \end{aligned}$$

where X_1, \dots, X_t are the n -th order generators of the argument $(E_1, E_2, \dots, E_p; S_1, \dots, S_q, I_\alpha)$, to within a permutation.

Proof. According to Theorem 6 the functional basis of $(E_1, \dots, E_p; S_1, \dots, S_q, N_3, T_{n,n})$ consists of the functional basis of tensors $(E_1, \dots, E_p; S_1, \dots, S_q, N_3)$ and the invariants (4.3) where G_1, \dots, G_t are the n -th order generators of the argument $(E_1, E_2, \dots, E_p; S_1, \dots, S_q, N_3, I_\alpha)$, to within a permutation. As mentioned before, if n is an even number, have n -th order generators G_t , to within a permutation, have the general form

$$(4.31) \quad G_i = A_{i1} \otimes A_{i2} \otimes \dots \otimes A_{ik},$$

where A_{ij} are the second-order symmetric tensor-valued generators of the tensors $(E_1, \dots, E_p; S_1, \dots, S_q, N_3, I_\alpha)$ and $k = n/2$. If we check the generating set given by WANG [9] and SMITH [18], we can easily find that the last equation can be rewritten as follows:

$$(4.32) \quad G_i = (\mathbf{N}_3^a \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \blacksquare (\mathbf{B}_{i1} \otimes \mathbf{B}_{i2} \otimes \dots \otimes \mathbf{B}_{ik}),$$

where a, b, \dots, c are integers with values 0, 1 or 2 and \mathbf{B}_{ij} are the second-order symmetric tensor-valued generators of the tensors $(E_1, \dots, E_p; S_1, \dots, S_q, I_\alpha)$. As a result, we have

$$\begin{aligned}
 (4.33) \quad \mathbf{T}_{n,n} \cdot \mathbf{G}_i &= (\hat{\mathbf{T}}_{n,n} + \frac{1}{2} [\mathbf{k}_1^{n/2} \otimes \mathbf{k}_2^{n/2}]) \cdot ((\mathbf{N}_3^a \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \\
 &\quad \blacksquare (\mathbf{B}_{i_1} \otimes \mathbf{B}_{i_2} \otimes \dots \otimes \mathbf{B}_{i_k})) = (-1)^{a+b+\dots+c} ((\mathbf{N}_3^a \otimes \mathbf{N}_3^b \otimes \dots \otimes \mathbf{N}_3^c) \blacksquare \hat{\mathbf{T}}_{n,n}) \\
 &\quad \cdot (\mathbf{B}_{i_1} \otimes \mathbf{B}_{i_2} \otimes \dots \otimes \mathbf{B}_{i_k}) + \frac{1}{2} [\mathbf{k}_1^{n/2} \otimes \mathbf{k}_2^{n/2}] \cdot \mathbf{G}_i \\
 &= (-1)^{a+b+\dots+c} ((\mathbf{N}_3^{a+b+\dots+c} \otimes \mathbf{I}^{\otimes(n-1)}) \mathbf{I}) \blacksquare \hat{\mathbf{T}}_{n,n} \cdot (\mathbf{B}_{i_1} \otimes \mathbf{B}_{i_2} \otimes \dots \otimes \mathbf{B}_{i_k}) \\
 &\quad + \frac{1}{2} [\mathbf{k}_1^{n/2} \otimes \mathbf{k}_2^{n/2}] \cdot \mathbf{G}_i \\
 &= (\hat{\mathbf{T}}_{n,n} + \frac{1}{2} [\mathbf{k}_1^{n/2} \otimes \mathbf{k}_2^{n/2}]) (\mathbf{N}^{a+b+\dots+c} \mathbf{B}_{i_1} \otimes \mathbf{B}_{i_2} \otimes \dots \otimes \mathbf{B}_{i_k}) \\
 &\quad + \frac{1}{2} [\mathbf{k}_1^{n/2} \otimes \mathbf{k}_2^{n/2}] \cdot (\mathbf{G}_i - \mathbf{N}^{a+b+\dots+c} \mathbf{B}_{i_1} \otimes \mathbf{B}_{i_2} \otimes \dots \otimes \mathbf{B}_{i_k}).
 \end{aligned}$$

