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# Phonon gas hydrodynamics based on the maximum entropy principle and the extended field theory of a rigid conductor of heat(*)

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THE GENERAL CASE of extended field theory of a rigid conductor of heat, consistent with the entropy principle in the sense of Müller, is introduced and the equivalent symmetric conservative system with respect to the fields of Liu multipliers is constructed. Then the extended hydrodynamics of a phonon gas based on the maximum entropy principle and governed by the system of field equations analogous to the extended field theory of a rigid conductor of heat is considered. It is shown that the equations of phonon gas hydrodynamics can be rearranged to the symmetric conservative system with respect to Lagrange multipliers of the variational problem of entropy maximization; moreover, the additional conservation equation implied by this symmetric conservative system was shown to correspond to the result of substitution of the distribution function which maximizes the entropy into the kinetic balance of entropy. This provides a direct and simpler proof than that given by DREYER [12], concerning the fact that the distribution function maximizing the entropy leads to the field equations of hydrodynamics consistent with the entropy principle after substitution into the respective system of moment equations and that, in this case, Liu multipliers of the entropy principle correspond to the Lagrange multipliers of the variational problem of entropy maximization. It is also shown that, due to the symmetry of moments, the field equations of the phonon gas hydrodynamics considered can be rearranged to the form of the symmetric conservative system generated by a single potential.

Wprowadza się ogólny przypadek rozszerzonej teorii pola sztywnego przewodnika ciepła, zgodnej z zasadą entropijną Müllera. Odpowiadający układ równań pola sprowadza się następnie do równoważnego układu symetrycznego konserwatywnego względem pól mnożników Liu. Jako odniesienie fizyczne rozpatruje się rozszerzoną hydrodynamikę gazu fononów opartą na zasadzie maksimum entropii, której układ równań pola ma analogiczną strukturę do układu równań rozszerzonej teorii pola sztywnego przewodnika ciepła. Równania rozpatrywanej hydrodynamiki gazu fononów zapisują się jako układ symetryczny konserwatywny ze względu na pola mnożników Lagrange'a zagadnienia wariacyjnego na maksimum entropii, a dodatkowe równanie zachowania implikowane przez ten układ symetryczny konserwatywny odpowiada podstawieniu funkcji rozkładu maksymalizującej entropię do kinetycznego bilansu entropii. Wykorzystując te fakty pokazuje się w odmienny, znacznie prostszy sposób niż podany wcześniej przez DREYERA [12], że funkcja rozkładu maksymalizująca entropię, po wstawieniu do odpowiedniego układu równań momentowych, prowadzi do układu równań pola zgodnego z zasadą entropijną, a mnożniki Liu korespondują z mnożnikami Lagrange'a zagadnienia wariacyjnego. Ponadto pokazuje się, że dzięki symetrii momentów występujących w rozpatrywanej hydrodynamice gazu fononów, jej układ równań pola można przedstawić jako układ symetryczny konserwatywny o lewej stronie określonej jedną funkcją generującą.

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Вводится общий случай расширенной теории поля жесткого теплопроводника, согласной с энтропический принципом Мюллера. Отвечающая система уравнений поля сводится затем к эквивалентной системе симметричной консервативной по отношению к полям множителей Лю. Как физическое отнесение рассматривается расширенная гидродинамика газа фононов, опирающаяся на принцип максимума энтропии, которой система уравнений поля имеет аналогичную структуру как система уравнений расширенной теории поля жесткого теплопроводника. Оказывается, что уравнения рассматриваемой гидродинамики газа фононов записываются как симметричная консервативная система по отношению к полям множителей Лагранжа, а вариационная задача для максимума энтропии и дополнительное уравнение сохранения, вызыванные через эту симметричную консервативную систему, отвечают подстановке функции распределения, максимизируюжей энтропию, в кинетический баланс энтропии. Используя эти факты, показывается, отличным значительно более простым способом, чем приведенный раньше Дреером [12], что функция распределения, максимизирующая энтропию, после подстановки в соответствующую систему моментных уравнений, приводит к системе уравнений поля совпадающей с энтропическим принципом, а множители Лю соответствуют множителям Лагранжа вариационной задачи. Кроме этого оказывается, что благодаря симметрии моментов, выступающих в рассматриваемой гидродинамике газа фононов, ее систему уравнений поля можно представить как симметричную консервативную систему с левой стороной определенной одной генерирующей функцией.

## 1. Introduction

In [1], Liu proposed a general method of phenomenological modelling of continuous media in terms of the extended set of balance equations. He assumed local constitutive relations to be restricted by the entropy inequality in the sense of Müller [2, 3]. According to this approach, it is possible to consider the sequence of models, obtained by employing successive extensions of the fundamental set of balance laws.

In this paper, the extended field theory of a rigid conductor of heat is introduced and compared with the extended hydrodynamics of a phonon gas based on the maximum entropy principle. We discuss the general case of the theory, described by the system of balance equations composed of the energy balance and the arbitrary finite set of its extensions. From the hyperbolicity condition, usually assumed in extended thermodynamics, it follows that the field equations of the extended field theory of a rigid conductor of heat can be written in the symmetric conservative form with respect to the fields of Liu multipliers.

A particular phenomenological model of the heat conduction based on the concepts of the extended thermodynamics was proposed by Ruggeri [4, 5]. In turn, Jou and Perez Garcia [6] and Bampi, Morro and Jou [7] considered the description of transport processes in a rigid conductor of heat, consistent with the extended irreversible thermodynamics (EIT). In [6] this description was related with the theory of fluctuations whereas in [7] it was compared with phonon gas hydrodynamics derived by Chester [8]. In [9] it was shown that, in the case of the linear isotropic dispersion curve, the low-temperature phonon gas hydrodynamics proposed by Nielsen and Shklovsky [10] and related to the phonon distribution function which maximizes the entropy is described by the system of field equations of the form of extended field theory of a rigid conductor of heat.

The general case of the extended hydrodynamics of degenerate material gases (corresponding to the completion of the fundamental set of balance laws with an arbitrary finite set of its extensions) was considered by DREYER [12]. The procedure applied by Dreyer is the following. He derives the set of moment equations for intrinsic moments from the Boltzmann kinetic equation. Besides the classical conservation equations this set contains the additional balance equations for the successive non-equilibrium moments up to the arbitrary finite order $N$. He introduces the corresponding extended thermodynamics motivated by this set of moment equations, such that the moments up to the order $N$ are taken as primitive fields, while the moment of the order $N+1$ and all the production terms are given by local constitutive relations. He postulates the consistency of this system with the Müller entropy principle, that is, he assumes the existence of the entropy and of the entropy flux depending on the same primitive fields. Then, by means of the Liu procedure, he formulates the consistency conditions and determines the entropy in terms of the primitive fields and the Liu multipliers.

On the other hand, he introduces the variational problem of the entropy maximization under the constraints corresponding to prescribed values of these moments which were taken as the primitive fields of extended thermodynamics. As a result, he obtains the distribution function dependent on the Lagrange multipliers of the variational problem. Hence, the kinetic expressions for moments, entropy and the entropy flux, taken for this distribution function, result in functions of Lagrange multipliers. Dreyer shows that the result of the insertion of the distribution function which maximizes the entropy into the kinetic expression for the entropy can be expressed in terms of Lagrange multipliers and moments in the form identical to that known from extended thermodynamics. From these observations he concludes that Liu multipliers can be identified with the Lagrange multipliers and that the description of the state of the system by the distribution function which maximizes the entropy is consistent with the extended thermodynamics.

In this paper a similar approach is applied to the phonon gas. In Sect. 3 the energy balance of the phonon gas is derived from the Boltzmann–Peierls kinetic equation. Then we derive a sequence of the moment identities which are the succesive extensions of the energy balance. It is composed of the balance equation for the heat flux, for the flux of the heat flux ,..., up to the moment equation for the arbitrary $N-1$-th moment of this kind. In that manner we obtain a system of moment equations for moments, which correspond to the primitive fields for the extended field theory of a rigid conductor of heat.

We formulate the variational problem for the phonon distribution function which maximizes the kinetic expression for the entropy density under the constraints corresponding to the fixed values of those $N$ moments, which were taken as the primitive fields of the extended field theory of a rigid conductor of heat. Thus we obtain the distribution function dependent on the Lagrange multipliers of the variational problem. We also show (similarly as in [13]), that this distribution function maximizes the flux of the kinetic entropy in the arbitrary direction under the constraints, corresponding to the fixed values of the projections of these moments on that direction.

We insert the distribution function obtained from the variational principle into the corresponding sequence of the moment equations and arrive at the closed system of equations with respect to the Lagrange multipliers. The same distribution function is also inserted into the kinetic balance of entropy. We show that the system of the field equations with respect to the Lagrange multipliers is symmetric conservative and the additional conservation law implied by this system corresponds to the result of the insertion of the distribution function which maximizes the entropy into the kinetic balance of entropy. The potentials (generators) of this symmetric conservative system are determined in the form of integral expressions, dependent on the phonon distribution function which maximizes the entropy.

The structure of this symmetric conservative system corresponds to the structure of the field equations of the extended field theory of a rigid conductor of heat, discussed in Sect. 2. As it was shown by RUGGERI and STRUMIA [14], RUGGERI [4], every symmetric conservative system is consistent with the Müller entropy principle. Hence the approach, proposed in this paper, gives the alternative derivation of the identities analogical to that obtained by DREYER [12]. Moreover, by applying the reasoning presented in [13] it can be shown that the distribution function which maximizes the entropy is the only distribution function assuring the consistency with the Müller entropy principle.

In [15], it was shown that the equations of the isoentropic flow of an ideal gas can be equivalently written in the form of the symmetric conservative system, the left-hand side of which is determined by a single generating function (potential). In turn, in [9] it has been shown that for the field equations of the low-temperature phonon gas hydrodynamics involving the linear isotropic approximation of the dispersion relation it is possible to find the whole family of equivalent symmetric conservative systems of such special form. In Sect. 5 we show that also the equations of the extended hydrodynamics of the phonon gas, obtained in this paper, can be equivalently written in the form of symmetric conservative systems with the left-hand sides determined by a single potential. In order to obtain these systems, we apply the group reduction of tensors and the scaling procedure (introduced previously in [9]).

Contrary to [9] and [15] where the potential for the symmetric conservative system was determined in terms of macroscopic quantities, in this paper we determine it in the form of the integral expression, involving the distribution function which maximizes the entropy.

## 2. Extended field theory of a rigid conductor of heat

By a rigid heat conductor we shall mean the undeformable (rigid) continuous material body $\mathscr{B}$. We shall assume, that for a certain inertial observer the body remains unmoved, and the material points of the body manifold will be identified with the corresponding points of a three-dimensional Euclidean point space $E^3$. For a rigid heat conductor, all balance laws of continuum mechanics are satisfied trivially except the energy balance.

