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On the stability of an evaporation viscous jet (*)

J. SKIEPKO (WARSZAWA)

THE STABILITY problem of an evaporating viscous jet is discussed. The flow in the surrounding medium is assumed to be potential. In equations for disturbances, the velocity of flow in basic flow is approximated by a constant value. The solutions in the region of a jet and in the vapour region are expressed by modified Bessel functions. Using the boundary conditions on the surface of a jet, the condition of the existence of disturbances growing in time is formulated. This condition is formulated in terms of the solutions of a certain algebraic equation, including modified Bessel functions.

Dyskutowany jest problem stabilności lepkiej strugi. Założono, że przepływ na zewnątrz strugi jest potencjalny. W równaniach dla zaburzeń, składowe prędkości podstwowego przepływu aproksymowano stałymi wartościami. Rozwiązania w obszarze strugi, jak też i na zewnątrz niej, wyrażają się za pośrednictwem zmodyfikowanych funkcji Bessela. Korzystając z warunków brzegowych na powierzchni strugi, sformułowano warunek istnienia rosnących w czasie zaburzeń. Warunek ten jest sformułowany w terminach istnienia pewnych rozwiązań algebraicznego równania zawierającego zmodyfikowane funkcje Bessela.

Исследуется проблема устойчивости вязкой струк. Предполагается, что течение вне струк является потенциальном. В уравнениях для возмущений составляющие скорости основного течения аппроксимированы постоянными значениями. Решения в области струк, как тоже и вне ее, выражаются посредством модифицированных функций Бесселя. С использованием граничных условий на поверхности струк, сформулировано условие, гарантирующее существование возрастающих во времени возмущений. Это условие сформулировано в терминах решений алгебраического уравнения, содержавшего модифицированные функции Бесселя.

1. Introduction

IN MANY TECHNICALLY important applications, due to high temperature or more often due to low pressure of the surrounding medium, liquid jets evaporate on their surface. The appearance of evaporation on the surface of a liquid jet produces an additional mechanism of instability. Its interactions with other mechanisms such as thermal conduction or surface tension can, in some cases, make flow more stable, in others can amplify the effects of instability. The motion in an evaporating jet and in a surrounding medium, which will be called a basic flow, does not have a simple mathematical description. It is necessary to simplify the description of basic flow; this stems not only from difficulties in solving the governing equations but also from the need to have a possibly simple form of system equations for the disturbances. Approximation of the parameters of basic flow by constant values leads to the equation for disturbances in the form of Bessel equations. Having the general form of the solution for disturbances and using the boundary conditions on the surface of the jet, a condition which guarantees the existence of the solution growing in time is formulated. This condition has the form of an algebraic equation dependent on wave number and wave speed. In the limited case it takes the well-known form of the Rayleigh condition.

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2. A basic flow and governing equations

A cylindrical infinite viscous jet of the radius a is immersed in an inviscid gas. The flow in the gas region is assumed to be potential. In the vicinity of the surface of the jet, the liquid evaporates in a narrow layer. Such a layer, called the thermal layer, was introduced by H. J. PALMER [2] in his investigation of the stability of the evaporating semispace. In Palmer's discussion on the boundary between the thermal layer and the liquid, the temperature and flux of heat are continuous. On the boundary between the thermal layer and vapour, the flux of energy has to be continuous. In the case of a cylindrical jet, the introduction of a thermal layer would involve too many calculations, so we assume that the evaporation process takes place only on the surface of the jet. This means that the thermal layer degenerates at the surface of the jet. Then the surface of the jet separates liquid particles from vapour particles, which just in the last moment were reproducing from neighbouring particles of the liquid jet. This means that the temperature of vapour particles on the surface of the jet is lower than the temperature of the neighbouring particles of the jet. This difference in temperature is adequate to the amount of heat needed for evaporation. The temperature after evaporation is a function of the temperature before evaporation. We denote

$$T_v = F(T_l)\,. \tag{2.1}$$

Expanding in the Taylor series in the vicinity of T_l^0, we obtain

$$T_v - T_v^0 = F'(T_l^0)(T_l - T_l^0) + \ldots \tag{2.2}$$

The last relation says that the disturbances of the vapour temperature are equal to the disturbances of temperature of the jet at the same point multiplied by $F'(T_l^0)$.

The mathematical description of the flow inside the liquid jet and in the vapour region is made possible by a suitable system of differential equations and boundary conditions. In the cylindrical coordinate system r, φ, z, with the axis z coinciding with the axis of a cylindrical jet, the liquid region corresponds to $r \le a$. Assuming low velocity in that region, the Navier–Stokes and energy equations with convectional terms omitted can be applied.