Since the only values the tensor $\mathbf{N}_3^{a+b+\dots+c}$ may take are $(\mathbf{I}_\alpha, -\mathbf{I}_\alpha, \mathbf{N}_3, -\mathbf{N}_3)$ and tensor $(\mathbf{B}_{i_1} \otimes \mathbf{B}_{i_2} \otimes \dots \otimes \mathbf{B}_{i_k})$ may be expressed by \mathbf{X}_i and their permutations, the first term in left-hand side of the last equality can be expressed by the functional basis of tensors $(\mathbf{E}_1, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{I}_\alpha)$ and invariants in (4.30). The second term is an invariant of tensors $(\mathbf{E}_1, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{I}_\alpha)$ and \mathbf{N}_3 . *QED*

The last theorem does not hold when n is odd. The main reason is that we usually have no decomposition formula similar to Eq. (4.32) in this case. Instead of Theorem 8, we have the following more restrictive theorem.

THEOREM 9. *Let $(\mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{N}_3, \mathbf{T}_{n,n})$ be a structural tensor sequence for some group \mathcal{G} and let $(\mathbf{E}_1, \dots, \mathbf{E}_p)$ be a sequence containing only second-order symmetric or/and anti-symmetric tensors. If all \mathbf{S}_i are of second-order, then the functional basis of $(\mathbf{E}_1, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{N}_3, \mathbf{T}_{n,n})$ consists of the functional basis of tensors $(\mathbf{E}_1, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{N}_3)$ and the invariants*

$$(4.34) \quad \begin{aligned}
 &\mathbf{T}_{2n,2n} \cdot \mathbf{Y}_1, \dots, \mathbf{T}_{2n,2n} \cdot \mathbf{Y}_s, \\
 &(\mathbf{N}_3 \mathbf{T}_{2n,2n}) \cdot \mathbf{Y}_1, \dots, (\mathbf{N}_3 \mathbf{T}_{2n,2n}) \cdot \mathbf{Y}_s,
 \end{aligned}$$

where $\mathbf{Y}_1, \dots, \mathbf{Y}_s$ are the $2n$ -th order generators of the argument $(\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_p; \mathbf{S}_1, \dots, \mathbf{S}_q, \mathbf{I}_\alpha)$, to within a permutation.

The proof of this theorem is completely similar to Theorem 8. The only fact we must take special care of is that the n -th order generator in this case is $\mathbf{0}$.

Theorems 8 and 9 may not hold when “functional basis” is replaced by “integrity basis”. Again, the reason is that the decomposition formula (4.32) does not hold for integrity basis in general.

5. Integrity bases for a single symmetric tensor

Now, to show how our approach can be used, we will briefly examine the integrity bases of a single symmetric tensor, relative all crystal symmetry groups except for those of cubic system. According to our results in Sect. 4, all we should do is to examine whether there exist redundant invariants among the small number of invariants of forms (4.3) and (4.4). We will not attempt to present here the details of the examinations for every crystal group. Instead of that, we will analyze several typical examples, though all results are given out below. Applications of our results in Sect. 4 to the case of two symmetric tensors will be reported in a separate paper.

5.1. Transversely isotropic

First, let us analyze the integrity bases for transverse isotropy which is classified into five subclasses [6]. It has been mentioned before that the integrity bases for most crystal symmetries contain transversely isotropic invariants.