In this paper we shall consider a uniform rigid heat conductor without heat supply and, therefore, it will be convenient to use the energy density referred to the unit volume $\varepsilon$. The local form of the energy balance takes then the form

$$\partial_t \varepsilon + \operatorname{div} \mathbf{q} = 0, \tag{2.1}$$

where $\mathbf{q}$ denotes the heat flux. In order to formulate the thermodynamic theory of a rigid conductor of heat, the entropy balance should be taken into account. The assumption that the heat supply vanishes implies vanishing of the entropy supply. Then the local form of the balance of entropy reads

$$\partial_t \eta + \operatorname{div} \mathbf{h} = \sigma, \tag{2.2}$$

where $\eta$ is the volume density of the entropy, $\mathbf{h}$ is the entropy flux and $\sigma$ is the entropy production, where additionally $\sigma \geqslant 0$.

According to the approach of Liu [1] we can write formally the extended system of balance equations of the order $L-1$, $L>2$ for the rigid heat conductor. This system has the form

$$\begin{aligned}
&\partial_t \varepsilon + \operatorname{div} \mathbf{q} = 0,\\
&\partial_t \mathbf{q} + \operatorname{div} \overset{(2)}{\mathbf{Q}} = \overset{(1)}{\mathbf{P}},\\
&\partial_t \overset{(2)}{\mathbf{Q}} + \operatorname{div} \overset{(3)}{\mathbf{Q}} = \overset{(2)}{\mathbf{P}},\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ,\\
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&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad ,\\
&\partial_t \overset{(L-1)}{\mathbf{Q}} + \operatorname{div} \overset{(L)}{\mathbf{Q}} = \overset{(L-1)}{\mathbf{P}} .
\end{aligned} \tag{2.3}$$

By $\overset{(2)}{\mathbf{Q}}, \overset{(3)}{\mathbf{Q}}, \ldots, \overset{(L)}{\mathbf{Q}}$ we denote the tensors of the order two, three,...,up to the order $L$, correspondingly, which do not have any direct physical meaning, and which play the role of the fluxes in the system (2.3) (we employ the convention that the index in brackets denotes the order of the tensor). Similarly, by $\overset{(1)}{\mathbf{P}}, \overset{(2)}{\mathbf{P}}, \ldots \overset{(L-1)}{\mathbf{P}}$ we denote the tensor quantities of the order one,two, ..., $L-1$, which are interpreted as the corresponding productions. As the primitive fields of the extended field theory of a rigid heat conductor (of the order $L-1$) we take the set of fields $\varepsilon, \mathbf{q}, \overset{(2)}{\mathbf{Q}}, \ldots, \overset{(L-1)}{\mathbf{Q}}$ whereas the flux in the last equation and productions $\overset{(1)}{\mathbf{P}}, \overset{(2)}{\mathbf{P}}, \ldots, \overset{(L-1)}{\mathbf{P}}$ should be given by the constitutive relations. In the approach of Liu, the local constitutive relations are assumed together with the usual principles of continuum mechanics, that is, the principle of material objectivity, the invariance with respect to the symmetry group of the considered material and the principle of equipresence. For the extended field theory of a rigid heat conductor, local constitutive relations take the form

$$
\begin{aligned}
\overset{(L)}{\mathbf{Q}} &= \overset{(L)}{\hat{\mathbf{Q}}}(\varepsilon, \mathbf{q}, \ldots, \overset{(L-1)}{\mathbf{Q}}),\\
\overset{(1)}{\mathbf{P}} &= \overset{(1)}{\hat{\mathbf{P}}}(\varepsilon, \mathbf{q}, \ldots, \overset{(L-1)}{\mathbf{Q}}),\\
&\;\cdots\cdots\cdots\\
&\;\cdots\cdots\cdots\\
&\;\cdots\cdots\cdots\\
\overset{(L-1)}{\mathbf{P}} &= \overset{(L-1)}{\hat{\mathbf{P}}}(\varepsilon, \mathbf{q}, \ldots, \overset{(L-1)}{\mathbf{Q}}).
\end{aligned}
\tag{2.4}
$$

According to the Müller entropy principle, the thermodynamic restrictions on the form of the constitutive functions (2.4) are formulated by means of the Liu procedure applied to the field equations (2.3), (2.4) and to the entropy inequality

$$\partial_t \eta + \operatorname{div} \mathbf{h} > 0. \tag{2.5}$$

For the entropy density and the entropy flux, given by the constitutive functions of the form

$$
\begin{aligned}
\eta &= \hat{\eta}(\varepsilon, \mathbf{q}, \ldots, \overset{(L-1)}{\mathbf{Q}}),\\
\mathbf{h} &= \hat{\mathbf{h}}(\varepsilon, \mathbf{q}, \ldots, \overset{(L-1)}{\mathbf{Q}}),
\end{aligned}
\tag{2.6}
$$

the Liu procedure leads to the following relations

$$(2.7)\qquad d\hat{h} = \sum_{I=0}^{L-2} \underset{(1,I+1)}{Tr} \ldots \underset{(I,2I)}{Tr} \left[ D_{\overset{(I)}{\mathbf{Q}}} \hat{\eta} \right] \otimes d \overset{(I+1)}{\mathbf{Q}}$$
$$+ \underset{(1,L)}{\mathrm{Tr}} \ldots \underset{(L-1,2L-2)}{\mathrm{Tr}} \left[ D_{\overset{(I-1)}{\mathbf{Q}}} \hat{\eta} \right] \otimes d\overset{(L)}{\hat{\mathbf{Q}}},$$

$$(2.8)\qquad \sum_{I=1}^{L-1} \left[ D_{\overset{(I)}{\mathbf{Q}}} \hat{\eta} \right] \odot \overset{(I)}{\hat{\mathbf{P}}} \geqslant 0,$$

where $\otimes$ is the tensor product, $\underset{(i,j)}{\mathrm{Tr}}$ denotes the trace operation taken with respect to the indices $(i,j)$ (in order to simplify the notation, we introduce the additional symbols $\underset{(0,0)}{\mathrm{Tr}}$ and $\underset{(1,1)}{\mathrm{Tr}}$ understood as identities), $\odot$ denotes the total contraction of tensors and the notation $\overset{(0)}{\mathbf{Q}} = \varepsilon$, $\overset{(1)}{\mathbf{Q}} = \mathbf{q}$ has been applied.

The quantities

$$(2.9)\qquad \overset{(J)}{\boldsymbol{\lambda}} := D_{\overset{(J)}{\mathbf{Q}}} \hat{\eta}, \qquad J = 0,1,\ldots,L-1,$$

which occur in Eqs. (2.7), (2.8), are called the Liu multipliers. In extended thermodynamics, hyperbolicity of the field equations, satisfying the thermodynamical restrictions (2.7), (2.8), is assumed [16, 17].

BOILLAT [18], RUGGERI and STRUMIA [14], RUGGERI [4, 5] have shown that the field equations of extended thermodynamics can be transformed to the form of symmetric conservative systems of partial differential equations with respect to the fields of Liu multipliers. Such transformation is a consequence of the bijective relation between the primitive fields and the Liu multipliers, which can be written as

$$(2.10)\qquad \begin{aligned} \overset{(J)}{\mathbf{Q}} &= \overset{(J)}{\mathbf{Q}} \left( \overset{(0)}{\boldsymbol{\lambda}}, \overset{(1)}{\boldsymbol{\lambda}}, \ldots, \overset{(L-1)}{\boldsymbol{\lambda}} \right), \\ \overset{(K)}{\boldsymbol{\lambda}} &= \overset{(K)}{\boldsymbol{\lambda}} \left( \overset{(0)}{\mathbf{Q}}, \overset{(1)}{\mathbf{Q}}, \ldots, \overset{(L-1)}{\mathbf{Q}} \right). \end{aligned}$$

In the case of Eqs. (2.3), (2.4), the potentials (generators) of the symmetric conservative system have the following form

$$(2.11)\qquad \begin{aligned} \varphi &= \sum_{J=0}^{L-1} \overset{(J)}{\boldsymbol{\lambda}} \odot \overset{(J)}{\hat{\mathbf{Q}}} - \hat{\eta}, \\ \boldsymbol{\psi} &= \sum_{J=0}^{L-2} \underset{(1,J+1)}{\mathrm{Tr}} \dots \underset{(J,2J)}{\mathrm{Tr}}\, \overset{(J)}{\boldsymbol{\lambda}} \otimes \overset{(J+1)}{\hat{\mathbf{Q}}} + \underset{(1,L)}{\mathrm{Tr}} \dots \underset{(L-1,2L-2)}{\mathrm{Tr}}\, \overset{(L-1)}{\boldsymbol{\lambda}} \otimes \overset{(L)}{\hat{\mathbf{Q}}} - \hat{\mathbf{h}}. \end{aligned}$$

Taking into account Eqs. (2.10), (2.7) and differentiating Eqs. (2.11) with respect to the tensors $\overset{(J)}{\boldsymbol{\lambda}}$ we arrive at

$$(2.12)\qquad \begin{aligned} D_{\overset{(J)}{\lambda}} \varphi &= \overset{(J)}{\hat{\mathbf{Q}}}, \\ \sigma_{J+1} \times \left[ D_{\overset{(J)}{\lambda}} \boldsymbol{\psi} \right] &= \overset{(J+1)}{\hat{\mathbf{Q}}}, \end{aligned}$$

where $\sigma_J$ denotes the permutation of the tensor of the order $J$, which intercheanges the first index with the last $J$-th index.

From Eq. (2.11) it follows that the system of Eqs. (2.3), (2.4) can be written in the symmetric conservative form

$$(2.13)\qquad \partial_t \left[ D_{\overset{(J)}{\lambda}} \varphi \right] + \operatorname{div} \left[ \sigma_{J+1} \times D_{\overset{(J)}{\lambda}} \boldsymbol{\psi} \right] = \overset{(J)}{\hat{\mathbf{P}}}.$$

As it has been shown by GODUNOV [19], the symmetric conservative systems imply an additional balance equation. In the case of the system (2.13), this additional balance equation has the form

$$(2.14)\qquad \partial_t \left[ \sum_{J=0}^{L-1} \overset{(J)}{\boldsymbol{\lambda}} \odot D_{\overset{(J)}{\lambda}} \varphi - \varphi \right] + \operatorname{div} \left[ \sum_{J=0}^{L-1} \overset{(J)}{\boldsymbol{\lambda}} \odot \left( \sigma_{J+1} \times D_{\overset{(J)}{\lambda}} \boldsymbol{\psi} \right) - \boldsymbol{\psi} \right] = \sum_{J=0}^{L-1} \overset{(J)}{\boldsymbol{\lambda}} \odot \overset{(J)}{\hat{\mathbf{P}}} = \sigma.$$

From Eq. (2.11) it follows, that Eq. (2.13) is identical with the entropy balance (2.2) with $\eta$ and $\mathbf{h}$ given by Eq. (2.6).

