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BRIEF NOTES

On flow rules in plastic deformation

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THE PURPOSE of the present paper is to compare some consequences of the classical flow-rule of plasticity due to Levy and Mises with those of the flow equation recently proposed by DUBEY and BEDI [1]. The kinematics of a thin plate in large simple shear deformation is used for the purpose of comparison. The prescribed continued deformation is used to calculate the finite values of distortion and rotation. The prescribed continued deformation is used to extract information about (Eulerian) strain-rate and spins of the various axes defined in the body.

1. Introduction

IN THE CLASSICAL flow equation of plasticity, which can be traced to LEVY and MISES [2], the plastic component of increment in strain is assumed proportional to the deviatoric stress component. The constant of proportionality used in the flow equation is assumed to depend on the loading condition and hardening parameter. This parameter is determined from the stress-strain relationship obtained from experiments which may involve either uniaxial stressing as in tension test, or pure twisting as in torsion test, or a combination of tension-torsion loading. The type of experiment one should use for finding the hardening parameter is immaterial if the behaviour of the body is isotropic, in which case all loading histories must result in the same stress-strain equation. If, for example, the two curves obtained from the uniaxial and the torsion test are identical, the assumption of isotropic behaviour may be considered valid. If this is not the case, the assumption of isotropy is invalid and the material behaviour is anisotropic. In this situation, the hardening parameter obtained from one experimental data may not be used for situation involving some other type of loading, thus severely restricting the applicability of Mises flow rule.

It is now common to recast the flow equation in rate-form. The usual practice is to replace the increments in strain by (Eulerian) strain-rate, also known as the rate of deformation tensor. The implicit reason for this procedure is that it allows one to treat the integrated values of strain-rate components as strain. It is known, nevertheless, that the strain-rate is not a flux [3]; that is, strain-rate is not a rate of change of a strain-measure. The kinematics of a plate in simple shear will be used to show the error introduced in treating strain-rate as rate of logarithmic strain.

The modified flow rule proposed by DUBEY and BEDI [1] is obtained by replacing the strain-rate by an objective rate of strain. The two flow rules are of course identical in the reference configuration and also when deformation involves no rotation. In general cases of finite deformation though, their difference may be significant. One of the aims of the present work is to compare the two flow rules in those cases in which the deformation is not so restrictive.

In order not to introduce unnecessary complications, the elastic part of the deformation will be ignored in the following presentation and the material behaviour will be assumed rigid-plastic.

2. Kinematics

The finite deformation of a body is described in terms of the gradient

$$F_{ij} = \frac{\partial x_i}{\partial X_j}, \tag{2.1}$$

where x_i is the current position of a typical particle initially at X_i. In order to extract information about distortion and rotation from the above equation (2.1) it is expressed as a product of second order symmetric stretch tensors, **V** or **U**, and a proper rotation tensor, **R**. Thus

$$\mathbf{F} = \mathbf{V} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{U}, \tag{2.2}$$

where the symbol $(\cdot)$ denotes inner product; that is $(\mathbf{V} \cdot \mathbf{R})_{ij} = V_{ik} R_{kj}$. Note that V_{ij} represent the x_i-components of **V** and repeated suffix implies summation.

It is known that every second-order symmetric tensor has three principal values and directions. Let λ_i denote the principal values of the stretch tensor V_{ij} and let y_i be the principal directions. The axes which momentarily coincide with the principal directions are called the Eulerian axes. Note that in the principal axes technique used here, V_{ij} are the components of stretch on the fixed or x_i-axes; whereas U_{ij} are the components of the same stretch tensor on rotating or z_i-axes. The z_i-axes initially coincide with x_i and undergo the same rotation **R** as the body. Since z_i are associated with the body rotation, hence they may be called the body or **R**-rotated axes. A significant consequence of this interpretation is that it leads to a single set of principal directions on which the stretch components are denoted by either Λ_{ij} or λ_i, whereas U_{ij} and V_{ij} are its components on the rotated and the fixed axes , respectively. The principal and rotated-axes components retain their values on their respective axes during further rigid rotation of the body which causes these axes to rotate by the same amount. They are therefore objective.