For groups C_∞ and $C_{\infty h}$, the extended argument is $(\mathbf{E}, \mathbf{N}_3)$ whose integrity basis consists of six invariants as follows:

$$(5.1) \quad \text{trE} \quad \text{trE}^2 \quad \text{trE}^3 \quad \text{trEN}_3^2 \quad \text{trE}^2\mathbf{N}_3^2 \quad \text{trE}^2\mathbf{N}_3^2\mathbf{EN}_3.$$

For groups $C_{\infty v}$ and $D_\infty, D_{\infty h}$, the extended argument is $(\mathbf{E}, \mathbf{e}_3 \otimes \mathbf{e}_3)$ whose integrity basis consists of five invariants as follows:

$$(5.2) \quad \text{trE} \quad \text{trE}^2 \quad \text{trE}^3 \quad \mathbf{e}_3 \cdot \mathbf{Ee}_3 \quad \mathbf{e}_3 \cdot \mathbf{E}^2 \mathbf{e}_3.$$

5.2. Triclinic system

For group C_1 , the extended argument is $(\mathbf{E}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ whose integrity basis consists of six invariants as follows:

$$(5.3) \quad \mathbf{e}_i \cdot \mathbf{Ee}_j.$$

For group C_i , the structural tensor sequence consists of $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$ from which the tensors $\mathbf{e}_i \otimes \mathbf{e}_j$, $ij = 1,2,3$, can be obtained by dot multiplications, permutatons. Therefore the integrity basis for C_i is the same as for C , i.e., (5.3).

5.3. Monoclinic system

The extended arguments for the three groups in this system are

$$\begin{aligned} C_s &: (\mathbf{E}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \otimes \mathbf{e}_3) = (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{N}_3), \\ C_2 &: (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{e}_3, \mathbf{Z}) = (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{N}_3), \\ C_{2h} &: (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{N}_3). \end{aligned}$$

According to Theorems 1, 2 and 8, the integrity bases for all the three groups are the same and consist of seven invariants.

$$(5.4) \quad \text{trE}, \text{trE}^2, \mathbf{e}_3 \cdot \mathbf{Ee}_3, \mathbf{e}_3 \cdot \mathbf{E}^2 \mathbf{e}_3, \mathbf{E} \cdot \mathbf{T}_{2,2}, (\mathbf{N}_3 \mathbf{E}) \cdot \mathbf{T}_{2,2}, (\mathbf{N}_3 \mathbf{E}^2) \cdot \mathbf{T}_{2,2}.$$

The other three invariants trE^3 , $\mathbf{E}^2 \cdot \mathbf{T}_{2,2}$ and $\mathbf{N}_3^2 \cdot \mathbf{EN}_3 \mathbf{E}^3$ specified by Theorem 8 can be algebraically expressed by invariants in (5.4) and hence have been omitted here.

5.4. Orthorhombic system

The extended arguments for the three groups in this system are

$$\begin{aligned} C_{2v} &: (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{e}_3) = (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{e}_3 \otimes \mathbf{e}_3), \\ C_2 &: (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{Z}) = (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{e}_3 \otimes \mathbf{e}_3), \\ D_{2h} &: (\mathbf{E}, \mathbf{T}_{2,2}, \mathbf{e}_3 \otimes \mathbf{e}_3). \end{aligned}$$

Therefore there exists same integrity basis for them. According to Theorem 6, it consists of seven invariants

$$(5.5) \quad \text{trE}, \text{trE}^2, \text{trE}^3, \mathbf{e}_3 \cdot \mathbf{Ee}_3, \mathbf{e}_3 \cdot \mathbf{E}^2 \mathbf{e}_3, \mathbf{e}_1 \cdot \mathbf{Ee}_1, \mathbf{e}_1 \cdot \mathbf{E}^2 \mathbf{e}_1.$$