For $r < a$

$$\nabla \cdot \mathbf{U}_l = 0\,, \tag{2.3}$$

$$-\frac{1}{\varrho_l}\nabla P_l + \nu_l \nabla^2 \mathbf{U}_l = 0\,, \tag{2.4}$$

$$k_l \nabla^2 T_l^0 = 0\,. \tag{2.5}$$

Neglecting the viscous effects in the vapour region leads to the following system of equations for $r > a$:

$$\nabla \cdot \mathbf{U}_v = 0\,, \tag{2.6}$$

$$(\mathbf{U}_v \nabla)\mathbf{U}_v = -\frac{\nabla P_v}{\varrho_v}\,, \tag{2.7}$$

$$k_v \nabla^2 T_v^0 = 0\,, \tag{2.8}$$

where $\mathbf{U}_i$ — velocity vector, P_i — pressure, ϱ_i — density, c_{vi} — specific heat at constant volume, T_i^0 — temperature, k_i — coefficient of thermal conductivity, $i = l, v$, l refers to the liquid region, v refers to the vapour region ($r \ge a$).

On the limit between the liquid and the vapour regions, the conditions of conservation of mass, momentum and energy take the form for $r = a$

$$V_v \varrho_v = V_l \varrho_l = \eta_0 , \tag{2.9}$$

$$-P_l + 2\mu_l \frac{\partial V_l}{\partial r} - \eta_0^2 \varrho_l^{-1} = -P_v - \eta_0^2 \varrho_v^{-1} - \sigma G , \tag{2.10}$$

$$\mu_l \frac{\partial U_l}{\partial r} + \eta_0 U_l = \eta_0 U_v , \tag{2.11}$$

$$\eta_0 \lambda_{\text{vap}} + \frac{1}{2}\eta_0^3(\varrho_v^{-2} - \varrho_l^{-2}) + k_l \frac{\partial T_l^0}{\partial r} = 0 , \tag{2.12}$$

$$T_v^0 = F(T_l^0) , \tag{2.13}$$

where V_i — radial component of velocity, U_i — axial component of velocity, σ — coefficient of a surface tension, $G = (1/R_1) + (1/R_2)$, where R_1, R_2 — the radii of a curvature of a normal section of a surface. $F(T_l^0)$ — known function determining the temperature of vapour produced by the liquid at temperature T_l^0, η_0 — evaporation rate in steady conditions, λ_{vap} — the latent heat of evaporation for the liquid.

To determine the evaporation rate η, the formula

$$\eta = E\left(\frac{M}{2\pi R}\right)^{1/2}\left(\frac{P^0}{T_s^{1/2}} - \frac{P_v}{(T_v)^{1/2}}\right) , \tag{2.14}$$

will be used, where E is the evaporation coefficient, R is the gas constant, M is the molecular weight of the liquid, P^0 is its vapour pressure at the surface temperature T_s, and P_v and T_v are the pressure and temperature of the gas phase above the liquid (cf. MAA [3], PALMER [2]).

3. An evolution of small disturbances

Investigation of the stability is equivalent to an analysis of the behavior in time of the small disturbances imposed on the basic flow. The differential equations describing the evolution of the small disturbances are obtained by linearization of Navier–Stokes equations for the liquid region or Euler equation for the vapour region in the vicinity of a basic flow. In that linearization the components of the velocity vector are approximated by the constant values. Denoting the disturbance of the vector of velocity by u_i, $i = l, v$, we can put

$$\mathbf{u}_v = \nabla \varphi_v , \tag{3.1}$$

φ_v — the scalar function.

This is so because we assumed the potentiality of the flow in the vapour region. In the liquid region the flow is viscous, but there we can represent velocity as a sum of the potential and vortical parts:

$$\mathbf{u}_l = \nabla \varphi_l + \nabla \times \mathbf{B}, \quad \text{where} \quad \mathbf{B} = \left(0, -\frac{\psi}{r}, 0\right) . \tag{3.2}$$

Using the representation (3.2) in the linearized form of Navier–Stokes equations and a continuity equation, the following equations are obtained:

$$\frac{\partial^2 \psi_l}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_l}{\partial r} + \frac{\partial^2 \psi_l}{\partial z^2} - \frac{\varrho_l}{\mu_l}\frac{\partial \psi_l}{\partial t} = 0 , \tag{3.3}$$

$$\frac{\partial \varphi_l}{\partial t} = -\frac{p_l}{\varrho_l}, \tag{3.4}$$

$$\frac{\partial^2 \varphi_l}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_l}{\partial r} + \frac{\partial^2 \varphi_l}{\partial z^2} = 0, \tag{3.5}$$

$$\kappa_l \frac{\partial T_l}{\partial t} = \frac{\partial^2 T_l}{\partial r^2} + \frac{1}{r}\frac{\partial T_l}{\partial r} + \frac{\partial^2 T_l}{\partial z^2}, \quad \kappa_l = \frac{\varrho_l c_{vl}}{k_l}. \tag{3.6}$$