Suppose that the components of the deformation gradient are prescribed:

$$\begin{aligned} F_{11} &= 1, \quad & F_{12} &= \gamma, \\ F_{21} &= 0, \quad & F_{22} &= 1. \end{aligned} \tag{2.3}$$

Substitute these values of the deformation gradient in (2.2) to obtain the rotation angle

$$\theta = \arctan(-\gamma/2) \tag{2.4}$$

and the x_i-components of stretch

$$\begin{aligned} V_{11} &= \frac{2+\gamma^2}{\sqrt{4+\gamma^2}}, \\ V_{12} &= \frac{\gamma}{\sqrt{4+\gamma^2}}, \\ V_{22} &= \frac{2}{\sqrt{4+\gamma^2}}. \end{aligned} \tag{2.5}$$

The principal values of stretch are obtained from Eqs. (2.5) in the form

$$\lambda_1 = \frac{\gamma + \sqrt{4+\gamma^2}}{2}, \qquad \lambda_2 = \frac{-\gamma + \sqrt{4+\gamma^2}}{2}. \tag{2.6}$$

The orientation of the principal axis is also obtained from Eqs. (2.5) which yields the following value for the angle between y_1 and x_1:

$$\beta = \frac{\pi}{4} + \frac{\theta}{2}. \tag{2.7}$$

Use the logarithmic strain measure to obtain the following principal values for strain:

$$E_i = \ln \lambda_i . \tag{2.8}$$

It follows from Eqs. (2.6) and (2.8) that $E_2 = -E_1$. Hence the strain components on the fixed axes can be obtained in the form

$$e_{11} = -e_{22} = E_1 \cos 2\beta, \qquad e_{12} = E_1 \sin 2\beta . \tag{2.9}$$

Note that the normal components of strain on x_i have *non-zero* values.

During the continued deformation, the rate of deformation gradient is prescribed as follows:

$$\dot F_{11} = \dot F_{22} = 0, \qquad \dot F_{21} = 0, \quad \dot F_{12} = \dot\gamma . \tag{2.10}$$

Use the chain rule of differentiation to express the rate of deformation gradient in terms the symmetric and anti-symmetric parts of the velocity gradient, D_{ij} and W_{ij}, respectively:

$$\dot F_{ij} = (D_{ik} + W_{ik}) F_{kj} . \tag{2.11}$$

The following values for strain-rate and spin are obtained from Eqs. (2.3), (2.10) and (2.11).

$$D_{11} = D_{22} = 0, \qquad D_{12} = W_{12} = \dot\gamma/2 . \tag{2.12}$$

The significance of these results must be carefully noted. For example, note that the normal components of strain in Eqs. (2.9) are non-zero, whereas the normal components of strain-rate are zero (2.12). The expression for strain in Eqs. (2.9) suggest that they are not constant. In fact, they depend on E_1 and, hence, on the value of principal stretch and angle. The obvious conclusion is that $\dot e_{ij} \neq D_{ij}$. The implication is that the strain-rate, D_{ij}, is not a flux [3]. In other words, the time rate of change of a strain may not be equated with the strain-rate. Yet, a common practice in the classical plasticity is to express the flow rule in terms of the strain-rate and the value obtained as a result of its integration is routinely interpreted as strain. It seems that such an interpretation is not always justified.

Note also that the continued finite deformation yields a value

$$\dot\beta = \dot\theta/2 \tag{2.13}$$

for the spin of the principal axes. The interpretation applied to the integrated values of Eqs. (2.12) in the theory of plasticity, on the other hand, suggests that the angle β must remain constant and hence the spin of the principal strain axis must vanish, which clearly is not the case here. This is a serious deficiency and warrants a careful investigation of the existing practices in the application of the rate-theory of plasticity.