5.5. Tetragonal system

The extended arguments for the seven groups in this system are

$$\begin{aligned} C_4 &: (\mathbf{E}, \mathbf{T}_{4,4}, \mathbf{e}_3, \mathbf{Z}) = (\mathbf{E}, \mathbf{T}_{4,4}, \mathbf{N}_3), \\ S_4 &: (\mathbf{E}, \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_3, \mathbf{N}_3) = (\mathbf{E}, \mathbf{T}_{4,4}, \mathbf{N}_3), \\ C_{4h} &: (\mathbf{E}, \mathbf{T}_{4,4}, \mathbf{N}_3), \\ C_{4v} &: (\mathbf{E}, \mathbf{T}_{4,4}, \mathbf{e}_3) = (\mathbf{E}, \mathbf{T}_{4,4}), \\ D_{2d} &: (\mathbf{E}, \mathbf{T}_d, \mathbf{e}_3 \otimes \mathbf{e}_3) = (\mathbf{E}, \mathbf{T}_{4,4}), \\ D_4 &: (\mathbf{E}, \mathbf{T}_{4,4}, \mathbf{Z}) = (\mathbf{E}, \mathbf{T}_{4,4}), \\ D_{4h} &: (\mathbf{E}, \mathbf{T}_{4,4}). \end{aligned}$$

According to Theorems 1 – 3 the extended arguments for the last four groups are equivalent and, according to Theorem 6, the integrity basis for them consists of eight invariants

$$(5.6) \quad \begin{aligned} & \text{trE}, \text{trE}^2, \text{trE}^3, \mathbf{e}_3 \cdot \mathbf{Ee}_3, \mathbf{e}_3 \cdot \mathbf{E}^2\mathbf{e}_3, \\ & (\mathbf{E} \otimes \mathbf{E}) \cdot \mathbf{T}_{4,4}, (\mathbf{E} \otimes \mathbf{E}^2) \cdot \mathbf{T}_{4,4}, (\mathbf{E}^2 \otimes \mathbf{E}^2) \cdot \mathbf{T}_{4,4}. \end{aligned}$$

The extended arguments for the first three groups are equivalent to each other and, according to Theorem 8, the integrity basis for them consists of twelve invariants

$$(5.7) \quad \begin{aligned} & \text{trE}, \text{trE}^2, \text{trE}^3, \mathbf{e}_3 \cdot \mathbf{Ee}_3, \mathbf{e}_3 \cdot \mathbf{E}^2\mathbf{e}_3, \mathbf{N}_3^2 \cdot \mathbf{EN}_3\mathbf{E}^2, \\ & (\mathbf{E} \otimes \mathbf{E}) \cdot \mathbf{T}_{4,4}, (\mathbf{E}^2 \otimes \mathbf{E}) \cdot \mathbf{T}_{4,4}, (\mathbf{E}^2 \otimes \mathbf{E}^2) \cdot \mathbf{T}_{4,4}, \\ & (\mathbf{N}_3 \mathbf{E} \otimes \mathbf{E}) \cdot \mathbf{T}_{4,4}, (\mathbf{N}_3 \mathbf{E} \otimes \mathbf{E}^2) \cdot \mathbf{T}_{4,4}, (\mathbf{N}_3 \mathbf{E}^2 \otimes \mathbf{E}^2) \cdot \mathbf{T}_{4,4}. \end{aligned}$$

For this system, all the invariants involving $\mathbf{T}_{4,4}$ specified in Theorems 6 and 8 are irreducible.

5.6. Rhombohedral system

The extended arguments for the five groups in this system are

$$\begin{aligned} C_3 & : (\mathbf{E}, \mathbf{T}_{3,3}, \mathbf{e}_3, \mathbf{Z}) = (\mathbf{E}, \mathbf{T}_{3,3} \otimes \mathbf{e}_3, \mathbf{N}_3), \\ C_{3i} & : (\mathbf{E}, \mathbf{T}_{3,3} \otimes \mathbf{e}_3, \mathbf{N}_3), \\ C_{3v} & : (\mathbf{E}, \mathbf{T}_{3,3}, \mathbf{e}_3), \\ D_3 & : (\mathbf{E}, \mathbf{T}_{3,3}, \mathbf{Z}) = (\mathbf{E}, \mathbf{T}_{3,3}, \mathbf{e}_3), \\ C_{3d} & : (\mathbf{E}, \mathbf{T}_{3,3} \otimes \mathbf{e}_3) = (\mathbf{E}, \mathbf{T}_{3,3}, \mathbf{e}_3). \end{aligned}$$