## 3. Kinetic description of a phonon gas. Moment equations and balance of entropy

In this Section we briefly review some properties of the kinetic description of phonon systems. For simplicity, we shall restrict the considerations to a single branch of phonon excitations, neglecting all interactions between different branches. In this case, the kinetic equation takes the form

$$\frac{\partial}{\partial t} f(x^j, k^m, t) + \sum_{i=1}^{3} \frac{\partial \omega(k^m)}{\partial k^i} \frac{\partial f(x^j, k^m, t)}{\partial x^i} = J(f(x^l, k^m, t)), \tag{3.1}$$

where $f(x^j, k^m, t)$ is the phonon distribution function, $\mathbf{k}$ denotes the phonon wave-vector with the components $k^1, k^2, k^3$, $\mathbf{x} = [x^1, x^2, x^3]$ is the spatial variable, $\omega(k)$ denotes the dispersion curve of excitations and $J(f)$ is the collision integral [20].

The variable $\mathbf{k}$ belongs to the First Brillouin Zone, which can be represented as a parallepiped with correspondingly identified opposite walls. This identification endows the First Brillouin Zone with the topology of a torus [21]. With the accuracy to the set of measure zero, the First Brillouin Zone can be identified with the set of Euclidean vectors from the interior of the parallepiped. The quantities $\hbar\omega(\mathbf{k})$ and $\hbar\mathbf{k}$ describe the energy and the quasimomentum of a single phonon, respectively ($2\pi\hbar$ is Planck's constant). Similarly as in the case of kinetic theory of material gases, the macroscopic quantities are defined as moments of the distribution function, that is, as integrals of the products of the distribution function and the appropriate tensor fields, defined on the First Brillouin Zone. These tensor fields describe the contribution of a single phonon of wave-vector $\mathbf{k}$ to the given physical quantity.

Let $\mathcal{W}(\mathbf{k})$ be the function defined on the interior of the parallepiped representing the First Brillouin Zone and with values in the set of Euclidean tensors. The physical quantity, corresponding to $\mathcal{W}(\mathbf{k})$, is given by

$$\mathbf{W}(\mathbf{x}, t) = \int \mathcal{W}(\mathbf{k}) f(\mathbf{x}, \mathbf{k}, t) \frac{d^3\mathbf{k}}{(2\pi)^3}. \tag{3.2}$$

It follows from the physical meaning of the quantities $\hbar\omega(\mathbf{k})$ and $\hbar\mathbf{k}$ that the fields defined as integrals

$$\begin{aligned} \varepsilon(\mathbf{x}, t) &= \int \hbar\omega(\mathbf{k}) f(\mathbf{x}, \mathbf{k}, t) \frac{d^3\mathbf{k}}{(2\pi)^3}, \\ \mathbf{P}(\mathbf{x}, t) &= \int \hbar\mathbf{k} f(\mathbf{x}, \mathbf{k}, t) \frac{d^3\mathbf{k}}{(2\pi)^3}, \end{aligned} \tag{3.3}$$

have a meaning of the macroscopic spatial densities of the energy and quasimomentum, correspondingly. In Eqs. (3.2), (3.3) we confine ourselves to the normalization convention for the distribution function usually applied in solid state physics and the integration is carried over the set of Euclidean vectors belonging to the domain of $\mathcal{W}(\mathbf{k})$. Moment equations are obtained as in the case of kinetic theory of material gases, that is, the moment equation for the quantity $\mathbf{W}(\mathbf{x},t)$ is obtained by computing the time derivative of Eq. (3.2) and substituting the kinetic equation into the integrand

$$(3.4) \qquad \partial_t \mathbf{W}(\mathbf{x},t) = \int \mathcal{W}(\mathbf{k})\, \partial_t f(\mathbf{x},\mathbf{k},t) \frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$= -\operatorname{div}\left\{\int f(\mathbf{x},\mathbf{k},t)\,[\mathcal{W}(\mathbf{k})\otimes\nabla_{\mathbf{k}}\omega(\mathbf{k})]\frac{d^3\mathbf{k}}{(2\pi)^3}\right\}$$

$$+\int \mathcal{W}(\mathbf{k}) J(f(\mathbf{x},\mathbf{k},t))\frac{d^3\mathbf{k}}{(2\pi)^3}.$$

The field $\mathcal{W}(\mathbf{k})$ is called a summational invariant if, for every distribution function $f(\mathbf{x},\mathbf{k},t)$, the following integral vanishes

$$(3.5) \qquad \int \mathcal{W}(\mathbf{k})\, J(f(\mathbf{x},\mathbf{k},t))\frac{d^3\mathbf{k}}{(2\pi)^3} = 0.$$

The phonon energy $\hbar\omega(\mathbf{k})$ is a summational invariant and, therefore, the energy balance takes the form

$$(3.6) \qquad \partial_t \varepsilon + \operatorname{div}\mathbf{q} = 0,$$

where $\varepsilon$ is given by Eq. $(3.3)_1$ and the heat flux $\mathbf{q}$ is defined as

$$(3.7) \qquad \mathbf{q} = \int \hbar\omega(\mathbf{k}) f(\mathbf{x},\mathbf{k},t)\; \nabla_{\mathbf{k}}\omega(\mathbf{k})\frac{d^3\mathbf{k}}{(2\pi)^3}.$$

Hence, the energy balance of the phonon gas results from the moment equation for the field $\mathcal{W}(\mathbf{k}) = \hbar\omega(\mathbf{k})$.

Since in every equation of the system (2.3), except the first one, the time derivative acts on the flux from the previous equation then, in order to obtain the system of moment equations corresponding to the system (2.3), we should introduce the tensor fields of the form

$$(3.8) \qquad \overset{(I)}{\mathbf{N}}(\mathbf{k}) = \hbar\omega(\mathbf{k})\left\{\overset{I}{\otimes}[\nabla_{\mathbf{k}}\omega(\mathbf{k})]\right\}, \qquad I = 0,1,\ldots,L.$$

The corresponding moment equations have the form

$$(3.9)\qquad \partial_t \int \overset{(I)}{\mathbf{N}}(\mathbf{k}) f(\mathbf{x},\mathbf{k},t)\frac{d^3\mathbf{k}}{(2\pi)^3} + \operatorname{div}\int \overset{(I+1)}{\mathbf{N}}(\mathbf{k}) f(\mathbf{x},\mathbf{k},t)\frac{d^3\mathbf{k}}{(2\pi)^3} = \int \overset{(I)}{\mathbf{N}}(\mathbf{k})\, J(f(\mathbf{x},\mathbf{k},t))\frac{d^3\mathbf{k}}{(2\pi)^3}, \qquad I = 0, 1, 2, \ldots, L-1.$$

For convenience, we shall apply the notation

$$(3.10)\qquad \begin{gathered} \overset{(M)}{\mathbf{Q}} := \int \overset{(M)}{\mathbf{N}}(\mathbf{k}) f(\mathbf{x},\mathbf{k},t)\frac{d^3\mathbf{k}}{(2\pi)^3}, \\ \overset{(M)}{\mathbf{P}} := \int \overset{(M)}{\mathbf{N}}(\mathbf{k}) J(f(\mathbf{x},\mathbf{k},t))\frac{d^3\mathbf{k}}{(2\pi)^3}, \\ M = 0, 1, \ldots, L. \end{gathered}$$

It follows from Eqs. (3.10), (3.6) that $\varepsilon = \overset{(0)}{\mathbf{Q}}$, $\mathbf{q} = \overset{(1)}{\mathbf{Q}}$ and all the remaining tensors $\overset{(M)}{\mathbf{Q}}$ and $\overset{(M)}{\mathbf{P}}$, $M = 2, 3, \ldots, L$ are totally symmetric tensors of the order $M$. Taking into account the notation (3.10), we write the system (3.9) in the form

$$(3.11)\qquad \begin{gathered} \partial_t \overset{(0)}{\mathbf{Q}} + \operatorname{div}\overset{(1)}{\mathbf{Q}} = 0, \\ \partial_t \overset{(1)}{\mathbf{Q}} + \operatorname{div}\overset{(2)}{\mathbf{Q}} = \overset{(1)}{\mathbf{P}}, \\ \partial_t \overset{(1)}{\mathbf{Q}} + \operatorname{div}\overset{(3)}{\mathbf{Q}} = \overset{(2)}{\mathbf{P}}, \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \partial_t \overset{(L-1)}{\mathbf{Q}} + \operatorname{div}\overset{(L)}{\mathbf{Q}} = \overset{(L-1)}{\mathbf{P}}. \end{gathered}$$

Hence, the structure of the system of moment equations (3.11) obtained by integrating the kinetic equation with the tensors $\overset{(I)}{\mathbf{N}}(\mathbf{k})$ is formally identical to that of the extended system of balance equations for a rigid conductor of heat.

The kinetic entropy of the gas of Bose particles governed by the kinetic equation (3.1) is given by

$$(3.12)\quad \eta(\mathbf{x},t) = \int \{[f(\mathbf{x},\mathbf{k},t)+1]\ln[f(\mathbf{x},\mathbf{k},t)+1] - f(\mathbf{x},\mathbf{k},t)\ln f(\mathbf{x},\mathbf{k},t)\}\,\frac{d^3\mathbf{k}}{(2\pi)^3}.$$

For every solution of the kinetic equation, the balance of entropy holds

$$(3.13)\qquad \partial_t \eta + \operatorname{div}\mathbf{h} = \sigma$$

with the entropy flux $\mathbf{h}$ given by

$$(3.14)\quad \mathbf{h}(\mathbf{x},t) = \int \{[f(\mathbf{x},\mathbf{k},t)+1]\ln[f(\mathbf{x},\mathbf{k},t)+1] - f(\mathbf{x},\mathbf{k},t)\ln f(\mathbf{x},\mathbf{k},t)\}\,\nabla_{\mathbf{k}}\omega(\mathbf{k})\,\frac{d^3\mathbf{k}}{(2\pi)^3},$$

and the entropy production $\sigma$ related to the collision integral of the kinetic equation given by the formula

$$(3.15)\quad \sigma(\mathbf{x},t) = \int \{\ln[f(\mathbf{x},\mathbf{k},t)+1] - \ln f(\mathbf{x},\mathbf{k},t)\}\,J(f(\mathbf{x},\mathbf{k},t))\,\frac{d^3\mathbf{k}}{(2\pi)^3}.$$

If for the given collision integral (3.15) is nonnegative for every distribution function then we say that the considered kinetic equation satisfies the Boltzmann $H$-Theorem.