3. Foundation

The stress-strain law for a material is usually obtained from a uniaxial tension test and is expressed in the form

$$e = A\sigma^n , \tag{3.1}$$

where A and n are material constants. This equation may be used as a basis for generalization of plasticity law to three-dimensional deformation. For example, the equivalent stress

$$\sigma_M = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} \tag{3.2}$$

is a 3-dimensional generalization of the uniaxial stress. Here,

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{3.3}$$

is the deviatoric stress component and δ_{ij} is Kronecker's delta.

A generalization of strain in the form

$$e_H = \sqrt{\frac{2}{3}e_{ij}e_{ij}} , \tag{3.4}$$

which is attributed to Hencky, has been less successful so far as its use in the plasticity theory is concerned. A generalization form of strain which is widely used can be obtained by integration from the following definition of the equivalent strain-rate:

$$\varepsilon_M = \sqrt{\frac{2}{3}D_{ij}D_{ij}} . \tag{3.5}$$

But it is not clear how to interpret the integrated value of ε_M, especially in view of the fact that strain-rate used in Eq. (3.5) is not generally a flux of a strain.

For the simple shear case considered in the previous section,

$$e_H = \frac{2}{\sqrt{3}}E_1 , \tag{3.6}$$

$$e_M = \int \varepsilon_M \, dt = \frac{\gamma}{\sqrt{3}} . \tag{3.7}$$

It is clearly seen that $\dot{e}_H \neq \varepsilon_M$ and hence the former cannot be obtained from ε_M by integration. This observation poses a dilemma as to which of the two strains is the proper generalization of the uniaxial strain in the stress-strain law (3.1). Yet another difficulty created due to this dilemma is as follows.

Consider the material derivative of stress-strain law (3.1) and rewrite it in the form

$$\dot{e} = \frac{2}{3}\frac{\dot{\sigma}}{h} , \tag{3.8}$$

where the parameter h is given by

$$\frac{2}{3h} = nA\sigma^{n-1} = (ne/\sigma) . \tag{3.9}$$

The stress-strain law (3.1) has been used to derive the second relationship Eq. (3.9).

A three-dimensional generalization of Eq. (3.8) can be expressed in the form

$$\dot{e}_{ij} = \sigma'_{ij}\frac{\dot{\sigma}}{h\sigma}. \tag{3.10}$$

It is now possible to generalize the above equation by replacing the material rate of strain on the left-hand side by an objective rate. Thus

$$e^{0}_{ij} = \sigma'_{ij}\frac{\dot{\sigma}}{h\sigma}, \tag{3.11}$$

where the superscript 0 denotes an objective rate. This flow rule was proposed by DUBEY and BEDI [1].

The classical flow rule can be obtained from Eq. (3.10) by following the common practice in plasticity of replacing the rate of strain by D_{ij}. The consequence is the flow rule,

$$D_{ij} = \sigma'_{ij}\frac{\dot{\sigma}}{h\sigma}. \tag{3.12}$$

The difficulty one faces now is this: which of the two expressions for the hardening parameter in Eq. (3.9) must one use in the flow rule (3.12). This point is not even mentioned in the literature because, perhaps, of the fact that the hardening parameter used in the flow rule is calculated in terms of $(nA\sigma^{n-1})$ and no attention is paid to the value of (ne/σ) which it must equal if Eq. (3.1) is correct. Indeed, the point is irrelevant if these two expressions yield the same value as they must according to Eq. (3.1) and the assumption of isotropy.

In order to address this paradox, introduce a parameter s as a measure of the length of the trajectory of plastic deformation. For the classical flow rule, choose

$$\dot{s} = \sqrt{\frac{2}{3}D_{ij}D_{ij}} = \varepsilon_M. \tag{3.13}$$

At the same time, retain e_H as a generalized strain measure. Since it is obvious from Eqs. (3.6), (3.7) and (3.13) that $s \neq e_H$, therefore, introduce a factor g such that

$$\dot{e}_H = g\dot{s}. \tag{3.14}$$

The value of $\dot{e}_H$ in terms of the rate of strain is obtained from Eq. (3.4) in the form