Similarly, according to Theorems 1 – 3 and Theorem 8, the extended arguments for the last three groups are equivalent and the integrity basis for them consists of the following nine invariants

$$(5.8) \quad \begin{aligned} & \text{trE}, \text{trE}^2, \text{trE}^3, \mathbf{e}_3 \cdot \mathbf{Ee}_3, \mathbf{e}_3 \cdot \mathbf{E}^2\mathbf{e}_3, (\mathbf{E} \otimes \mathbf{Ee}_3) \cdot \mathbf{T}_{3,3}, \\ & (\mathbf{E}^2 \otimes \mathbf{Ee}_3) \cdot \mathbf{T}_{3,3}, (\mathbf{E} \otimes \mathbf{E}^2\mathbf{e}_3) \cdot \mathbf{T}_{3,3}, (\mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E}) \cdot \mathbf{T}_{6,6}. \end{aligned}$$

There are another four invariants $(\mathbf{E}_2 \otimes \mathbf{E}^2\mathbf{e}_3) \cdot \mathbf{T}_{3,3}$, $(\mathbf{E}^2 \otimes \mathbf{E} \otimes \mathbf{E}) \cdot \mathbf{T}_{6,6}$, $(\mathbf{E}^2 \otimes \mathbf{E}^2 \otimes \mathbf{E}) \cdot \mathbf{T}_{6,6}$, $(\mathbf{E}^2 \otimes \mathbf{E}^2 \otimes \mathbf{E}^2) \cdot \mathbf{T}_{6,6}$ among all the invariants obtained from Eqs. (4.3) and (4.4) which can be algebraically expressed by invariants in (5.8) and hence have been omitted here.

The extended arguments for the first two groups are equivalent to each other and, according to Theorem 6, the integrity basis for them is

$$(5.9) \quad \begin{aligned} & \text{tr}E, \text{tr}E^2, \text{tr}E^3, e_3 \cdot Ee_3, e_3 \cdot E^2e_3, N_3^2 \cdot EN_3 E^2, \\ & (E \otimes Ee_3) \cdot T_{3,3}, (E^2 \otimes Ee_3) \cdot T_{3,3}, (E \otimes E^2e_3) \cdot T_{3,3}, \\ & (E \otimes E \otimes E) \cdot T_{6,6}, (N_3 E \otimes Ee_3) \cdot T_{3,3}, (N_3 E^2 \otimes Ee_3) \cdot T_{3,3}, \\ & (N_3 E \otimes E^2e_3) \cdot T_{3,3}, (N_3 E \otimes E \otimes E) \cdot T_{6,6}. \end{aligned}$$

5.7. Hexagonal system

The extended arguments for the seven groups in this system are

$$\begin{aligned} C_{3h} &: (E, T_{3,3}, N_3) = (E, T_{6,6}, N_3), \\ C_6 &: (E, T_{6,6}, e_3, Z) = (E, T_{6,6}, N_3), \\ C_{6h} &: (E, T_{6,6}, N_3), \\ D_{3h} &: (E, T_{3,3}) = (E, T_{6,6}), \\ C_{6v} &: (E, T_{6,6}, e_3) = (E, T_{6,6}), \\ C_6 &: (E, T_{6,6}, Z) = (T, T_{6,6}), \\ D_{6h} &: (E, T_{6,6}). \end{aligned}$$

The extended arguments for the last four groups in this system are equivalent according to Theorems 1 and 2. The integrity basis for them, according to Theorem 6, consists of the following nine invariants

$$(5.10) \quad \begin{aligned} & \text{tr}E, \text{tr}E^2, \text{tr}E^3, e_3 \cdot Ee_3, e_3 \cdot E^2e_3, (E \otimes E \otimes E) \cdot T_{6,6}, \\ & (E^2 \otimes E \otimes E) \cdot T_{6,6}, (E^2 \otimes E^2 \otimes E) \cdot T_{6,6}, (E^2 \otimes E^2 \otimes E^2) \cdot T_{6,6}. \end{aligned}$$

