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Reflection and refraction of harmonic shear waves in incompressible viscoelastic fluids

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IN OUR PREVIOUS papers certain properties of small-amplitude transverse and longitudinal harmonic waves in simple fluids were widely discussed. In the present paper we consider the case in which the direction of shear wave propagation forms some angle with a plane interface between two non-mixing fluids with different viscoelastic response. The properties of reflected and refracted harmonic shear waves are discussed in greater detail for very low and very high (ultrasonic) frequencies. In particular, the amplitude indices (ratios) for reflected and refracted waves, the damping effects, the phase angles etc. are compared with those for purely viscous fluids.

W naszych poprzednich pracach obszernie przedyskutowano niektóre własności poprzecznych i podłużnych fal harmonicznych o małej amplitudzie, propagujących się w cieczach prostych. W niniejszej pracy rozważono przypadek, w którym kierunek propagacji fal ścinania tworzy pewien kąt z płaską powierzchnią między dwiema niemieszającymi się cieczami o różnych własnościach lepkosprężystych. Własności odbitych i załamanych harmonicznych fal ścinania przedyskutowano bardziej szczegółowo dla bardzo niskich i bardzo wysokich (naddźwiękowych) częstości. Porównano w szczególności wskaźniki amplitud fal odbitych i załamanych, efekty tłumienia, kąty przesunięcia fazowego itp. z odpowiednimi wielkościami dla cieczy czysto lepkich.

В наших предыдущих работах широко обсуждались некоторые свойства поперечных и продольных гармонических волн с малой амплитудой, распространяющихся в простых жидкостях. В настоящей работе рассматривается случай в котором направление распространения волн сдвига образует некоторый угол с плоской поверхностью между двумя несмешивающимися жидкостями с разными вязкоупругими свойстваим. Свойства отраженных и преломленных гармонических волн сдвига обсуждались более подробно для очень низких и очень высоких (ультразвуковых) частот. Сравнивались эффекты демпфирования, углы фазового перемещения и др. с соответствующими величинами для чисто вязких жидкостей.

1. Introduction

IN A SERIES of papers [1, 2, 3, 4, 5] various properties of plane harmonic waves, propagating or standing in compressible and incompressible simple fluids (cf. [6]), were discussed in greater detail. The governing motions were treated as particular cases of motions with superposed proportional stretch histories [7] or commutative motions according to GODDARD's definition [8]. The closed solutions could be presented under the assumptions either of linear shear response (cf. [1, 2]) or small amounts of deformation (small amplitudes). The longitudinal waves were discussed only in the linearized case (cf. [4, 5]). Certain general properties of waves were expressed by the generalized dynamic viscosities or moduli and the parameters similar to those introduced by TRUESDELL [9] for second order fluids.

In the present paper, being a continuation of previous considerations, we investigate the case of small-amplitude transverse (shear) waves progressing through a viscoelastic fluid composed of two non-mixing parts separated by a plane interface. Certain properties of reflected and refracted waves, for example the amplitude indices (ratios), the phase angles etc. are discussed in the full range of frequencies and for various material responses.

In Sect. 2 we briefly outline the useful results of previous considerations, paying particular attention to the Maxwell-like or Kelvin-like behaviour of fluids at very high frequencies (cf. [4, 5]). Section 3 is concerned with the kinematics of plane waves in the case in which the direction of propagation forms some angle of incidence with a plane interface between two materially distinct fluids. To this end a formal approach based on that proposed in the fundamental treatise of RAYLEIGH [10] is widely used. In Sect. 4 the cases of simultaneous reflection and refraction as well as full reflection and refraction without reflection are discussed in greater detail for fluids sliding at the interface. Section 5 is devoted on the whole to the limit behaviour of waves for very low and very high frequencies. It is assumed that, at very low frequencies, the fluid considered behaves like a Newtonian one, while at very high frequencies its viscoelastic behaviour may be of the Maxwell-like or Kelvin-like type. In Sect. 6 certain limit properties of waves are discussed and some results are illustrated in a graphical form.

