Analysis of separation of blood flow in constricted arteries

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THE PRESENT investigation represents the analysis of blood flow through an axisymmetric constricted artery and the constriction is cosine-shaped. The perturbation method has been applied to study the flow field in the tube and the theoretical results obtained in this analysis are given for the velocity distribution, wall shearing stress and the separation phenomena. Numerical solution of the theoretical results have been shown graphically and the pattern of separation is compared with that of Morgan and Young [4].

Przeprowadzono analizę procesu przepływu krwi przez osiowo-symetryczną arterię z przewężeniem o profilu cosinusowym. Zastosowano metodę perturbacji dla określenia pola prędkości przepływu w arterii, rozkładu naprężeń stycznych na ściankach i zjawisk oderwania. Na rysunkach przedstawiono wyniki obliczeń numerycznych, a postać oderwania strumienia porównano z wynikami podanymi przez Morgana i Younga [4].

Проведен анализ процесса течения крови через осесимметричную артерию с сужением с косинусным профилем. Применен метод пертурбации для определения поля скорости течения в артерии, распределения касательных напряжений на стенках и явлений отрыва. На рисунках представлены результаты численных разчетоб, а вид отрыва потока сравнен с результатами, приведенными Морганом и Юнгом [4].

1. Introduction

LOCAL CONSTRICTION in an artery disturbs normal blood flow and frequently results in arterial disease. There is much evidence that the importance of hydrodynamic factors can play an important role in the development and progression of this disease. The flow characteristics which may have medical importance require some alterations.

The actual cause of stenosis are known but its effects on the flow characteristics have been studied by many authors. In one of the earliest papers YOUNG [1] considered a highly simplified linear model to analyse blood flow through a mildly constricted tube. Then FORRESTER and YOUNG [2] extended the work of YOUNG [1] to study the effects of flow separation on a mild constriction. LEE and FUNG [3] investigated the problem of blood flow through a constricted tube numerically and the solution was applicable to both mild and severe contrictions. MORGAN and YOUNG [4] extended and modified the work of FORRESTER and YOUNG [2] by making use of both the integral-momentum and integral-energy equations. The solution obtained in their analysis was valid for both mild and severe constrictions.

The present investigation deals with the problem of blood flow through a locally constricted tube by means of the perturbation technique. The solution obtained in this analysis by considering blood as a Newtonian fluid is shown graphically for the wall shearing stress and separation phenomenon.

2. Mathematical model

The nondimensional equation of motion that governs the flow field in the constricted tube, is given in terms of a stream functions as

$$(2.1)\qquad R_e\delta\left[\frac{1}{\gamma}\frac{\partial(\nabla^2\psi,\psi)}{\partial(r,x)}-\frac{2}{r^2}\cdot\frac{\partial\psi}{\partial x}\cdot\nabla^2\psi\right]=\nabla^4\psi$$

with the following transformations:

$$(2.2)\qquad \begin{aligned} u_x &= \frac{u}{U_0}, & u_r &= \frac{v}{U_0},\\ r &= \frac{\overline{r}}{R_0}, & x &= \frac{z}{l_0},\\ \delta &= \frac{R_0}{l_0}, & \psi &= \overline{\psi}/U_0R_0^2, \end{aligned}$$

where Re $= U_0R_0/\nu$ is the Reynolds number, (u,v) and (u_x,u_r) are the dimensional and nondimensional velocity components in the axial and radial directions, respectively, R_0 is the radius of the normal tube and $l_0/2$ is the length of the constriction (Fig. 1), u_0 is the characteristic velocity; the stream function $\overline{\psi}$ is defined by

$$(2.3)\qquad \begin{aligned} u &= -\frac{1}{\overline{r}}\frac{\partial\overline{\psi}}{\partial\overline{r}},\\ v &= \frac{1}{\overline{r}}\frac{\partial\overline{\psi}}{\partial z}, \end{aligned}$$

and the operator is given by

$$(2.4)\qquad \nabla^2=\delta^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}.$$

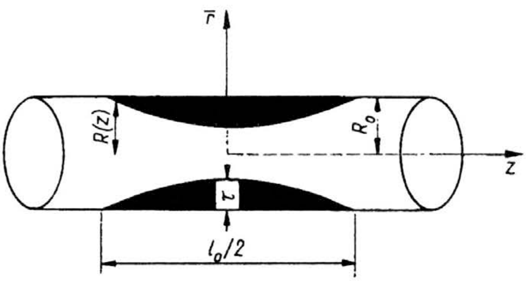

FIG. 1. Geometry of constriction.