## 4. Field equations based on the maximum entropy principle

One of the fundamental tools of statistical physics is the variational procedure of determining the microstate corresponding to the given macroscopic data. Such a microstate is obtained as a solution of the variational problem of the entropy maximization under the constraints corresponding to the values of macroscopic quantities, ZUBAREV [22]. Hence, it is possible to determine the phonon distribution function which maximizes the entropy (3.12) for fixed values of moments. Taking the values of the moments $\overset{(0)}{\mathbf{Q}}, \overset{(1)}{\mathbf{Q}}, \ldots, \overset{(L-1)}{\mathbf{Q}}$ as the constraints in the following variational problem

$$(4.1)\quad \delta\left\{\int [(f+1)\ln(f+1) - f\ln f]\,\frac{d^3\mathbf{k}}{(2\pi)^3} - \sum_{I=0}^{L-1} \overset{(I)}{\mathbf{\Lambda}} \odot \int \overset{(I)}{\mathbf{N}}(\mathbf{k}) f\,\frac{d^3\mathbf{k}}{(2\pi)^3}\right\} = 0,$$

we obtain the distribution function of the form

$$(4.2)\qquad f(\xi(\overset{(M)}{\Lambda},\mathbf{k})) = [\exp\xi(\overset{(M)}{\Lambda},\mathbf{k})-1]^{-1}, \qquad \xi(\overset{(M)}{\Lambda},\mathbf{k}) = \sum_{I=0}^{L-1} \overset{(I)}{\Lambda} \odot \overset{(I)}{\mathbf{N}}(\mathbf{k}),$$

which satisfies the necessary condition for the extremum of the functional. From the reasoning analogical to that given by DREYER [12, Eqs. (4.10), (4.11)] it follows that the function (4.2) corresponds to the maximum and therefore it maximizes the entropy under the assumed constraints. It can be observed that the distribution function (4.2) also maximizes the functional in the variational problem

$$(4.3)\qquad \delta\left\{\int[(f+1)\ln(f+1)-f\ln f]\,[\nabla_{\mathbf{k}}\omega(\mathbf{k})]\cdot\mathbf{n}\,\frac{d^3\mathbf{k}}{(2\pi)^3} - \sum_{I=0}^{L-1}\overset{(I)}{\Lambda}\odot\int \overset{(I+1)}{\mathbf{N}}(\mathbf{k})\mathbf{n}f\frac{d^3\mathbf{k}}{(2\pi)^3}\right\}=0,$$

where $\mathbf{n}$ is an arbitrary unit vector and $\overset{(I)}{\Lambda}$ are the Lagrange multipliers. The variational problem (4.3) determines the maximal value of the entropy flux in the arbitrary direction $\mathbf{n}$ under the constraints corresponding to the fixed values of projections of the fluxes $\overset{(1)}{\mathbf{Q}},\overset{(2)}{\mathbf{Q}},...\overset{(L)}{\mathbf{Q}}$ onto the same direction $\mathbf{n}$. As it follows from Eqs. (3.10), (3.11), the moments $\overset{(1)}{\mathbf{Q}},\overset{(2)}{\mathbf{Q}},...,\overset{(L)}{\mathbf{Q}}$ are fluxes in the considered system of moment equations. Hence, the distribution function (4.2) not only maximizes the entropy for the fixed values of $\overset{(0)}{\mathbf{Q}},\overset{(1)}{\mathbf{Q}},..., \overset{(L-1)}{\mathbf{Q}}$, but also maximizes the entropy flux in the arbitrary direction $\mathbf{n}$, for the fixed values of projections of the fluxes of these moments $\overset{(1)}{\mathbf{Q}}\cdot\mathbf{n},\overset{(2)}{\mathbf{Q}}\cdot\mathbf{n},...,\overset{(L)}{\mathbf{Q}}\cdot\mathbf{n}$ onto the same direction $\mathbf{n}$. In [13] the same maximizing property has been shown in the case of the low-temperature phonon gas hydrodynamics.

Assuming that the Lagrange multipliers $\overset{(I)}{\Lambda}$ are functions of the independent variables $\mathbf{x},t$ and inserting the distribution function (4.2) into the system (3.11) we obtain the closed system of equations with respect to the fields $\overset{(I)}{\Lambda}(\mathbf{x},t)$

$$(4.4)\qquad \partial_t\overset{(K)}{\tilde{\mathbf{Q}}}(\overset{(I)}{\Lambda}) + \operatorname{div}\overset{(K+1)}{\tilde{\mathbf{Q}}}(\overset{(I)}{\Lambda}) = \overset{(K)}{\tilde{\mathbf{P}}}(\overset{(I)}{\Lambda}), \qquad K=0,1,...,L-1,$$

where

$$\text{(4.5)}\qquad \begin{aligned} \overset{(K)\,(I)}{\tilde{\mathbf{Q}}}(\Lambda) &= \int \overset{(K)}{\mathbf{N}}(\mathbf{k}) f(\xi(\overset{(I)}{\Lambda},\mathbf{k}))\, \frac{d^3\mathbf{k}}{(2\pi)^3}, \\ \overset{(K)\,(I)}{\tilde{\mathbf{P}}}(\Lambda) &= \int \overset{(K)}{\mathbf{N}}(\mathbf{k})\, J(f(\xi(\overset{(I)}{\Lambda},\mathbf{k})))\, \frac{d^3\mathbf{k}}{(2\pi)^3}. \end{aligned}$$

According to the kinetic expressions (3.12), (3.13) and (3.15), entropy, entropy flux and the entropy production in a phonon gas described by the distribution function (4.2) are given by the following integrals

$$\text{(4.6)}\qquad \begin{aligned} \tilde{\eta}(\overset{(I)}{\Lambda}) &= \int \Big\{ [f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) + 1] \ln [f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) + 1] \\ &\qquad - f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) \ln f(\xi(\overset{(I)}{\Lambda},\mathbf{k}) \Big\} \frac{d^3\mathbf{k}}{(2\pi)^3}, \\ \tilde{\mathbf{h}}(\overset{(I)}{\Lambda}) &= \int \Big\{ [f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) + 1] \ln [f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) + 1] \\ &\qquad - f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) \ln f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) \Big\} \nabla_{\mathbf{k}}\omega(\mathbf{k}) \frac{d^3\mathbf{k}}{(2\pi)^3}, \\ \tilde{\sigma}(\overset{(I)}{\Lambda}) &= \int \Big\{ \ln [f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) + 1] - \ln f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) \Big\} J(f(\xi(\overset{(I)}{\Lambda},\mathbf{k}))) \frac{d^3\mathbf{k}}{(2\pi)^3}. \end{aligned}$$

The integrals (4.5) and (4.6) are convergent for all physically reasonable dispersion relations. If the domain of phonon wave-vectors $\mathbf{k}$ is approximated by the whole three-dimensional Euclidean vector space, the requirement of the convergence of Eqs. (4.5), (4.6) imposes the restrictions on the admissible despersion relations.

With the aid of Eqs. $(4.5)_1$ and $(4.6)_{1,2}$, the following scalar and vector functions of the arguments $\overset{(I)}{\Lambda}$ can be introduced

$$\text{(4.7)}\qquad \begin{aligned} \Phi(\overset{(I)}{\Lambda}) &= \sum_{J=0}^{L-1} \overset{(J)}{\Lambda} \odot \overset{(J)\,(I)}{\tilde{\mathbf{Q}}}(\Lambda) - \tilde{\eta}(\overset{(I)}{\Lambda}) = -\int \ln [f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) + 1] \frac{d^3\mathbf{k}}{(2\pi)^3}, \\ \boldsymbol{\Psi}(\overset{(I)}{\Lambda}) &= \sum_{J=0}^{L-1} \underset{(1,J+1)}{\mathrm{Tr}} \dots \underset{(J,2J)}{\mathrm{Tr}} \overset{(J)}{\Lambda} \otimes \overset{(J+1)\,(I)}{\tilde{\mathbf{Q}}}(\Lambda) - \tilde{\mathbf{h}}(\overset{(I)}{\Lambda}) = \\ &= -\int \ln [f(\xi(\overset{(I)}{\Lambda},\mathbf{k})) + 1] \nabla_{\mathbf{k}}\omega(\mathbf{k}) \frac{d^3\mathbf{k}}{(2\pi)^3}. \end{aligned}$$

In view of Eqs. $(4.5)_1$, $(4.6)_{1,2}$ and (4.2), differentiation of Eqs. (4.7) leads to the identities

$$\text{(4.8)} \qquad \overset{(J)}{\tilde{\mathbf{Q}}}(\overset{(I)}{\mathbf{\Lambda}}) = D_{(I)}\underset{\mathbf{\Lambda}}{}\Phi(\overset{(I)}{\mathbf{\Lambda}}), \qquad \overset{(J+1)}{\tilde{\mathbf{Q}}}(\overset{(I)}{\mathbf{\Lambda}}) = D_{(J)}\underset{\mathbf{\Lambda}}{}\mathbf{\Psi}(\overset{(I)}{\mathbf{\Lambda}}),$$

where the derivatives are understood as the derivatives of tensor functions with respect to tensor arguments (Frechet derivatives). It is apparent that substitution of Eqs. (4.8) into (4.4) results in the symmetric conservative system with respect to the fields of Lagrange multipliers

$$\text{(4.9)} \qquad \partial_t\left[D_{(M)}\underset{\mathbf{\Lambda}}{}\Phi(\overset{(I)}{\mathbf{\Lambda}})\right] + \operatorname{div}\left[D_{(M)}\underset{\mathbf{\Lambda}}{}\mathbf{\Psi}(\overset{(I)}{\mathbf{\Lambda}})\right] = \overset{(M)}{\tilde{\mathbf{P}}}(\overset{(I)}{\mathbf{\Lambda}}).$$

Taking scalar products of the subsequent equations of the system (4.9) with the corresponding subsequent Lagrange multipliers $\overset{(0)}{\mathbf{\Lambda}}, \overset{(1)}{\mathbf{\Lambda}}, \ldots, \overset{(L-1)}{\mathbf{\Lambda}}$ and adding the results, we obtain the folowing conservation equation

$$\text{(4.10)} \qquad \partial_t\left\{\sum_{J=0}^{L-1}\overset{(J)}{\mathbf{\Lambda}} \odot \left[D_{(J)}\underset{\mathbf{\Lambda}}{}\Phi(\overset{(I)}{\mathbf{\Lambda}})\right] - \Phi(\overset{(I)}{\mathbf{\Lambda}})\right\}$$

$$+ \operatorname{div}\left\{\sum_{J=0}^{L-1}\underset{(1,J+1)}{\operatorname{Tr}} \ldots \underset{(J,2J)}{\operatorname{Tr}}\overset{(J)}{\mathbf{\Lambda}} \otimes \left[D_{(J)}\underset{\mathbf{\Lambda}}{}\mathbf{\Psi}(\overset{(I)}{\mathbf{\Lambda}})\right] - \mathbf{\Psi}(\overset{(I)}{\mathbf{\Lambda}})\right\} = \sum_{J=0}^{L-1}\overset{(J)}{\mathbf{\Lambda}} \odot \overset{(J)}{\tilde{\mathbf{P}}}(\overset{(I)}{\mathbf{\Lambda}}).$$

On account of Eqs. (4.2) and (4.5), the integral formula (4.6) can be rearranged to the form

$$\text{(4.11)} \qquad \tilde{\sigma}(\overset{(I)}{\mathbf{\Lambda}}) = \int \ln(\exp\xi(\overset{(I)}{\mathbf{\Lambda}},\mathbf{k}))\, J(f(\xi(\overset{(I)}{\mathbf{\Lambda}},\mathbf{k})))\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$

$$= \int\left[\sum_{J=0}^{L-1}\overset{(J)}{\mathbf{\Lambda}} \odot \overset{(J)}{\mathbf{N}}(\mathbf{k})\right] J(f(\xi(\overset{(I)}{\mathbf{\Lambda}},\mathbf{k})))\,\frac{d^3\mathbf{k}}{(2\pi)^3} = \sum_{J=0}^{L-1}\overset{(J)}{\mathbf{\Lambda}} \odot \overset{(J)}{\tilde{\mathbf{P}}}(\overset{(I)}{\mathbf{\Lambda}}).$$

If it is assumed that the $H$-Boltzmann theorem holds for the kinetic equation (3.1) then the expression (4.11) must be nonnegative since in this case, the integral formula (3.15) is nonnegative for all distribution functions.