2. Small-amplitude transverse waves

In our previous papers [3, 4] it was shown that oscillatory motions of the type

$$x = X+\chi(Z)\exp i\omega t, \quad y = Y, \quad z = Z, \tag{2.1}$$

lead to plane transverse harmonic waves in incompressible viscoelastic fluids if the corresponding amounts of shear (amplitudes) are sufficiently small. In the above equations only the real parts are meaningful; x, y, z denote the Cartesian coordinates of a particle at time t, X, Y, Z are the Cartesian coordinates in a reference configuration, $\chi(Z)$ are complex functions of Z only, and ω denotes an angular frequency.

It should be added, however, that transverse waves, being particular cases of isochoric motions, can propagate in incompressible as well as in compressible simple fluids, while longitudinal waves can be considered only in compressible media under the assumption of adiabatic or isothermal processes (cf. [3, 5]).

The functions $\chi(Z)$ resulting from the following linearized wave equation:

$$i\omega\eta^*(\omega)\chi''+\varrho\omega^2\chi = 0, \tag{2.2}$$

can be written in the form

$$\chi(Z) = A\exp(\beta+i\gamma)Z+B\exp(-\beta-i\gamma)Z, \tag{2.3}$$

where A and B are integration constants, and

$$(\beta+i\gamma)^2 = \frac{i\varrho\omega}{\eta^*(\omega)}, \quad \beta \geqslant 0, \quad \gamma > 0, \tag{2.4}$$

with $\eta^*(\omega)$ — the generalized dynamic viscosity and ϱ — the density of a fluid in the reference configuration. The frequency-dependent parameters β and γ can be considered as the coefficient of damping (attenuation) and the phase shift (wave number), respectively. They are simply related to the active and passive part of the mechanical impedance used in dynamical or acoustical measurements (cf. [11]).

From Eq. (2.4) the following relations can be derived:

$$\beta^2 = \frac{\varrho\omega}{2\eta'}\left[\frac{1}{\sqrt{1+\xi^2}} - \frac{\xi}{1+\xi^2}\right] = \frac{\varrho\omega}{2\eta''}\left[\frac{\xi}{\sqrt{1+\xi^2}} - \frac{\xi^2}{1+\xi^2}\right], \tag{2.5}$$

$$\gamma^2 = \frac{\varrho\omega}{2\eta'}\left[\frac{1}{\sqrt{1+\xi^2}} + \frac{\xi}{1+\xi^2}\right] = \frac{\varrho\omega}{2\eta''}\left[\frac{\xi}{\sqrt{1+\xi^2}} + \frac{\xi^2}{1+\xi^2}\right], \tag{2.6}$$

where

$$\xi = \frac{\eta''(\omega)}{\eta'(\omega)} = \frac{1}{\tan\delta}, \tag{2.7}$$

and the second forms on the right hand sides of Eqs. (2.5) and (2.6) are valid only for $\xi \neq 0$. We also introduce the following notations:

$$\eta' = \eta'(\omega) = G''(\omega)/\omega, \quad \eta'' = \eta''(\omega) = G'(\omega)/\omega \tag{2.8}$$

In the above relations η' and η'' (or G' and G'') denote real and imaginary parts of the generalized dynamic shear viscosity (or dynamic shear modulus), respectively. Here δ may be considered as the generalized loss angle.

Various properties of the waves considered were discussed in our previous papers [3, 4] in terms of parameters ξ. For the present purposes it is sufficient to note that numerous viscoelastic fluids like dilute and uncrosslinked polymer solutions, light oils with polymeric additives etc. behave in such a way that

$$\lim_{\omega\to 0}\xi(\omega) = 0, \quad \lim_{\omega\to\infty}\xi(\omega) = \infty. \tag{2.9}$$

This means that at very low frequencies the fluids considered behave like purely Newtonian fluids, and we have

$$\lim_{\omega\to 0}\beta^2 = \lim_{\omega\to 0}\gamma^2 = 0. \tag{2.10}$$

At very high frequencies, i.e. for ω tending to infinity, we can distinguish (cf. [3, 4, 5]) either the Kelvin-like behaviour, if

$$\lim_{\omega\to\infty}\frac{\omega^2}{G'(\omega)} = \text{const}, \tag{2.11}$$

or the Maxwell-like behaviour, if

$$\lim_{\omega\to\infty} G'(\omega) = \text{const}. \tag{2.12}$$

In the first case we obtain

$$\lim_{\omega\to\infty}\beta^2 = 0, \quad \lim_{\omega\to\infty}\gamma^2 = \text{const}, \tag{2.13}$$

while in the second case

$$\lim_{\omega \to \infty} \beta^2 = \text{const or } \infty, \quad \lim_{\omega \to \infty} \gamma^2 = \infty. \tag{2.14}$$

The existence of a finite limit in Eq. (2.14) highly depends on the rate with which $\xi(\omega)$ tends to infinity for increasing ω.