The boundary of constricted tube is described by

$$(2.5)\qquad \begin{aligned} \frac{R(z)}{R_0} &= 1-\frac{\tau}{2R_0}\left(1+\cos\frac{4\pi z}{l_0}\right), \quad -\frac{l_0}{4}\le z\le\frac{l_0}{4},\\ &= 1, \quad \text{otherwise}, \end{aligned}$$

whose dimensionless form is

$$(2.6)\qquad \begin{aligned} f(x) &= 1-\frac{\varepsilon}{2}(1+\cos 4\pi x), \quad -\frac{1}{4}\le x\le\frac{1}{4},\\ &= 1, \quad \text{otherwise}, \end{aligned}$$

where $R(x)$ is the radius of the tube in the constricted region, ε is the maximum height of the constriction, $f = R/R_0$ and $\varepsilon = \tau/R_0$.

The boundary conditions in terms of nondimensional quantities are

$$(2.7)\qquad u_x = 0\,, \qquad u_r = 0 \quad \text{at } r = f\,;$$

$$(2.8)\qquad u_x = 0 \quad \text{finite at } r = 0\,;$$

$$(2.9)\qquad \int\limits_0^{f(x)} r u_x \, dr = -\frac{1}{2}\,.$$

To solve Eq. (2.1), we need some simplifications and use the perturbation technique to build up differential equations of different orders together with their boundary conditions. In order to do this we expand the stream function ψ in ascending powers of δ as

$$(2.10)\qquad \psi(x, r, \delta) = \psi_0(x, r) + \delta\psi_1(x, r) + \delta^2\psi_2(x, r) + \ldots\,,$$

where ψ_0, ψ_1 and ψ_2 are the perturbation quantities of order 0, 1 and 2. Introducing Eq. (2.10) into Eq. (2.1) and then equating terms with equal powers of δ, we have the following set of perturbation equations together with their corresponding boundary conditions to the second order:

Zero-th order

$$(2.11)\qquad L^2\psi_0 = 0\,, \qquad L = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\,;$$

$$(2.12)\qquad -\frac{1}{r}\frac{\partial \psi_0}{\partial r} = \text{finite}\,, \qquad \psi_0 = 0 \quad \text{at} \quad r = 0\,;$$

$$(2.13)\qquad -\frac{1}{r}\frac{\partial \psi_0}{\partial r} = 0\,, \qquad \psi_0 = -\frac{1}{2} \quad \text{at } r = f\,.$$

First order

$$(2.14)\qquad L^2\psi_1 = \mathrm{Re}\cdot\frac{1}{r}\frac{\partial \psi_0}{\partial x}\left(\frac{\partial^3\psi_0}{\partial r^3} - \frac{3}{r}\frac{\partial^2\psi_0}{\partial r^2} + \frac{3}{r^2}\frac{\partial \psi_0}{\partial r}\right) - \frac{\mathrm{Re}}{r}\cdot\frac{\partial \psi_0}{\partial r}L\frac{\partial \psi_0}{\partial x}\,,$$

$$(2.15)\qquad -\frac{1}{r}\frac{\partial \psi_1}{\partial r} = \text{finite}\,, \qquad \psi_1 = 0 \quad \text{at} \quad r = 0\,;$$

$$(2.16)\qquad \psi_1 = -\frac{1}{r}\frac{\partial \psi_1}{\partial r} = 0 \quad \text{at} \quad r = f\,.$$

Second order

$$(2.17)\qquad L^2\psi_2 = -2L\frac{\partial^2\psi_0}{\partial x} - \frac{\mathrm{Re}}{r}\left[\frac{\partial \psi_0}{\partial r}\cdot L\left(\frac{\partial \psi_1}{\partial x}\right) + \frac{\partial \psi_1}{\partial r}\cdot L\left(\frac{\partial \psi_0}{\partial x}\right)\right]$$

(2.17) [cont.]