$$\dot{e}_H = \frac{2}{3}\frac{e_{ij}\dot{e}_{ij}}{e_H} = \frac{2}{3}\frac{e_{ij}e^{0}_{ij}}{e_H}, \tag{3.15}$$

where 0 is an objective co-axial tensor-rate. In order to calculate the left-hand side of the above equation, one of the two options may be adopted: (a) use the flow rule (3.11) proposed by DUBEY and BEDI [1], or (b) use the classical flow rule in the form Eq. (3.12). If the choice is in favour of the option (b), it is then necessary to find a relationship between the rate of strain components and the strain-rate components in order to calculate the value of e_H. The principal axes technique is used for this purpose.

4. Principal axes technique

In order to use the principal axes technique, all tensor components must be transformed and expressed in terms of its components on the principal axes. In view of the interpretation used here, the principal directions of V_{ij} at the current instant define the principal or Eulerian axes. For the simple shear problem, the Eulerian components of strain rate (these are the components of the strain-rate on the principal or y_i-axes), denoted by superscript E, can be obtained from Eqs. (2.7) and (2.12) in the form

$$D_{11}^E = -D_{22}^E = D_{12}\sin 2\beta = \frac{\dot\gamma}{2}\cos\theta\,, \qquad D_{12}^E = D_{12}\cos 2\beta = -\frac{\dot\gamma}{2}\sin\theta\,. \tag{4.1}$$

The y_i-components of the rate of strain are

$$\dot E_{11} = \frac{\dot\lambda_1}{\lambda_1} = D_{11}^E\,, \qquad \dot E_{22} = \frac{\dot\lambda_2}{\lambda_2} = D_{22}^E\,, \qquad \dot E_{12} = \dot\beta(E_1 - E_2)\,, \tag{4.2}$$

where $\dot\beta$ is the spin of the Eulerian or principal axis. This spin also can be expressed in terms of the strain-rate [4]:

$$\dot\beta = \frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2-\lambda_2^2}D_{12}^E - W_{12}^E\,. \tag{4.3}$$

These relationships are next expressed in terms of Jaumann-rate, which is denoted by superscript J and defined as follows:

$$e_{ij}^J = \dot e_{ij} - W_{ik}e_{kj} + e_{ik}W_{kj}\,. \tag{4.4}$$

Thus, the components of the J-rate of strain on the Eulerian axes are

$$E_{11}^J = \dot E_1 = \dot E_{11} = D_{11}^E\,, \qquad E_{22}^J = \dot E_2 = \dot E_{22} = D_{22}^E\,, \qquad E_{12}^J = \frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2-\lambda_2^2}(E_1-E_2)D_{12}^E\,. \tag{4.5}$$

Rewrite the shear component of the rate of strain in the form

$$E_{12}^J = p_3 D_{12}^E \tag{4.6}$$

where the coefficient p_3, given by

$$p_3 = \frac{\lambda_1^2+\lambda_2^2}{\lambda_1^2-\lambda_2^2}(E_1-E_2)\,, \tag{4.7}$$

depends on the value of the principal strains and stretches; that is, it depends on the history of loading.

A transformation of these equations to the fixed axes results in the following relationships:

$$\begin{aligned} e_{11}^J &= -(p_3-1)D_{12}^E\sin 2\beta = -\frac{p_3-1}{p_3}E_{12}^J\sin 2\beta\,,\\ e_{22}^J &= -(p_3-1)D_{12}^E\sin 2\beta = \frac{p_3-1}{p_3}E_{12}^J\sin 2\beta\,,\\ e_{12}^J &= D_{12} + (p_3-1)D_{12}^E\cos 2\beta = D_{12} + \frac{p_3-1}{p_3}E_{12}^J\cos 2\beta\,. \end{aligned} \tag{4.8}$$

It is easy to invert the above equations and write D_{ij} in terms of e^J_{ij}.