It follows from Eqs. (4.6), (4.7), (4.11), (3.12), (3.13) and (3.14) that the additional conservation equation (4.10) implied by the system (4.9) corresponds to the kinetic balance of entropy of the phonon gas which is described by the distribution function (4.2)

$$\partial_t \tilde{\eta}(\overset{(I)}{\Lambda}) + \operatorname{div}\tilde{\mathbf{h}}(\overset{(I)}{\Lambda}) = \tilde{\sigma}(\overset{(I)}{\Lambda}). \tag{4.12}$$

The Liu procedure of extended thermodynamics can be applied to the system (4.9) written in the form Eq. (4.4) together with the additional conservation equation (4.10) written in the form of Eq. (4.12). Then, Eq. (4.11) plays the role of the residual equality and it implies that $\overset{(I)}{\Lambda}$, $I = 0,1,\ldots,L-1$ correspond to Liu multipliers. As a consequence, the Liu procedure yields the relations

$$\begin{aligned} \underset{\Lambda}{D_{(J)}}\tilde{\eta}(\overset{(I)}{\Lambda}) &= \sum_{M=0}^{L-1} \overset{(M)}{\Lambda} \odot \underset{\Lambda}{D_{(J)}} \overset{(M)}{\tilde{\mathbf{Q}}}(\overset{(I)}{\Lambda}), \\ \underset{\Lambda}{D_{(J)}}\tilde{\mathbf{h}}(\overset{(I)}{\Lambda}) &= \sum_{M=0}^{L-1} \underset{(1,M+1)}{\mathrm{Tr}} \ldots \underset{(M,2M)}{\mathrm{Tr}} \overset{(M)}{\Lambda} \otimes \underset{\Lambda}{D_{(J)}} \overset{(M+1)}{\tilde{\mathbf{Q}}}(\overset{(I)}{\Lambda}) \end{aligned} \tag{4.13}$$

and the differentials $d\tilde{\eta}$ and $d\tilde{\mathbf{h}}$

$$\begin{aligned} d\tilde{\eta} &= \sum_{I=0}^{L-1} \overset{(I)}{\Lambda} \odot d\overset{(I)}{\tilde{\mathbf{Q}}}, \\ d\tilde{\mathbf{h}} &= \sum_{I=0}^{L-1} \underset{(1,I+1)}{\mathrm{Tr}} \ldots \underset{(I,2I)}{\mathrm{Tr}} \overset{(I)}{\Lambda} \otimes d \overset{(I+1)}{\tilde{\mathbf{Q}}}, \end{aligned} \tag{4.14}$$

assume the form analogous to Eqs. (2.7), (2.9), which follows from the application of the entropy principle to the extended thermodynamics of a rigid conductor of heat. In Eq. (4.14), the Lagrange multipliers $\overset{(I)}{\Lambda}$ stand in place of the Liu multipliers $\overset{(I)}{\lambda}$. The relations (4.14) are the counterparts of the relations [12, Eqs. (4.15), (4.16)] derived for the case of degenerate material gases. With the aid of the reasoning analogous to that applied in [13], it can be proved that the distribution function (4.2) is the only distribution function parametrized by the tensor fields $\overset{(0)}{\Lambda}, \overset{(1)}{\Lambda}, \ldots, \overset{(L-1)}{\Lambda}$ which, after simultaneous substitution into the system of moment equations (3.9) and into the kinetic balance of entropy (3.12), (3.13), (3.14), gives the consistent system of $L+1$ tensor field equations (4.9), (4.10) with respect to the tensor fields $\overset{(0)}{\Lambda}, \overset{(1)}{\Lambda}, \ldots, \overset{(L-1)}{\Lambda}$. Due to the unique correspondence between the distribution function which maximizes the entropy and the symmetric conservative system of equations of the extended hydrodynamics of phonon gas, and due to the fact that the symmetric conservative systems are consistent with the entropy principle in the sence of Müller, it was possible to obtain the results analogous to that of DREYER [12] in a much simpler way. It should be noted that the integral identities analogous to Eqs. (4.7) and (4.8) have been also derived by DREYER [12, Eqs $(4.13)_1$, $(4.14)_1$, (A.1.1), (A.1.2)] but they have not been used to obtain the symmetric conservative system of field equations.

## 5. Symmetric conservative system generated by single potential

The group decomposition of symmetric Euclidean tensors of arbitrary order is well known [23]. The symmetric tensor of even order $N$ can be represented as a sum of $1/2N+1$ following tensors: the symmetric tensor of the order $N$ transforming under rotations like a scalar and the symmetric tensors of order $N$ transforming under rotations like traceless symmetric tensors of the orders $2,4,...,N$. Analogously, the symmetric tensor of odd order $M$ can be decomposed into the sum of $1/2(M+1)$ tensors of order $M$ with the following components: the symmetric tensor transforming under rotations like a vector, and the symmetric tensors transforming under rotations like traceless symmetric tensors of the orders $3,5,...,M$. Components of this decomposition are mutually orthogonal and symmetric tensors of the order $P$ transforming under rotations like traceless symmetric tensors of the order $\Sigma$ form a vector space of the dimension $2\Sigma+1$. In this space, the orthonormal basis can be introduced and the tensors, which are its components, satisfy the relations

$$\mathbf{E}^{P\Sigma}{}_{\alpha} \odot \mathbf{E}^{P\Sigma}{}_{\beta} = \delta_{\alpha\beta}, \qquad \alpha,\beta = 1,2,...,2\Sigma+1, \tag{5.1}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker symbol. With the aid of the group decomposition, an arbitrary symmetric tensor $\overset{(R)}{\mathbf{A}}$ of the order $R$ can be written in the following form

$$\overset{(R)}{\mathbf{A}} = \mathrm{A}_{R\Sigma}{}^{\gamma}\, \mathbf{E}^{R\Sigma}{}_{\gamma}, \tag{5.2}$$

where $\Sigma = 0,2,...,R$ for $R$ even and $\Sigma = 1,3,...,R$ for $R$ odd and $\gamma = 1,2,...,2\Sigma+1$. The usual summational convention is understood over repeated lower and upper indices.

Applying the group decomposition (5.1), (5.2) to the fields of multipliers $\overset{(I)}{\Lambda}$, $I = 0,1,...,L-1$, we obtain the following representations

$$(5.3)\qquad \overset{(\Gamma)}{\mathbf{\Lambda}}(x^i,t) = \Lambda_{\Gamma\Sigma}{}^{\gamma}(x^i,t)\mathbf{E}^{\Gamma\Sigma}{}_{\gamma}.$$

The expressions of the form div $\overset{(S)}{\tilde{\mathbf{Q}}}(\overset{(\Gamma)}{\mathbf{\Lambda}})$ appear in the left hand side of the system (4.4). The symmetric tensors $\overset{(S)}{\tilde{\mathbf{Q}}}(\overset{(\Gamma)}{\mathbf{\Lambda}})$, $S = 1,2,\ldots,L$ are given by the integral formulae $(4.5)_1$ and, at the same time, they can be represented as the derivatives of the vector potential $\mathbf{\Psi}(\overset{(\Gamma)}{\mathbf{\Lambda}})$ with respect to the subsequent multipliers $\overset{(S-1)}{\mathbf{\Lambda}}$. Introducing the orthonormal basis $\{\mathbf{e}_i\}$ of the Cartesian coordinate system $\{x^i\}$, $i = 1,2,3$ for the representation of the vector $\mathbf{\Psi}$ and employing the group decomposition (5.3) of the multiplier $\overset{(S-1)}{\mathbf{\Lambda}}$, we obtain

$$(5.4)\qquad \underset{\mathbf{\Lambda}}{D}{}_{(S-1)}\mathbf{\Psi}(\overset{(\Gamma)}{\mathbf{\Lambda}}) = \frac{\partial \Psi^i}{\partial \Lambda_{S-1\Sigma}{}^{\gamma}}\,\mathbf{e}_i \otimes \mathbf{E}_{S-1\Sigma}{}^{\gamma} = \frac{\partial \Psi^i}{\partial \Lambda_{S-1\Sigma}{}^{\gamma}}\,\mathbf{E}_{S-1\Sigma}{}^{\gamma} \otimes \mathbf{e}_i.$$

In Eq. (5.4), the symmetry of $\underset{\mathbf{\Lambda}}{D}{}_{(S-1)}\mathbf{\Psi}(\overset{(\Gamma)}{\mathbf{\Lambda}})$ has been taken into account. Hence, we obtain from Eq. (5.4)

$$(5.5)\qquad \operatorname{div}\overset{(S)}{\tilde{\mathbf{Q}}}(\overset{(\Gamma)}{\mathbf{\Lambda}}) = \operatorname{div}\left[\underset{\mathbf{\Lambda}}{D}{}_{(S-1)}\mathbf{\Psi}(\overset{(\Gamma)}{\mathbf{\Lambda}})\right] = \left[\frac{\partial}{\partial x^i}\left(\frac{\partial \Psi^i}{\partial \Lambda_{S-1\Sigma}{}^{\gamma}}\right)\right]\mathbf{E}_{S-1\Sigma}{}^{\gamma}.$$