$$+ \operatorname{Re} \frac{\partial \psi_0}{\partial x} \cdot \frac{1}{r} \left[\frac{\partial^3 \psi_1}{\partial r^3} - \frac{3}{r} \frac{\partial^2 \psi_1}{\partial r^2} + \frac{3}{r^2} \frac{\partial \psi_1}{\partial r} \right]$$

$$+ \operatorname{Re} \frac{\partial \psi_1}{\partial x} \cdot \frac{1}{r} \left[\frac{\partial^3 \psi_0}{\partial r^3} - \frac{3}{r} \frac{\partial^2 \psi_0}{\partial r^2} + \frac{3}{r^2} \frac{\partial \psi_0}{\partial r} \right];$$

$$\text{(2.18)} \qquad -\frac{1}{r}\frac{\partial \psi_2}{\partial r} = \text{finite}, \qquad \psi_2 = 0 \quad \text{at} \quad r = 0;$$

$$\text{(2.19)} \qquad \psi_2 = -\frac{1}{r}\frac{\partial \psi_2}{\partial r} = 0 \quad \text{at} \quad r = f.$$

3. Solution of the problem

The solution of the stream functions ψ_0, ψ_1 and ψ_2 can be obtained by solving their respective equations under the corresponding boundary conditions. The solution ψ_0 of the zero-th order problem described by Eqs. (2.11), (2.12), (2.13) is

$$\text{(3.1)} \qquad \psi_0 = \frac{1}{2}(y^4 - 2y^2).$$

After having the zero-th order solution, the first and second order problems can be easily solved and their solutions are

$$\text{(3.2)} \qquad \psi_1 = \frac{\operatorname{Re}}{36f}\frac{df}{dx}(y^8 - 6y^6 + 9y^4 4y^2),$$

$$\text{(3.3)} \qquad \psi_2 = -\frac{1}{6}\left[5\left(\frac{df}{dx}\right)^2 - f\frac{d^2 f}{dx^2}\right](y^2-1)^2 y^2 - \frac{\operatorname{Re}}{8}\left(\frac{df}{dx}\right)^2 \cdot F(y),$$

where

$$y = r/f,$$

$$\text{(3.4)} \qquad F(y) = \frac{1}{1350}(16y^{12} - 165y^{10} + 600y^8 - 1100y^6 + 960y^4 - 331y^2).$$

The dimensionless expressions for the velocity components in the axial and radial directions are

$$\text{(3.5)} \qquad u_x = \frac{2}{f^2}(1-y^2) - \frac{\delta \operatorname{Re} f'}{9f^3} \cdot (2y^6 - 9y^4 + 9y^2 - 2) + \delta^2 \left[\frac{1}{3f^2}(5f'^2 - ff'')(3y^4 - 4y^2 + 1) + \frac{\operatorname{Re}^2 f'^2}{5400 f^4}(96y^{10} - 825y^8 + 2400y^6 - 3300y^4 + 1960y^2 - 331) \right],$$

$$\text{(3.6)} \qquad u_r = -\frac{2\delta}{f^2} \cdot f'(y^3 - y) - \frac{\operatorname{Re}\delta^2}{36f^3}[f^2(9y^7 - 42y^5 + 45y^3 - 12y) - ff''(y^7 - 6y^5 + 9y^3 - 4y)].$$

The expressions (3.5) and (3.6) show that the leading term of u_x is independent of δ whereas it contains δ in u_r i.e., the radial velocity component is one order higher than the axial velocity component.

The wall shearing stress T_w in the form of dimensionless variables retaining the terms up to the second order is

$$(3.7) \qquad T_w = -\frac{1}{f^3} + \delta\frac{\mathrm{Re}\, f'}{6f^4} + \delta^2\left[-\frac{1}{f^3}f'^2 + \frac{1}{3f^3}(5f'^2 - ff'') - \frac{\mathrm{Re}^2}{16f^5}\cdot f'^2\right].$$

The points of separation and reattachement at the wall are given by $T_w = 0$, i.e.

$$(3.8) \qquad \mathrm{Re} = \frac{12f}{\delta(67f'^2 - 15ff'')}\Big[2\sqrt{5}\{3(32f'^2 - 15ff'') - \delta^2(2f'^2 - ff'') \times(67f'^2 - 15ff'')\}\frac{1}{2} - 15f'\Big].$$

4. Discussion

The theoretical distribution of the shearing stress along the wall of the tube is given in Fig. 2 for different values of the Reynolds number. The figure shows that the shearing stress for any given Reynolds number attains its maximum value just ahead of the throat of the tube and then decreases rapidly in the diverging section. At a low Reynolds number the stress becomes negative over some length of the tube in the diverging section. The reason for such negative distribution of shearing stress indicates the occurrence of separation which involves circulation with back flow near the wall. As a result, a high velocity core surrounded by the separated region is formed with low shear at the wall.