In a similar way, from the Euler equations, the continuity equation and the energy equation, the equations describing the evolution of disturbances in the vapour region, can be derived in the form

$$\frac{\partial \varphi_v}{\partial t} + U\frac{\partial \varphi_v}{\partial z} = -\frac{p_v}{\varrho_v}, \tag{3.7}$$

$$\frac{\partial^2 \varphi_v}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_v}{\partial r} + \frac{\partial^2 \varphi_v}{\partial z^2} = 0, \tag{3.8}$$

$$\kappa_v \left(\frac{\partial t_v}{\partial t} + U\frac{\partial T_v}{\partial z}\right) = \frac{\partial^2 T_v}{\partial r^2} + \frac{1}{r}\frac{\partial T_v}{\partial r} + \frac{\partial^2 T_v}{\partial r} + \frac{\partial^2 T_v}{\partial z^2}, \quad \kappa_v = \frac{\varrho_v c_{vv}}{k_v}, \tag{3.9}$$

U — the mean velocity in the jet region.

From the conservation laws of mass, momentum and energy on the surface of the jet, the following conditions hold: for $r = a$

$$\begin{aligned} \frac{\partial \varphi_v}{\partial r} &= \frac{\partial s}{\partial t} + U\frac{\partial s}{\partial z} + \frac{\eta_1}{\varrho_v}, \\ \frac{\partial \varphi_l}{\partial r} + \frac{1}{r}\frac{\partial \psi_l}{\partial z} &= \frac{\partial s}{\partial t} + \frac{\eta_1}{\varrho_l}, \end{aligned} \tag{3.10}$$

$$\mu_l \frac{1}{r}\frac{\partial^2 \psi_l}{\partial z^2} - \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \psi_l}{\partial r} + 2\frac{\partial^2 \varphi_l}{\partial r \partial z} + \eta_0\left(\frac{\partial \psi_l}{\partial z} - \frac{1}{r}\frac{\partial \psi_l}{\partial z}\right) = \eta_0 \frac{\partial \varphi_v}{\partial z} + \eta_1 U, \tag{3.11}$$

$$-p_l - 2\eta_0\eta_1\varrho_l^{-1} + 2\mu_l\left(\frac{\partial^2 \psi_l}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi_l}{\partial r}\right)\right) = -p_v + \frac{\sigma}{a^2}(1 - k^2a^2)s - 2\eta_0\eta_1\varrho_v^{-1}, \tag{3.12}$$

$$\eta_1 \lambda_{\text{vap}} + \frac{3}{2}\eta_0^2\eta_1(\varrho_v^{-2} - \varrho_l^{-2}) + k_l\frac{\partial T_l}{\partial r} = 0, \tag{3.13}$$

$$T_v = \tau T_l, \quad \text{where} \quad \tau = F'(T_l^0). \tag{3.14}$$

In the above expressions, the disturbance of the surface of the jet has been assumed in the form

$$s(z, r, t) = \overline{s}(r)e^{\beta t + ikz}. \tag{3.15}$$

A similar form for the other parameters is imposed

$$\begin{aligned} \psi_l(r, z, t) &= \overline{\psi}_l(r)e^{\beta t + ikz}, \\ \varphi_l(r, z, t) &= \overline{\varphi}_l(r)e^{\beta t + ikz}, \\ T_l(r, z, t) &= \overline{T}_l(r)e^{\beta t + ikz}, \\ \varphi_v(r, z, t) &= \overline{\varphi}_v(r)e^{\beta t + ikz}, \\ T_v(r, z, t) &= \overline{T}_v(r)e^{\beta t + ikz}, \end{aligned} \tag{3.16}$$

where k — wave number, β — wave speed.