3. Progressive shear waves in incompressible viscoelastic fluids separated by a plane interface

To achieve more simplicity of description we restrict our further attention to the case of progressive (propagating) waves, retaining only the first term in Eq. (2.3). If the direction of wave propagation (or the wave vector) forms some angle of incidence with the normal to a plane interface separating the fluid considered into two parts with different viscoelastic properties, the well-known phenomena of wave reflection and refraction must be taken into account (Fig. 1).

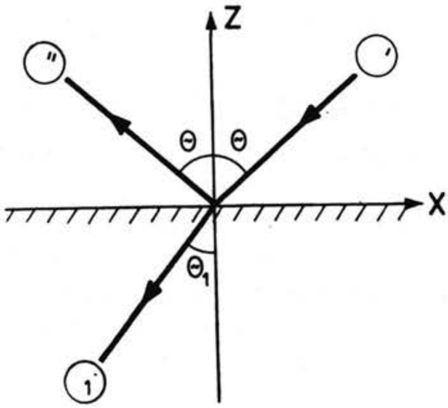

FIG. 1.

To this end Eqs. (2.1) can be transformed to the form

$$\begin{aligned} x &= X + \bar{A}\cos\Theta\exp[(\beta+i\gamma)(X\sin\Theta + Z\cos\Theta) + i\omega t],\\ y &= Y,\\ z &= Z - \bar{A}\sin\Theta\exp[(\beta+i\gamma)(X\sin\Theta + Z\cos\Theta) + i\omega t]. \end{aligned} \tag{3.1}$$

In the above equations Θ denotes the angle of incidence (cf. Fig. 1), $\bar{A}$ is a constant, and β, γ are defined in Eqs. (2.4) or Eqs. (2.5) and (2.6).

If we introduce the stream function Φ (potential) defined as follows,

$$u = \frac{\partial \Phi}{\partial Z}, \quad w = -\frac{\partial \Phi}{\partial X}, \tag{3.2}$$

where u and w are the velocity components in the directions of x and z, respectively, we have

$$\Phi = \alpha(\omega)\exp[(\beta+i\gamma)(X\sin\Theta + Z\cos\Theta) + i\omega t], \tag{3.3}$$

where $\alpha(\omega) = i\omega\bar{A}/(\beta+i\gamma)$.

Therefore, we arrive at the formal equations (cf. [10])

$$\Phi' = \alpha'(\omega)\exp[(\beta+i\gamma)Z\cos\Theta+(\beta+i\gamma)X\sin\Theta+i\omega t],$$
$$(3.4)\qquad \Phi'' = \alpha''(\omega)\exp[-(\beta+i\gamma)Z\cos\Theta+(\beta+i\gamma)X\sin\Theta+i\omega t],$$
$$\Phi_1 = \alpha_1(\omega)\exp[(\beta_1+i\gamma_1)Z\cos\Theta_1+(\beta+i\gamma)X\sin\Theta+i\omega t],$$

describing in order of precedence the primary, reflected and refracted shear waves. The primed quantities refer to primary waves, those with double primes — to reflected waves, and the subscript 1 is reserved for quantities connected with refracted (transmitted) waves (Fig. 1). According to these notations, α', α'', α_1 denote the complex amplitudes, Θ_1 — is the angle of refraction, and β_1, γ_1 characterize the viscoelastic properties of the lower fluid, where only refracted shear waves may be present. On writing Eqs. (3.4), we tacitly assumed that the angle of incidence is equal to the angle of reflection and the traces of wave vectors at the plane $Z = 0$ are the same (cf. [10]). The latter condition is equivalent to the following refraction law:

$$(3.5)\qquad \frac{\sin\Theta}{\sin\Theta_1} = \frac{\beta_1+i\gamma_1}{\beta+i\gamma},$$

which results directly from the appropriate boundary conditions (cf. Appendix A).