Similarly, according to Theorem 8, the integrity basis for the first three groups which have equivalent extended argument consists of fourteen invariants

$$(5.11) \quad \begin{aligned} & \text{tr}E, \text{tr}E^2, \text{tr}E^3, e_3 \cdot Ee_3, e_3 \cdot E^2e_3, N_3^2 \cdot EN_3 E^2, \\ & (E \otimes E \otimes E) \cdot T_{6,6}, (E^2 \otimes E \otimes E) \cdot T_{6,6}, (E^2 \otimes E^2 \otimes E) \cdot T_{6,6}, \\ & (E^2 \otimes E^2 \otimes E^2) \cdot T_{6,6}, (N_3 E \otimes E \otimes E) \cdot T_{6,6}, (N_3 E^2 \otimes E \otimes E) \cdot T_{6,6}, \\ & (N_3 E^2 \otimes E^2 \otimes E) \cdot T_{6,6}, (N_3 E^2 \otimes E^2 \otimes E^2) \cdot T_{6,6}. \end{aligned}$$

For this system, all the invariants involving $T_{6,6}$ specified in Theorems 6 and 8 are irreducible.

5.8. Cubic system

The extended arguments for the five groups in this system are

$$\begin{aligned} T &: (E, T_h, Z) = (E, T_h), \\ T_h &: (E, T_h), \\ T_d &: (E, T_d) = (E, O_h), \\ O &: (E, O_h, Z) = (E, O_h), \\ O_h &: (E, O_h). \end{aligned}$$

Our results in the previous sections are not valid for crystals in cubic system. Up to now, the author has no idea how to limit our search for integrity basis to a small number of homogeneous polynomial invariants. Let us quote the results by SMITH and RIVLIN [1]. For the first two groups in this system the integrity basis consists of fourteen invariants

$$(5.12) \quad \begin{aligned} & \text{trE}, \text{trE}^2, \text{trE}^3, (\mathbf{E} \otimes \mathbf{E})_h \cdot \mathbf{T}, \\ & (\mathbf{E}^2 \otimes \mathbf{E}) \cdot \mathbf{T}_h, (\mathbf{E} \otimes \mathbf{E}^2) \cdot \mathbf{T}_h, \lambda [\mathbf{E}^2 \otimes \mathbf{E}] \cdot \mathbf{T}_h, \\ & (\mathbf{E}^2 \otimes \mathbf{E}^2) \cdot \mathbf{T}_h, (\mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E}) \cdot (\mathbf{T}_h \mathbf{T}_h), (\mathbf{E}^2 \otimes \mathbf{E}^2 \otimes \mathbf{E}^2) \cdot (\mathbf{T}_h \mathbf{T}_h), \\ & \xi [\mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E}] \cdot (\mathbf{T}_h \otimes \mathbf{T}_h), \phi [\mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E}] \cdot (\mathbf{T}_h \otimes \mathbf{T}_h), \\ & (\mathbf{E}^2 \otimes \mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E}) \cdot (\mathbf{T}_h \otimes \mathbf{T}_h), \phi [\mathbf{E}^2 \otimes \mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E}] \cdot (\mathbf{T}_h \otimes \mathbf{T}_h), \end{aligned}$$

where

$$(5.13) \quad \lambda = [1324], \quad \xi = [12345768], \quad \phi = [12354678].$$