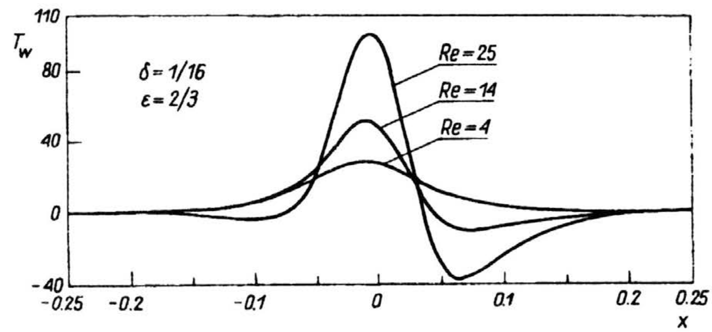

FIG. 2. Distribution of wall shearing stress.

At a high Reynolds number the negative shearing stress distribution is also observed over some upstream length of the tube. The spread of this negative shear in the upstream direction is much smaller than that in diverging section indicating large down-stream circulation.

The theoretical locations of the zero shearing stress are plotted against the Reynolds number in Figs. 3 and 4. The separation and reattachment points can be obtained from the relation (3.8). Figure 3 shows that separation occurs when a critical Reynolds number is reached. If the Reynolds number increases beyond this critical Reynolds number, the upstream branch of the curve gives the separation points and the downstream brand corresponds to the reattachment points. The separated region is located between these two points. At a higher Reynolds number the separation point asymptotically reaches a

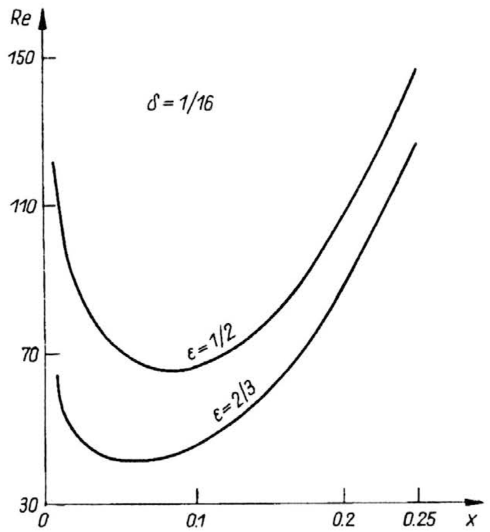

FIG. 3. Separation and reattachment points in the diverging region.

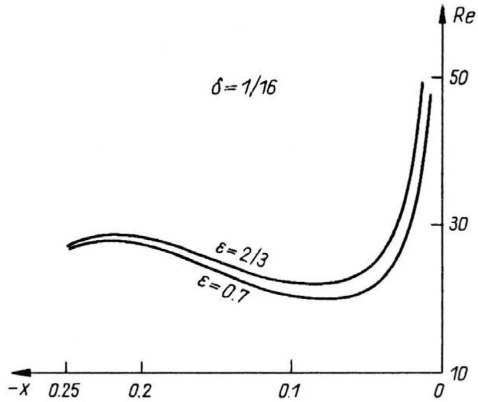

FIG. 4. Separation and reattachment points in the converging region.

point somewhere between the original separation point and the throat of the tube, and at the same time the reattachement point is nearly proportional to the Reynolds number. The result obtained above is qualitative because the solution has been considered up to the second order of δ and the general pattern of the solution is the same as that of MORGAN and YOUNG [4]. It is also observed that for any given Reynolds number or tube constriction the separation point moves towards the throat of the tube and the reattachment point moves downstream with the enlargement of the region of separation which is physiologically unfavorable.

Similarly, Fig. 4 clearly explains the separation phenomenon in the converging section of the tube. The downstream solution of the curve corresponds to the reattachment points and has the same nature as that of the upstream solution of the divergent section. But the upstream solution of the convergent section which gives the position of separation points slightly differs from downstream solution of the divergent section of the tube.

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