Using these expressions, Eqs. (3.3), (3.5), (3.6), (3.8) and (3.9) can be converted into for $r > a$

$$\overline{\varphi}_v'' + \frac{1}{r}\overline{\varphi}_v' - k^2\overline{\varphi}_v = 0, \tag{3.17}$$

$$\kappa_v(\beta + ikU)\overline{T}_v = \overline{T}_v'' + \frac{1}{r}\overline{T}_v' - k^2\overline{T}_v, \tag{3.18}$$

for $r < a$

$$\overline{\psi}_l'' - \frac{1}{r}\psi_l'(r) - k^2\overline{\psi}_l - \frac{\beta\varrho_l}{\mu_l}\overline{\psi}_v = 0, \tag{3.19}$$

$$\overline{\varphi}_l''(r) + \frac{1}{r}\overline{\varphi}_l'(r) - k^2\overline{\varphi}_l = 0, \tag{3.20}$$

$$\kappa_l\beta\overline{T}_l = \overline{T}_l'' + \frac{1}{r}T_l'(r) - k^2T_l(r). \tag{3.21}$$

The most general solution of the above system which vanishes at infinity and is finite at $r = 0$ has the form

$$\overline{\psi}_l(r) = C_5 r l_4 I_1(r l_4), \tag{3.22}$$

$$\overline{\varphi}_l(r) = C_1 I_0(kr), \tag{3.23}$$

$$\overline{\varphi}_v(r) = C_3 K_0(kr), \tag{3.24}$$

$$\overline{T}_l(r) = C_2 I_0(kr), \tag{3.25}$$

$$\overline{T}_v(r) = C_4 K_0(l_3 r), \tag{3.26}$$

where C_1, C_2, C_3, C_4, C_5 are constant, $l_2^2 = k^2 + \kappa_l\beta$, $l_3^2 = k^2 + \kappa_v(\beta + ikU)$, $l_4^2 = k^2 + \frac{\beta\varrho_l}{\mu_l}$, and I_0, K_0, I_1 are modified Bessel functions.

4. Solvability conditions

Assuming that the disturbances of the pressure and the temperature are small, the disturbance of the evaporation rate, after using Eq. (2.14), can be written in the form

$$\eta_1 = A_1 T_l + A_2 T_v + B_1 p_v, \tag{4.1}$$

where

$$A_1 = E\left(\frac{M}{2\pi R}\right)^{1/2}\left[\frac{\partial P^0}{\partial T_l}\frac{1}{(T_l^0)^{1/2}} - \frac{1}{2}\frac{P^0}{(T_l^0)^{3/2}}\right]_{r=a},$$

$$A_2 = \frac{1}{2}E\left(\frac{M}{2\pi R}\right)^{1/2}\frac{P_v}{(T_v^0)^{3/2}}\bigg|_{r=a},$$

$$B_1 = -E\left(\frac{M}{2\pi R}\right)^{1/2}\frac{1}{(T_v^0)^{1/2}}\bigg|_{r=a}.$$

The solutions (3.22) to (3.26) do not satisfy the boundary conditions (3.10) to (3.14) for all wave numbers and wave speeds. To find the condition under which these conditions can be satisfied, the solutions (3.22) to (3.26) are used in Eqs. (3.10) to (3.14). The elimination óf s and C_4 in the resulting equations leads to the following system:

$$
\begin{aligned}
&D_{11}C_1 + D_{12}C_2 + D_{13}C_3 + D_{15}C_5 = 0\,,\\
&D_{21}C_1 + D_{22}C_2 + D_{23}C_3 + D_{25}C_5 = 0\,,\\
&D_{31}C_1 + D_{32}C_2 + D_{33}C_3 + D_{35}C_5 = 0\,,\\
&D_{41}C_1 + D_{42}C_2 + D_{43}C_3 + D_{45}C_5 = 0\,,
\end{aligned} \tag{4.2}
$$

where

$$D_{11} = \frac{ik}{a}I_0(ka), \quad D_{12} = \left(\frac{\beta}{\varrho_v}A_1 - \frac{\beta + ikU}{\varrho_l}A_1\right) I_0(l_2a)\,,$$

$$D_{13} = B_2\frac{(\beta + ikU)^2}{\varrho_l}\varrho_v - \beta(\beta + ikU)K_0(ka) - \beta k K_0(ka)\,,$$

$$D_{14} = (\beta + ikU)al_4^2 I_1'(al_4)\,,$$

$$D_{21} = 2ik\mu_l I_0'(ka) + ik\eta_0 \cdot I_0 ka, \qquad D_{22} = -[UA_1I_0(l_2a) + \tau I_0(l_2a)UA_2]\,,$$

$$D_{23} = -\,UB_1\varrho_v(\beta + ikU)K_0(ka)\,,$$

$$D_{24} = \mu_l(-k^2l_4I_1(al_4) - ikI_1'(al_4) - \frac{\eta_0}{a}l_4I_1(al_4) - \eta_0 l_4^2 I_1'(al_4)\,,$$