The tensors $\overset{(P)}{\tilde{\mathbf{Q}}}(\overset{(S)}{\mathbf{\Lambda}})$, $p = 0,1,\ldots,L-1$ are differentiated with respect to time in the left-hand side of the system (4.4). These tensors can be represented as derivatives of the potential $\Phi(\overset{(\Gamma)}{\mathbf{\Lambda}})$ with respect to multipliers $\overset{(P)}{\mathbf{\Lambda}}$ and, as symmetric tensors, they can be decomposed according to Eq. (5.2). Hence, the time derivatives in the system (4.4) can be written in the following form

$$(5.6)\qquad \partial_t \overset{(P)}{\tilde{\mathbf{Q}}} = \partial_t\, \underset{\mathbf{\Lambda}}{D}{}_{(P)}\Phi(\overset{(\Gamma)}{\mathbf{\Lambda}}) = \left[\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \Lambda_{P\Sigma}{}^{\gamma}}\right)\right]\mathbf{E}_{P\Sigma}{}^{\gamma}.$$

Taking into account Eqs. (5.5) and (5.6) and applying the group decomposition (5.2) to the right hand side, which is a symmetric tensor given by Eq. $(4.5)_2$, we rearrange any equation of the system (4.4) to the form

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \Lambda_{R\Sigma}{}^{\gamma}}\right)+\frac{\partial}{\partial x^{i}}\left(\frac{\partial \Psi^{i}}{\partial \Lambda_{R\Sigma}{}^{\gamma}}\right)-P^{R\Sigma}{}_{\gamma}\right]\mathbf{E}_{R\Sigma}{}^{\gamma}=\mathbf{0}. \tag{5.7}$$

Since the base tensors $\mathbf{E}_{R\Gamma}{}^{\gamma}$ are linearly independent, Eq. (5.7) is equivalent to the following system of equations with respect to the components $\Lambda_{R\Sigma}{}^{\gamma}$ of the group decomposition (5.2) of the multiplier $\overset{(R)}{\mathbf{\Lambda}}$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \Lambda_{R\Sigma}{}^{\gamma}}\right)+\frac{\partial}{\partial x^{i}}\left(\frac{\partial \Psi^{i}}{\partial \Lambda_{R\Sigma}{}^{\gamma}}\right)=P^{R\Sigma}{}_{\gamma}, \tag{5.8}$$

where $\Sigma=0, 2, ..., R$ for $R$ even and $\Sigma=1, 3, ..., R$ for $R$ odd. Applying this procedure to all $L$ equations of the system (4.4), we obtain the equivalent system with respect to the componets of the group decomposition (5.2) of all multipliers $\overset{(I)}{\mathbf{\Lambda}}$

$$\begin{gathered}\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial \Lambda_{I\Sigma}{}^{\gamma}}\right)+\frac{\partial}{\partial x^{i}}\left(\frac{\partial \Psi^{i}}{\partial \Lambda_{I\Sigma}{}^{\gamma}}\right)=P^{I\Sigma}{}_{\gamma},\\ I=0, 1, 2, ..., L-1,\\ \Sigma=\begin{cases}0, 2, 4, ..., I & \text{for } I \text{ even},\\ 1, 3, 5, ..., I & \text{for } I \text{ odd},\end{cases}\\ \gamma=1, 2, 3, ..., 2\Sigma+1.\end{gathered} \tag{5.9}$$

Multiplying the obtained system by the row vector composed of components of the group decomposition of multipliers $[\Lambda_{00}{}^{1}, \Lambda_{11}{}^{1}, \Lambda_{11}{}^{2}, ..., \Lambda_{I\Sigma}{}^{\gamma}, ... \Lambda_{L-1\,L-1}{}^{2L-1}]$ we arrive at the additional conservation equation

$$\frac{\partial}{\partial_{t}}\left(\Lambda_{I\Sigma}{}^{\gamma}\frac{\partial \Phi}{\partial \Lambda_{I\Sigma}{}^{\gamma}}-\Phi\right)+\frac{\partial}{\partial x^{i}}\left(\Lambda_{I\Sigma}{}^{\gamma}\frac{\partial \Psi^{i}}{\partial \Lambda_{I\Sigma}{}^{\gamma}}-\Psi^{i}\right)=\Lambda_{I\Sigma}{}^{\gamma}P^{I\Sigma}{}_{\gamma}, \tag{5.10}$$

which, as it follows from Eqs. (5.3), (5.4), (5.5), (5.6), corresponds to the balance of entropy (4.10) written in components of the group decomposition of symmetric tensors. In Eq. (5.10), the usual summation convention is understood over repeated lower and upper indices.

The same group decomposition (5.1), (5.2) can be applied to the symmetric tensors $\overset{(I)}{\mathbf{N}}(\mathbf{k})$ given by

$$\overset{(I)}{\mathbf{N}}(\mathbf{k})=N_{I\Sigma}{}^{\alpha}(\mathbf{k})\,\mathbf{E}^{I\Sigma}{}_{\alpha}. \tag{5.11}$$

Emploing the decompositions (5.3) and (5.11) and taking into account the orthogonality of the base tensors, we express the argument $\xi(\overset{(\Gamma)}{\Lambda},\mathbf{k})$ of the distribution function as a function of the components of the group decomposition of $\overset{(\Gamma)}{\Lambda}$

$$(5.12)\qquad \xi(\overset{(\Gamma)}{\Lambda},\mathbf{k}) = \hat{\xi}(\Lambda_{I\Sigma}{}^{\gamma},\mathbf{k}) = \sum_{J=0}^{L-1} \overset{(J)}{\Lambda} \odot \overset{(J)}{\mathbf{N}}(\mathbf{k}) = \sum_{J=0}^{L-1} \sum_{\Delta\in\sigma(J)} \sum_{\alpha=1}^{2\Delta+1} \Lambda_{J\Delta}{}^{\alpha} N_{J\Delta}{}^{\alpha}(\mathbf{k}),$$

where $\sigma(J) = \{0, 2, ..., J\}$ for $J$ even and $\sigma(J) = \{1, 3, ..., J\}$ for $J$ odd. Let $\overset{(N)}{\Lambda}$ be an arbitrary Lagrange multiplier of even rank, except $\overset{(L-1)}{\Lambda}$ if $L$ is odd. In particular, $\overset{(N)}{\Lambda} = \overset{(0)}{\Lambda}$ can be taken. Substituting Eq. (5.12) into the distribution function (4.2) and differentiating the result with respect to the component $\Lambda_{N0}{}^{1}$ of $\overset{(N)}{\Lambda}$, we calculate the derivatives of components $\Psi^{i}$ of potential $\Psi(\overset{(\Gamma)}{\Lambda})$ with respect to $\Lambda_{N0}{}^{1}$

$$(5.13)\qquad \frac{\partial \Psi^{i}(\Lambda_{M\Sigma}{}^{\gamma})}{\partial \Lambda_{N0}{}^{1}} = \frac{\partial}{\partial \Lambda_{N0}{}^{1}} \left\{ -\int \frac{\partial \omega(\mathbf{k})}{\partial \mathbf{k}_{i}} \ln\left[1 + f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k}))\right] \frac{d^{3}\mathbf{k}}{(2\pi)^{3}} \right\}$$
$$= \int \frac{\partial \omega(\mathbf{k})}{\partial k_{i}} N_{N0}{}^{1}(\mathbf{k}) f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k})) \frac{d^{3}\mathbf{k}}{(2\pi)^{3}}.$$

Analogously, the derivatives of the scalar potential $\Phi(\overset{(\Gamma)}{\Lambda})$ and the derivatives of the components $\Psi^{i}(\overset{(\Gamma)}{\Lambda})$ with respect to the components $\Lambda_{N+1\,1}{}^{j} = 1, 2, 3$ of $\overset{(N+1)}{\Lambda}$ can be determined

$$\frac{\partial \Phi(\Lambda_{M\Sigma}{}^{\gamma})}{\partial \Lambda_{N+1\,1}{}^{j}} = \frac{\partial}{\partial \Lambda_{N+1\,1}{}^{j}} \left\{ -\int \ln\left[1 + f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k}))\right] \frac{d^{3}\mathbf{k}}{(2\pi)^{3}} \right\}$$
$$= \int N_{N+1\,1}{}^{j}(\mathbf{k}) f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k})) \frac{d^{3}\mathbf{k}}{(2\pi)^{3}},$$

$$(5.14)\qquad \frac{\partial \Psi^{i}(\Lambda_{M\Sigma}{}^{\gamma})}{\partial \Lambda_{N+1\,1}{}^{j}} = \frac{\partial}{\partial \Lambda_{N+1\,1}{}^{j}} \left\{ -\int \frac{\partial \omega(\mathbf{k})}{\partial k_{i}} \ln\left[1 + f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k}))\right] \frac{d^{3}\mathbf{k}}{(2\pi)^{3}} \right\}$$
$$= \int \frac{\partial \omega(\mathbf{k})}{\partial k_{i}} N_{N+1\,1}{}^{j}(\mathbf{k}) f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k})) \frac{d^{3}\mathbf{k}}{(2\pi)^{3}}.$$

It follows from the properties of the group decomposition of symmetric tensors that, for symmetric tensors **A** and **B** of the even rank $I$ transforming under rotations like a scalar, the following relation holds

$$\mathbf{A}\odot\mathbf{B} = c_I(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{I}{2}}\mathbf{A})(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{I}{2}}\mathbf{B}) \tag{5.15}$$

and, for symmetric tensors **C** and **D** of the odd rank $J$ transforming under rotations like a vector, we have

$$\mathbf{C}\odot\mathbf{D} = c_J(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{J-1}{2}}\mathbf{C})\cdot(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{J-1}{2}}\mathbf{D}), \tag{5.16}$$

where $c_I$ and $c_J$ are the respective combinatorical constants.