It is worthwhile to notice that in the case considered no longitudinal waves can be generated at the interface since both fluids are assumed to be incompressible.

We should also take into account the appropriate continuity conditions at the interface $Z = 0$. To this end we distinguish either the case in which the fluids can slide freely at the interface or the case in which the fluids fully adhere together.

3.1. The case of fluids sliding freely at the interface

In this case only the normal velocity and normal stress components are equal on both sides of the interface $Z = 0$, viz.

$$(3.6)\qquad w'+w'' = w_1 \quad \text{for} \quad Z = 0,$$

and

$$(3.7)\qquad T'^{33}+T''^{33} = T_1^{33} \quad \text{for} \quad Z = 0.$$

Bearing in mind Eqs. (3.2) and the constitutive relation (cf. [3, 4])

$$(3.8)\qquad T^{33} = -p+2\eta^*\frac{\partial w}{\partial Z} = -p-2\eta^*\frac{\partial^2\Phi}{\partial X\partial Z},$$

where p denotes a hydrostatic pressure (the same on both sides of the interface), we arrive at the following conditions:

$$(3.9)\qquad \alpha'+\alpha'' = \alpha_1,$$

$$(3.10)\qquad \varrho(\beta_1+i\gamma_1)\cos\Theta(\alpha'-\alpha'') = \varrho_1(\beta+i\gamma)\cos\Theta_1\alpha_1,$$

where ϱ and ϱ_1 denote mass densities of the upper and lower fluid, respectively. Since on both sides of the interface the waves have the same angular frequency ω, we shall also use the well-known relationship (cf. [10, 11]):

$$\frac{c}{c_1} = \frac{k_1}{k} = \sqrt{\frac{\beta_1^2+\gamma_1^2}{\beta^2+\gamma^2}}, \tag{3.11}$$

where k, k_1 are the wave numbers and $c = \omega/k$, $c_1 = \omega/k_1$ — the real-valued wave velocities in both parts of the fluid.

Equations (3.9) and (3.10) lead to the amplitude indices of the form

$$\frac{\alpha''}{\alpha'} = \frac{1-\dfrac{\varrho_1}{\varrho}\dfrac{\beta+i\gamma}{\beta_1+i\gamma_1}\dfrac{\cos\Theta_1}{\cos\Theta}}{1+\dfrac{\varrho_1}{\varrho}\dfrac{\beta+i\gamma}{\beta_1+i\gamma_1}\dfrac{\cos\Theta_1}{\cos\Theta}}, \quad \frac{\alpha_1}{\alpha'}\frac{2}{1+\dfrac{\varrho_1}{\varrho}\dfrac{\beta+i\gamma}{\beta_1+i\gamma_1}\dfrac{\cos\Theta_1}{\cos\Theta}}. \tag{3.12}$$

3.2. The case of fully adhering fluids

In this case all the velocity and stress components should be equal on both sides of the interface $Z = 0$. To satisfy these continuity conditions we must take into account, apart from the waves described by Eqs. (3.4), the interface type waves in the following formal form (cf. Appendix B):

$$\begin{aligned}\tilde{\Phi} &= \tilde{\alpha}(\omega)\exp[-\nu Z+\mu X+i\omega t],\\ \tilde{\Phi}_1 &= \tilde{\alpha}_1(\omega)\exp[\nu_1 Z+\mu X+i\omega t].\end{aligned} \tag{3.13}$$

Either of the above functions satisfies the corresponding wave equation (cf. Eqs. (2.2), (B.1)) if

$$\begin{aligned}\nu^2+\mu^2 &= \frac{i\varrho\omega}{\eta^*} = (\beta+i\gamma)^2,\\ \nu_1^2+\mu^2 &= \frac{i\varrho\omega}{\eta_1^*} = (\beta_1+i\gamma_1)^2.\end{aligned} \tag{3.14}$$

On the basis of the result (B.8) obtained in Appendix B, we put

$$\mu^2 = \nu^2 = \frac{1}{2}(\beta+i\gamma)^2, \quad \nu_1^2 = (\beta_1+i\gamma_1)^2-\frac{1}{2}(\beta+i\gamma)^2. \tag{3.15}$$