The benefit of the development followed above is that it provides a technique for correlating the spin of the principal axes irrespective of whether the strain-rate or a rate of strain is used in the analysis. To be more specific, Eq. (4.3) provides an expression for the spin $\dot{\beta}$ in terms of the strain-rate. An expression in terms of the Jaumann-rate of strain follows by using Eq. (4.6) in (4.3). This observation can be exploited to yield a methodology for developing or correlating results obtained from different flow rules. Thus, one may choose any objective rate of strain in the flow rule, but they must result in the same value for the spin of the principal axes.

5. Application

(a) Consider first the classical flow rule (3.12), and use Eq. (3.13) to rewrite it in terms of $\dot{s}$:

$$D_{ij} = \frac{3}{2}\sigma'_{ij}\frac{\dot{s}}{\sigma} \tag{5.1}$$

(Here and in the following, σ denotes the equivalent stress. The subscript M has been dropped.) For the prescribed simple shear, $D_{11} = D_{22} = 0$. In view of the flow rule (3.12) or (5.1) therefore, normal component of stress on fixed axes must vanish: $\sigma_{11} = \sigma_{22} = 0$. Thus, the shear stress, denoted by τ, is the only non-zero component of stress. The equivalent stress defined by Eq. (3.2) can be expressed in terms of the shear stress:

$$\sigma = \sqrt{3}\tau\,. \tag{5.2}$$

A transformation to the principal axes yields the stress components:

$$\begin{aligned}\Sigma_{11} = -\Sigma_{22} &= \frac{2\tau}{\sqrt{4+\gamma^2}}\,,\\ \Sigma_{12} &= \frac{\gamma\tau}{\sqrt{4+\gamma^2}}\,.\end{aligned} \tag{5.3}$$

Thus the ratio

$$\frac{2\Sigma_{12}}{\Sigma_{11}-\Sigma_{22}} = \frac{\gamma}{2} = \cot 2\beta \tag{5.4}$$

can be used to find the relative orientation of the principal axes of strain-rate.

The length of the plastic trajectory and its rate, obtained from Eqs. (2.12), (3.5) and (3.13) have values

$$s = \gamma/\sqrt{3}\,, \qquad \dot{s} = \dot{\gamma}/\sqrt{3}\,. \tag{5.5}$$

The history-dependent factor g in Eq. (3.14) is calculated next with the help of Eqs. (3.6), (3.15), (4.1), (4.5) and (5.5). Its value is found to be

$$g = \frac{2}{\sqrt{4+\gamma^2}}\,, \tag{5.6}$$

which can also be obtained by differentiating e_H in Eq. (3.6) and substituting this value along with the value of $\dot{s}$ from Eqs. (5.5) in (3.14). The usual practice to find the principal direction from the values obtained from integration of strain-rate components must be abandoned, since the interpretation of such integrated values as strain has been shown to be generally incorrect for large deformation. The recommended procedure is to find the spin of the principal axes from Eq. (4.3). The value so obtained is found to be the same as that in Eq. (2.13) which shows consistency in the result. This value of spin can then be integrated to yield the angle β.

(b) Consider next the consequences of the proposed flow rule. For this purpose, use the J-rate as the objective coaxial rate of strain and the definition

$$\dot{s} = \sqrt{\frac{2}{3} e^J_{ij} e^J_{ij}} \tag{5.7}$$

for the rate of change of the plastic trajectory. In view of Eq. (5.7), the proposed flow rule is rewritten in terms of $\dot{s}$:

$$e^J_{ij} = \frac{3}{2} \sigma'_{ij} \frac{\dot{s}}{\sigma}. \tag{5.8}$$

Since the strain-rate and the J-rate of strain both have the same values for the normal components on the principal axes, the two theories result in identical values for the ratios of the normal y_i-components of stress. It is clear from Eq. (4.6) however, that the shear component of the rate of the strain is not the same as the shear component of the strain-rate. This is the reason for the difference between the two theories. Even this difference vanishes if it turns out that $p_3 = 1$ as it is in the reference configuration. The difference vanishes also in the case when the spin of the principal strain-axes vanishes.