On account of (5.15), (5.16), the base tensors $\mathbf{E}^{N0}{}_1$ and $\mathbf{E}^{N+1\,1}{}_i$ satisfy the relations

$$\begin{gathered}1 = \mathbf{E}^{N0}{}_1\odot\mathbf{E}^{N0}{}_1 = c_N(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\mathbf{E}^{N0}{}_1)^2,\\ \delta_{ij} = \mathbf{E}^{N+1\,1}{}_i\odot\mathbf{E}^{N+1\,1}{}_j = c_{N+1}(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\mathbf{E}^{N+1\,1}{}_i)(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\mathbf{E}^{N+1\,1}{}_j)\end{gathered} \tag{5.17}$$

and the group decomposition of $\overset{(N)}{\mathbf{N}}(\mathbf{k})$ and $\overset{(N+1)}{\mathbf{N}}(\mathbf{k})$ implies

$$\begin{gathered}\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\overset{(N)}{\mathbf{N}} = \underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}[N_{N0}{}^1\,\mathbf{E}^{N0}{}_1],\\ \underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\overset{(N+1)}{\mathbf{N}} = \underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}[N_{N+1\,1}{}^i\,\mathbf{E}^{N+1\,1}{}_i].\end{gathered} \tag{5.18}$$

In view of Eq. (3.8), it follows from Eqs. (5.11), (5.15), (5.16), (5.17) and (5.18) that

$$\begin{aligned}N_{N0}{}^1(\mathbf{k}) &= \overset{(N)}{\mathbf{N}}(\mathbf{k})\odot\mathbf{E}^{N0}{}_1 = [N_{N\Sigma}{}^\gamma(\mathbf{k})\,\mathbf{E}^{N\Sigma}{}_\gamma]\odot\mathbf{E}^{N0}{}_1\\ &= N_{N0}{}^1(\mathbf{k})\,\mathbf{E}^{N0}{}_1\odot\mathbf{E}^{N0}{}_1 = c_N(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\overset{(N)}{\mathbf{N}})(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\mathbf{E}^{N0}{}_1)\\ &= \sqrt{c_N}\,\hbar\,\omega(\mathbf{k})\,|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^N,\end{aligned} \tag{5.19}$$

(5.19) [cont.]
$$N_{N+1\,1}{}^{j}(\mathbf{k}) = \overset{(N+1)}{\mathbf{N}}(\mathbf{k}) \odot \mathbf{E}^{N+1\,1}{}_{j} = [N_{N+1\Sigma}{}^{\gamma}(\mathbf{k})\,\mathbf{E}^{N+1\Sigma}{}_{\gamma}] \odot \mathbf{E}^{N+1\,1}{}_{j}$$
$$= N_{N+1\,1}{}^{i}\,\mathbf{E}^{N+1\,1}{}_{i} \odot \mathbf{E}^{N+1\,1}{}_{j} = c_{N+1}(\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\overset{(N+1)}{\mathbf{N}}) \cdot (\underbrace{\mathrm{Tr}...\mathrm{Tr}}_{\frac{N}{2}}\mathbf{E}^{N+1\,1}{}_{j})$$
$$= \sqrt{c_{N+1}}\,\hbar\omega(\mathbf{k})\,|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^{N}\,\frac{\partial\omega(\mathbf{k})}{\partial k_j}.$$

Substitution of Eq. (5.19) into Eqs. (5.12) and (5.13) yields

$$\frac{\partial\Phi(\Lambda_{M\Sigma}{}^{\gamma})}{\partial\Lambda_{N+1\,1}{}^{j}} = \sqrt{c_{N+1}}\int\hbar\omega(\mathbf{k})|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^{N} f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k}))\,\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}^{j}}\,\frac{d^3\mathbf{k}}{(2\pi)^3},$$

(5.20)
$$\frac{\partial\Psi^{i}(\Lambda_{M\Sigma}{}^{\gamma})}{\partial\Lambda_{N0}{}^{1}} = \sqrt{c_{N}}\int\hbar\omega(\mathbf{k})|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^{N} f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k}))\,\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}_{i}}\,\frac{d^3\mathbf{k}}{(2\pi)^3},$$

$$\frac{\partial\Psi^{i}(\Lambda_{M\Sigma}{}^{\gamma})}{\partial\Lambda_{N+1\,1}{}^{j}} = \sqrt{c_{N+1}}\int\hbar\omega(\mathbf{k})|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^{N} f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k}))\,\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}_{i}}\,\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}^{j}}\,\frac{d^3\mathbf{k}}{(2\pi)^3},$$

and we obtain

(5.21)
$$\frac{\partial\Phi(\Lambda_{M\Sigma}{}^{\gamma})}{\partial\Lambda_{N+1\,1}{}^{j}} = \sqrt{\frac{c_{N+1}}{c_N}}\,\frac{\partial\Psi_{j}(\Lambda_{M\Sigma}{}^{\gamma})}{\partial\Lambda_{N0}{}^{1}},$$
$$\frac{\partial\Psi^{i}(\Lambda_{M\Sigma}{}^{\gamma})}{\partial\Lambda_{N+1\,1}{}^{j}} = \frac{\partial\Psi^{j}(\Lambda_{M\Sigma}{}^{\gamma})}{\partial\Lambda_{N+1\,1}{}^{i}}.$$

It follows from (5.21) that the transformation of fields $\Lambda_{I\Sigma}{}^{\gamma}$ should be introduced in order to rearrange the system (5.9) to the form of the symmetric conservative system generated by a single potential. Therefore, we define

(5.22)
$$\overline{\Lambda}_{I\Sigma}{}^{\gamma} = \begin{cases} \sqrt{\dfrac{c_{N+1}}{c_N}}\,\Lambda_{I\Sigma}{}^{\gamma} & \text{for} \quad I = N+1 \quad \text{and} \quad \Sigma = 1, \\ \Lambda_{I\Sigma}{}^{\gamma} & \text{for all remaining} \quad I,\Sigma, \end{cases}$$

and express the argument (5.12) of the distribution function as a function of new fields $\overline{\Lambda}_{I\Sigma}{}^{\gamma}$

$$\xi(\overset{(I)}{\mathbf{\Lambda}},\mathbf{k}) = \hat{\xi}(\Lambda_{I\Sigma}{}^{\gamma},\mathbf{k}) = \check{\xi}(\overline{\Lambda}_{I\Sigma}{}^{\gamma},\mathbf{k}) = \sum_{I=0}^{N}\sum_{\Sigma\in\sigma(I)}\sum_{\alpha=1}^{2\Sigma+1}\overline{\Lambda}_{I\Sigma}{}^{\alpha}N_{I\Sigma}{}^{\alpha}(\mathbf{k})$$
$$+\sum_{i=1}^{3}\overline{\Lambda}_{N+1\,1}{}^{i}\sqrt{\frac{\mathrm{c}_N}{\mathrm{c}_{N+1}}}N_{N+1\,1}{}^{i}(\mathbf{k})+\sum_{\Sigma\in\sigma(N+1)\setminus\{1\}}\sum_{\alpha=1}^{2\Sigma+1}\overline{\Lambda}_{N+1\Sigma}{}^{\alpha}N_{N+1\Sigma}{}^{\alpha}(\mathbf{k})$$
$$+\sum_{I=N+2}^{L-1}\sum_{\Sigma\in\sigma(I)}\sum_{\alpha=1}^{2\Sigma+1}\overline{\Lambda}_{I\Sigma}{}^{\alpha}N_{I\Sigma}{}^{\alpha}(\mathbf{k}). \tag{5.23}$$

It follows from Eq. (5.23) that the distribution function satisfies the condition

$$f(\xi(\overset{(I)}{\mathbf{\Lambda}}(x^i,t),\mathbf{k})) = f(\hat{\xi}(\Lambda_{I\Sigma}{}^{\alpha}(x^i,t),\mathbf{k})) = f(\check{\xi}(\overline{\Lambda}_{I\Sigma}{}^{\alpha}(x^i,t),\mathbf{k})) \tag{5.24}$$

under the transformation of the fields (5.22), for all $x^i,t$ and $\mathbf{k}$.

Substitution of Eq. (5.23) into Eq. (4.7) enables us to express the scalar potential $\Phi$ and the components $\Psi^i$ of the vector potential $\mathbf{\Psi}$ as functions of new variables $\overline{\Lambda}_{I\Sigma}{}^{\gamma}$

$$\begin{aligned}\Phi(\overset{(I)}{\mathbf{\Lambda}}) &= \Phi(\Lambda_{I\Sigma}{}^{\gamma}) = \check{\Phi}(\overline{\Lambda}_{I\Sigma}{}^{\gamma}),\\ \Psi^i(\overset{(I)}{\mathbf{\Lambda}}) &= \Psi^i(\Lambda_{I\Sigma}{}^{\gamma}) = \Psi^i(\overline{\Lambda}_{I\Sigma}{}^{\gamma}).\end{aligned} \tag{5.25}$$

On account of Eqs. (5.23), (5.19) and (5.24), the differentiation of integral formulae defining the potentials yields

$$\begin{aligned}\frac{\partial\check{\Phi}(\overline{\Lambda}_{I\Sigma}{}^{\gamma})}{\partial\overline{\Lambda}_{N+1\,1}{}^{j}} &= \sqrt{c_N}\int\hbar\omega(\mathbf{k})|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^N f(\check{\xi}(\overline{\Lambda}_{I\Sigma}{}^{\gamma},\mathbf{k}))\frac{\partial\omega(\mathbf{k})}{\partial k^j}\frac{d^3\mathbf{k}}{(2\pi)^3}\\ &\qquad\qquad\sqrt{\frac{c_N}{c_{N+1}}}\frac{\partial\Phi(\Lambda_{I\Sigma}{}^{\gamma})}{\partial\Lambda_{N+1\,1}{}^{j}},\\ \frac{\partial\check{\Psi}^i(\overline{\Lambda}_{I\Sigma}{}^{\gamma})}{\partial\overline{\Lambda}_{N0}{}^{1}} &= \sqrt{c_N}\int\hbar\omega(\mathbf{k})|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^N f(\check{\xi}(\overline{\Lambda}_{I\Sigma}{}^{\gamma},\mathbf{k}))\frac{\partial\omega(\mathbf{k})}{\partial k_i}\frac{d^3\mathbf{k}}{(2\pi)^3}\\ &\qquad\qquad = \frac{\partial\Psi^i(\Lambda_{I\Sigma}{}^{\gamma})}{\partial\Lambda_{N0}{}^{1}},\end{aligned} \tag{5.26}$$

(5.26) [cont.]
$$\frac{\partial \breve{\Psi}^i(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{N+1\,1}{}^j} = \sqrt{c_N}\int \hbar\omega(\mathbf{k})\left|\nabla_{\mathbf{k}}\omega(\mathbf{k})\right|^N f(\breve{\xi}(\overline{\Lambda}_{I\Sigma}{}^\gamma,\mathbf{k}))\,\frac{\partial\omega(\mathbf{k})}{\partial k_i}\,\frac{\partial\omega(\mathbf{k})}{\partial k^j}\,\frac{d^3\mathbf{k}}{(2\pi)^3}$$
$$= \sqrt{\frac{c_N}{c_{N+1}}}\,\frac{\partial \Psi^i(\Lambda_{I\Sigma}{}^\gamma)}{\partial \Lambda_{N+1\,1}{}^j},$$

and

(5.27)
$$\frac{\partial \breve{\Phi}(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{J\Delta}{}^\alpha} = \int N_{J\Delta}{}^\alpha(\mathbf{k}) f(\breve{\xi}(\overline{\Lambda}_{I\Sigma}{}^\gamma,\mathbf{k}))\,\frac{d^3\mathbf{k}}{(2\pi)^3} = \frac{\partial \Phi(\Lambda_{I\Sigma}{}^\gamma)}{\partial \Lambda_{J\Delta}{}^\alpha},$$
$$\frac{\partial \breve{\Psi}^i(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{J\Delta}{}^\alpha} = \int N_{J\Delta}{}^\alpha(\mathbf{k}) f(\breve{\xi}(\overline{\Lambda}_{I\Sigma}{}^\gamma,\mathbf{k}))\,\frac{\partial\omega(\mathbf{k})}{\partial k_i}\,\frac{d^3\mathbf{k}}{(2\pi)^3} = \frac{\partial \Psi^i(\Lambda_{I\Sigma}{}^\gamma)}{\partial \Lambda_{J\Delta}{}^\alpha},$$

for $J \neq N+1$ and $\Delta \neq 1$.