$$D_{31} = \varrho_l\beta I_0(ka) + 2\mu_l k^2 I_0''(ka) + \frac{\sigma}{a^2\beta}(1 - k^2a^2)kI_0'(ka)\,,$$

$$D_{32} = -\left\{\frac{2\eta_0}{\varrho_l}A_1I_0(l_2a) + \frac{\sigma}{\beta a^2\varrho_1}(1 - k^2a^2)A_1I_0(l_2a) + \tau\frac{I_0(l_2a)}{K_0(l_3a)}\left[2\eta_0\varrho_l^{-1}A_2K_0(l_3a) + \frac{\sigma}{a^2\beta\varrho_l}(1 - k^2a^2)A_2K_0(l_3a)\right]\right\},$$

$$D_{33} = \left[2\eta_0\varrho_l^{-1}B_1\varrho_v(\beta + ikU)K_0(ka) - \varrho_v(\beta + ikU)K_0(ka) - \frac{\sigma}{a^2\beta}(1 - k^2a^2)\frac{\varrho_v}{\varrho_l}B_1(\beta + ikU)K_0(ka)\right],$$

$$D_{34} = 2\mu_l ikl_4I_1(al_4) - \frac{\sigma}{a^2\beta}(1 - k^2a^2)ikl_4I_1(al_4)\,,$$

$$D_{41} = D_{44} = 0\,,$$

$$D_{42} = \lambda_{\text{vap}} + \frac{3}{2}(\eta_0)^2(\varrho_v^{-2} - \varrho_1^{-2})A_1I_0(l_2a) - kl_2I_0'(l_2a) - \tau\frac{I_0(l_2a)}{K_0(l_3a)}\left[\lambda_{\text{vap}} + \frac{3}{2}\eta_0^2(\varrho_v^{-2} - \varrho_l^{-2})A_2K_0(l_3a)\right.,$$

$$D_{43} = \lambda_{\text{vap}} + \frac{3}{2}\eta_0^2(\varrho_v^{-2} - \varrho_l^{-2})B_1\varrho_v(\beta + ikU)K_0(ka)\,.$$

The system of equations (4.2) has a nontrivial solution only if

$$
\begin{vmatrix}
D_{11} & D_{12} & D_{13} & D_{15}\\
D_{21} & D_{22} & D_{23} & D_{25}\\
D_{31} & D_{32} & D_{33} & D_{35}\\
D_{41} & D_{42} & D_{43} & D_{45}
\end{vmatrix} = 0\,. \tag{4.3}
$$

This equation determines the wave numbers and wave speeds for which the boundary conditions (3.10) to (3.14) can be satisfied. If, for some wave numbers k, there exist wave speeds β with positive real parts, then there are disturbances growing in time; this means that the investigated flow is unstable.

In the case of the lack of evaporation, the coefficients η_0, A_1, A_2 and B_1 are equal to zero and $D_{43} = 0$, $D_{42} = -kl_2 I_0(l_2 a) \neq 0$, and the last equation of the system (4.2) is satisfied only by $C_2 = 0$. Putting $C_2 = 0$ in three other equations of the system (4.2) transforms them to the system

$$D_{11}C_1 + D_{13}C_3 + D_{15}C_5 = 0\,,$$
$$D_{21}C_1 + D_{23}C_3 + D_{25}C_5 = 0\,,$$
$$D_{31}C_1 + D_{33}C_3 + D_{35}C_5 = 0\,.$$

The condition for the existence of nontrivial solutions of the system (4.4) has the form

$$\begin{vmatrix} D_{11} & D_{13} & D_{15} \\ D_{21} & D_{23} & D_{25} \\ D_{31} & D_{33} & D_{35} \end{vmatrix} = 0\,. \tag{4.5}$$

If $\mu_l = \varrho_v = 0$, the last condition takes the well-known form

$$\beta^2 = \frac{\sigma}{\varrho_l a^3}(1 - k^2a^2)ka\frac{I_1(ka)}{I_0(ka)}\,, \tag{4.6}$$

obtained by Rayleigh [5].

5. Final remarks

The result of the present analysis is a derivation of the condition (4.3) as a kind of tool which makes it possible to obtain, through numerical calculation, the main stability characteristics in every defined case. Unfortunately, in the general case that condition does not allow us to deduce even qualitative information concerning the destabilizing role of evaporation.

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DEPARTMENT OF MATHEMATICS, MECHANICS AND INFORMATICS
UNIVERSITY OF WARSAW, WARSZAWA.

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