For the last three groups in this system the integrity basis consists of nine invariants

$$\begin{aligned} & \text{trE}, \text{trE}^2, \text{trE}^3, (\mathbf{E} \otimes \mathbf{E}) \cdot \mathbf{O}_h, (\mathbf{E}^2 \otimes \mathbf{E}) \cdot \mathbf{O}_h, (\mathbf{E} \otimes \mathbf{E}^2) \cdot \mathbf{O}_h, (\mathbf{E}^2 \otimes \mathbf{E}^2) \cdot \mathbf{O}_h, \\ & (\mathbf{E} \otimes \mathbf{E} \otimes \mathbf{E}) \cdot (\mathbf{O}_h \mathbf{O}_h), (\mathbf{E}^2 \otimes \mathbf{E} \otimes \mathbf{E}) \cdot (\mathbf{O}_h \mathbf{O}_h), (\mathbf{E}^2 \otimes \mathbf{E}^2 \otimes \mathbf{E}) \cdot (\mathbf{O}_h \mathbf{O}_h). \end{aligned}$$

Acknowledgment

The discussions with Prof. J. RYCHLEWSKI are greatly acknowledged.

This research was supported partly by Chinese National Foundation of Science and partly by a postdoctoral research fellowship from IFTR.

References

1. G. F. SMITH and R. S. RIVLIN, *The strain-energy function for anisotropic elastic materials*, Trans. Amer. Math. Soc., **88**, pp. 175—193, 1958.
2. A. E. GREEN and J. E. ADKINS, *Large elastic deformations*, Oxford University Press (Clarendon), London and New York 1960.
3. G. F. SMITH, M. M. SMITH and R. S. RIVLIN, *Integrity bases for a symmetric tensor and a vector... The crystal classes*, Arch. Ratl. Mech. Anal., **12**, pp. 93—133, 1963.
4. G. F. SMITH and R. S. RIVLIN, *Integrity bases for vectors... The crystal classes*, Arch. Ratl. Mech. Anal., **15**, pp. 169—221, 1964.
5. J. P. BOEHLER, *Simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy*, ZAMM, **59**, pp. 157—169, 1979.
6. I. SHIH LIU, *On representations of anisotropic invariants*, Int. J. Engng. Sci., **20**, pp. 1099—1109, 1982.
7. J. RYCHLEWSKI, *Symmetry of causes and effects*, Shanghai Institute of Applied Mathematics and Mechanics, Research Report SM8706, 1987.
8. J. M. ZHANG and J. RYCHLEWSKI, *Structural tensors for anisotropic solids*, Arch. Mech., **42**, pp. 267—277, 1990.

9. C. C. WANG, *A new representation theorem for isotropic functions. An answer to Professor G.F. Smith's criticism of my papers on representations for isotropic functions, Part I and II*, Arch. Ratl. Mech. Anal., **36**, pp. 166—223, 1970.
10. A. J. M. SPENCER, *Theory of invariants*, in: Continuum Physics, ed. by A.C. ERINGEN, Academic Press, pp. 239—353, 1971.
11. C. A. TRUESDELL and W. NOLL, *The non-linear field theories of mechanics*, S. FLÜGGE [ed.], in: Handbuch der Physik III/3, Springer-Verlag, 1965.
12. V. V. LOKHIN, *Symmetry and non-linear tensor functions in Minkowski space*, Soviet Math. Dokl., **10**, pp. 623—625, 1969.
13. G. B. GUREVIFCH, *Foundations of the theory of algebraic invariants*, Noordhoff, Groningen 1964.
14. H. WEYL, *The classical groups*, Princeton Univ. Press. Princeton, New Jersey 1939.
15. J. BETTEN, *Invariants of fourth order tensors*, in: Applications of Tensor Functions in Solid Mechanics, pp 203—226, J.P. BOEHLER [ed.], Springer-Verlag, Wien, New York 1987.
16. J. RYCHLEWSKI, *On general form of isotropic transformation of tensor space*, Bull. Acad. Polon. Série Sci Techn., **7**, pp. 283—287, 1970.
17. J. RYCHLEWSKI, *Symmetry of tensor functions and spectral theorem*, Advances in Mech., **11**, pp. 77—124, 1988.
18. G. F. SMITH, *On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors*, Int. J. Engng. Sci., **9**, pp. 899—916, 1971.

DEPARTMENT OF MECHANICAL AND PRODUCTION ENGINEERING
NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE.

Received November 8, 1990.