It follows from Eqs. (5.26) that, under the transformation of variables (5.22), the relations (5.21) transform to the form

(5.28)
$$\frac{\partial \breve{\Phi}(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{N+1\,1}{}^j} = \frac{\partial \breve{\Psi}_j(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{N0}{}^1},$$
$$\frac{\partial \breve{\Psi}^i(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{N+1\,1}{}^j} = \frac{\partial \breve{\Psi}^j(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{N+1\,1}{}^i}.$$

For fixed values of the arguments $\overline{\Lambda}_{I\Sigma}{}^\alpha$, $I \neq N$ and $\Sigma \neq 0$, $I \neq N+1$ and $\Sigma \neq 1$ (that is except $\overline{\Lambda}_{N0}{}^1$ and $\overline{\Lambda}_{N+1\,1}{}^i$, $i = 1, 2, 3$), a closed differential form of the arguments $\overline{\Lambda}_{N0}{}^1$, $\overline{\Lambda}_{N+1\,1}{}^i$ can be introduced

(5.29)
$$\Omega = \breve{\Phi}(\overline{\Lambda}_{I\Sigma}{}^\gamma)d\overline{\Lambda}_{N0}{}^1 + \sum_{i=1}^{3} \breve{\Psi}^i(\overline{\Lambda}_{I\Sigma}{}^\gamma)d\overline{\Lambda}_{N+1\,1}{}^i.$$

For any simply connected region of the domain of admissible values of $\overline{\Lambda}_{N0}{}^1$, $\overline{\Lambda}_{N+1\,1}{}^i$, and for fixed remaining $\overline{\Lambda}_{I\Sigma}{}^\gamma$, it has a potential $\mathscr{H}(\overline{\Lambda}_{I\Sigma}{}^\gamma)$ which can be determined from the following system of four equations

(5.30)
$$\frac{\partial \mathscr{H}(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{N0}{}^1} = \breve{\Phi}(\overline{\Lambda}_{I\Sigma}{}^\gamma),$$
$$\frac{\partial \mathscr{H}(\overline{\Lambda}_{I\Sigma}{}^\gamma)}{\partial \overline{\Lambda}_{N+1\,1}{}^i} = \breve{\Psi}_i(\overline{\Lambda}_{I\Sigma}{}^\gamma), \qquad i = 1, 2, 3.$$

It can be deduced from the integral formulae (4.7) defining potentials, from the form of the distribution function (4.2) and from the form of its argument $\breve{\xi}(\overline{\Lambda}_{I\Sigma}{}^{\gamma};\mathbf{k})$ that, with the aid of an auxiliary function $\alpha(\cdot)$ defined on $(0,\infty)$ by the integral

(5.31)
$$\alpha(y) = \int_0^y \frac{\ln(x+1)}{x(x+1)}\, dx,$$

the solution of the system (5.30) can be written in the form

(5.32)
$$\mathscr{H}(\overline{\Lambda}_{M\Sigma}{}^{\gamma}) = \int \frac{\alpha(f(\breve{\xi}(\overline{\Lambda}_{M\Sigma}{}^{\gamma},\mathbf{k})))}{\sqrt{c_N}\hbar\,\omega(\mathbf{k})|\nabla_{\mathbf{k}}\omega(\mathbf{k})|^N} \frac{d^3\mathbf{k}}{(2\pi)^3}.$$

Introducing the relations (5.26), (5.27), (5.30) to the system (5.9) and employing the transformation of variables (5.22), (5.23) in the expressions $(4.5)_1$ defining the right-hand side, we arrive at the equivalent system with respect to the fields $\overline{\Lambda}_{I\Sigma}{}^{\gamma}(\mathbf{x},t)$ the left-hand side of which is completely determined by a single function $\mathscr{H}(\overline{\Lambda}_{I\Sigma}{}^{\gamma})$

(5.33)
$$\frac{\partial}{\partial t}\left[\frac{\partial}{\overline{\Lambda}_{N0}{}^{1}}\left(\frac{\partial\mathscr{H}}{\partial\overline{\Lambda}_{I\Sigma}{}^{\gamma}}\right)\right] + \frac{\partial}{\partial x^i}\left[\frac{\partial}{\partial\overline{\Lambda}_{N+1\,1}{}^{i}}\left(\frac{\partial\mathscr{H}}{\partial\overline{\Lambda}_{I\Sigma}{}^{\gamma}}\right)\right] = P^{I\Sigma}{}_{\gamma},$$
$$I = 0, 1, 2, ..., N-1, N, N+2, ..., L-1$$
$$\Sigma = \begin{cases} 0, 2, 4, ..., I & \text{for } I \text{ even}, \\ 1, 3, 5, ..., I & \text{for } I \text{ odd}, \end{cases}$$
$$\gamma = 1, 2, ..., 2\Sigma+1,$$
$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial\overline{\Lambda}_{N0}{}^{1}}\left(\frac{\partial\mathscr{H}}{\partial\overline{\Lambda}_{N+1\,1}{}^{j}}\right)\right] + \frac{\partial}{\partial x^i}\left[\frac{\partial}{\partial\overline{\Lambda}_{N+1\,1}{}^{i}}\left(\frac{\partial\mathscr{H}}{\partial\overline{\Lambda}_{N+1\,1}{}^{j}}\right)\right] = \sqrt{\frac{c_N}{c_{N+1}}}\, P^{N+1\,1}{}_{j},$$
$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial\overline{\Lambda}_{N0}{}^{1}}\left(\frac{\partial\mathscr{H}}{\partial\overline{\Lambda}_{N+1\Delta}{}^{\beta}}\right)\right] + \frac{\partial}{\partial x^i}\left[\frac{\partial}{\partial\overline{\Lambda}_{N+1\,1}{}^{i}}\left(\frac{\partial\mathscr{H}}{\partial\overline{\Lambda}_{N+1\Delta}{}^{\beta}}\right)\right] = P^{N+1\Delta}{}_{\beta},$$
$$\Delta = 3, 5, ..., N+1,$$
$$\beta = 1, 2, ..., 2\Delta+1.$$

Analogously as in the case of the additional conservation equation (5.10) implied by the system (5.9), the additional conservation equation implied by Eqs. (5.33) results from the multiplication of the system (5.33) by the row vector composed of the components $\overline{\Lambda}_{M\Sigma}{}^{\gamma}$ which should be arranged in the order corresponding to the order of equations in the system (5.33). It assumes the following form:

$$(5.34)\qquad \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial \overline{\Lambda}_{N0}{}^{1}}\left[\overline{\Lambda}_{J\Delta}{}^{\alpha}\frac{\partial \mathcal{H}}{\partial \overline{\Lambda}_{J\Delta}{}^{\alpha}} - 2\mathcal{H}\right]\right\} + \frac{\partial}{\partial x^{i}}\left\{\frac{\partial}{\partial \overline{\Lambda}_{N+1\,1}{}^{i}}\left[\overline{\Lambda}_{J\Delta}{}^{\alpha}\frac{\partial \mathcal{H}}{\partial \overline{\Lambda}_{J\Delta}{}^{\alpha}} - 2\mathcal{H}\right]\right\} = \overline{\Lambda}_{J\Delta}{}^{\alpha} P^{J\Delta}{}_{\alpha},$$

where the summational convention is assumed over indices $i$, $J$, $\Delta$, $\alpha$ with the respective ranges defined in Eqs. (5.33).

It follows from Eqs. (5.22), (5.26) and (5.30) that (5.34) is equivalent to (5.10) and therefore it is the balance of entropy. This conclusion also directly follows from the fact that the balance of entropy corresponds to the kinetic balance of entropy (4.10) taken for the distribution function (4.2) and the transformation of variables (5.22) does not change the distribution function.

The system (4.4) is a system of $L$ tensor equations. If $L$ is even, then each of the multipliers $\overset{(0)}{\mathbf{\Lambda}}, \overset{(2)}{\mathbf{\Lambda}}, \dots, \overset{(L-2)}{\mathbf{\Lambda}}$ can be chosen as $\overset{(N)}{\mathbf{\Lambda}}$, and therefore the construction of the system (5.33) can be performed in $\frac{1}{2}L$ ways. If $L$ is odd, then $\overset{(0)}{\mathbf{\Lambda}}, \overset{(2)}{\mathbf{\Lambda}}, \dots, \overset{(L-3)}{\mathbf{\Lambda}}$ can be chosen as $\overset{(N)}{\mathbf{\Lambda}}$ and the same construction can be done in $\frac{1}{2}(L-1)$ ways. Hence, $\frac{1}{2}L$ or $\frac{1}{2}(L-1)$ equivalent symmetric conservative systems generated by a single potential can be found for even or odd $L$, respectively. In particular, if $\overset{(0)}{\mathbf{\Lambda}}$ is taken as $\overset{(N)}{\mathbf{\Lambda}}$, then $\overset{(N+1)}{\mathbf{\Lambda}} = \overset{(1)}{\mathbf{\Lambda}}$ is a vector and the group decomposition reduces to one component $\Lambda_{00}{}^{1}$ of $\overset{(0)}{\mathbf{\Lambda}}$ and three components $\Lambda_{11}{}^{j}$, $j = 1,2,3$, of $\overset{(1)}{\mathbf{\Lambda}}$. Then $c_0 = c_1 = 1$ in Eqs. (5.17), (5.19) and, as a consequence, Eqs. (5.17) and (5.19) reduce to identities. In this case, the potential $\mathcal{H}$ assumes the form

$$(5.35)\qquad \mathcal{H}(\Lambda_{M\Sigma}{}^{\gamma}) = -\int \frac{\alpha(f(\hat{\xi}(\Lambda_{M\Sigma}{}^{\gamma},\mathbf{k})))}{\hbar\omega(\mathbf{k})}\frac{d^{3}\mathbf{k}}{(2\pi)^{3}},$$

and corresponding system (5.3) simplifies to the following form:

$$(5.36)\qquad \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \Lambda_{00}{}^{1}}\left(\frac{\partial \mathcal{H}}{\partial \Lambda_{I\Sigma}{}^{\gamma}}\right)\right] + \frac{\partial}{\partial x^{i}}\left[\frac{\partial}{\partial \Lambda_{11}{}^{i}}\left(\frac{\partial \mathcal{H}}{\partial \Lambda_{I\Sigma}{}^{\gamma}}\right)\right] = P^{I\Sigma}{}_{\gamma},$$

(5.36) [cont.]

$$I = 0, 1, 2, ..., L-1,$$

$$\Sigma = \begin{cases} 0, 2, 4, ..., I & \text{for } I \text{ even}, \\ 1, 3, 5, ..., I & \text{for } I \text{ odd}, \end{cases}$$

$$\gamma = 1, 2, ..., 2\Sigma+1.$$

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