In view of Eqs. (4.1) and (4.5), (5.8) yields the ratio

$$\frac{2\Sigma_{12}}{\Sigma_{11} - \Sigma_{22}} = \frac{p_3 \gamma}{2} = p_3 \cot 2\beta, \tag{5.9}$$

which is different from the ratio (5.4) obtained for the classical flow rule. The ratio of stresses obtained from Eq. (5.9) can be used in Eq. (3.2) to yield

$$\begin{aligned} \Sigma_{11} &= -\Sigma_{22} = \frac{2\sigma}{\sqrt{3(4 + p_3^2\gamma^2)}}, \\ \Sigma_{12} &= \frac{p_3\gamma\sigma}{\sqrt{3(4 + p_3^2\gamma^2)}}. \end{aligned} \tag{5.10}$$

The x_i-components of stress can now be obtained in the form

$$\begin{aligned} \sigma_{11} &= -\sigma_{22} = -\frac{2(p_3 - 1)\gamma\sigma}{\sqrt{3(4 + \gamma^2)(4 + p_3^2\gamma^2)}}, \\ \sigma_{12} &= \frac{(4 + p_3\gamma^2)\sigma}{\sqrt{3(4 + \gamma^2)(4 + p_3^2\gamma^2)}}. \end{aligned} \tag{5.11}$$

Thus, the proposed flow rule yields non-zero values for normal stresses σ_{11} and σ_{22}. They are small for small values of $(p_3 - 1)\gamma$, and in such situation, the proposed theory can be said to exhibit second-order effect. However, for $\gamma = 1$, their values are found to be approximately (1/10)-th of the shear stress. Note that the stresses in Eq. (5.11) reduce to those predicted by the classical theory provided $p_3 = 1$, which is true in the reference configuration.

In order to obtain the factor g, substitute Eqs. (5.8) and (3.15) with J-rate in Eq. (3.14) to obtain a general relation

$$g = \frac{\sigma'_{ij} e_{ij}}{\sigma e_H}. \tag{5.12}$$

Note that $g = 1$ for the case of uniaxial loading. For the simple shear problem

$$g = \frac{2}{\sqrt{4 + p_3^2\gamma^2}}. \tag{5.13}$$

The value of $\dot{s}$ can be obtained with the help of Eqs. (4.1), (4.5) and (5.7),

$$\dot{s} = \dot{\gamma}\sqrt{\frac{4 + 3p_3\gamma^2}{3(4 + \gamma^2)}}. \tag{5.14}$$

The value of $\dot{e}_H$ obtained by substituting (5.13) and (5.14) in (3.14) is the same as the value obtained directly by differentiating (3.6) which indicates a consistency in the development.

6. Conclusion

Some consequences of the flow rule proposed by DUBEY and BEDI [1] are examined and compared with the results obtained from classical flow rule attributed to Levy and Mises. For the purpose of comparison, the concept of the length of plasticity trajectory is introduced in terms of the flow variable, and its rate is related to the rate of strain via a history-dependent factor. The result of this approach is that it provides a methodology for the development of flow rules in terms of either the strain-rate or the rate of strain. The principal axes technique has been used to relate the flow variables used in the two types of flow rules. It is shown that the proposed flow rule predicts second-order effects unlike the classical flow rule which does not.

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References

1. R. N. DUBEY and S. BEDI, *Some implications of the MDB flow rules plasticity*, Proc. Adv. in Plast., pp. 291–295, Pergamon Press, 1989.
2. R. HILL, *The mathematical theory of plasticity*, Oxford Enging. Sci. Series, Oxford University Press, Oxford 1950.
3. R. HILL, *Basic aspects of invariance in solid mechanics*, Adv. Applied Mech., **18**, 1–75, 1978.
4. R. N. DUBEY, *Principal axes technique in large deformation*, Proc. Adv. Const. Laws for Engineer. Mat., Vol. II, pp. 962–968. Int. Academic Publishers, 1989.

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