Thus, Eqs. (3.13) take the following final form:

$$\begin{aligned}\tilde{\Phi} &= \tilde{\alpha}(\omega)\exp\left[-\frac{1}{\sqrt{2}}(\beta+i\gamma)Z+\frac{1}{\sqrt{2}}(\beta+i\gamma)X+i\omega t\right],\\ \tilde{\Phi}_1 &= \tilde{\alpha}_1(\omega)\exp\left[\nu_1 Z+\frac{1}{\sqrt{2}}(\beta+i\gamma)X+i\omega t\right].\end{aligned} \tag{3.16}$$

It is worthwhile to note that the above waves are similar to those describing the reflected and refracted waves in Eqs. (3.4). $\tilde{\Phi}$ as well as $\tilde{\Phi}_1$ are damped with increasing distance from the interface $Z = 0$.

The continuity conditions can be expressed symbolically in the form

$$
(3.17)\qquad
\begin{aligned}
w'+w''+\tilde{w} &= w_1+\tilde{w}_1 \quad \text{for} \quad Z=0,\\
u'+u''+\tilde{u} &= u_1+\tilde{u}_1 \quad \text{for} \quad Z=0,\\
T'^{33}+T''^{33}+\tilde{T}^{33} &= T_1^{33}+\tilde{T}_1^{33} \quad \text{for} \quad Z=0,\\
T'^{13}+T''^{13}+\tilde{T}^{13} &= T_1^{13}+\tilde{T}_1^{13} \quad \text{for} \quad Z=0,
\end{aligned}
$$

where symbols with tildas refer to functions defined in Eqs. (3.16). Bearing in mind Eqs. (3.2), (3.8) and (B.3) we arrive at the following set of linear algebraic equations:

$$
(3.18)\qquad
\begin{aligned}
&\alpha'+\alpha''+\frac{1}{\sqrt{2}}\tilde{\alpha} = \alpha_1+\frac{1}{\sqrt{2}}\tilde{\alpha}_1,\\
&(\beta+i\gamma)\cos\Theta(\alpha'-\alpha'')-\frac{1}{\sqrt{2}}(\beta+i\gamma)\tilde{\alpha} = (\beta_1+i\gamma_1)\cos\Theta_1\,\alpha_1+\nu_1\tilde{\alpha}_1,\\
&\varrho(\beta_1+i\gamma_1)^2\cos\Theta\sin\Theta(\alpha'-\alpha'')-\frac{1}{2}\varrho(\beta_1+i\gamma_1)^2\tilde{\alpha}\\
&\qquad = \varrho_1(\beta_1+i\gamma_1)(\beta+i\gamma)\cos\Theta_1\sin\Theta\,\alpha_1+\frac{1}{\sqrt{2}}\varrho_1(\beta+i\gamma)\nu_1\tilde{\alpha}_1,\\
&\varrho(\beta_1+i\gamma_1)^2\cos2\Theta(\alpha'+\alpha'')\\
&\qquad = \varrho_1[(\beta_1+i\gamma_1)^2\cos^2\Theta_1-(\beta+i\gamma)^2\sin^2\Theta]\alpha_1+\varrho_1\left[\nu_1^2-\frac{1}{2}(\beta+i\gamma)^2\right]\tilde{\alpha}_1,
\end{aligned}
$$

where we have made use of the fact that (cf. Eq. (3.14))

$$
(3.19)\qquad \frac{(\beta_1+i\gamma_1)^2}{(\beta+i\gamma)^2} = \frac{\varrho_1}{\varrho}\frac{\eta^*}{\eta_1^*}.
$$

The set of Eqs. (3.18) can also be written in the ordered form:

$$
(3.20)\qquad
\begin{aligned}
&-\sqrt{2}\frac{\alpha''}{\alpha'}+\sqrt{2}\frac{\alpha_1}{\alpha'}-\frac{\tilde{\alpha}}{\alpha'}+\frac{\tilde{\alpha}_1}{\alpha'} = \sqrt{2},\\
&(\beta+i\gamma)\cos\Theta\frac{\alpha''}{\alpha'}+(\beta_1+i\gamma_1)\cos\Theta_1\frac{\alpha_1}{\alpha'}+\frac{1}{\sqrt{2}}(\beta+i\gamma)\frac{\tilde{\alpha}}{\alpha'}+\nu_1\frac{\tilde{\alpha}_1}{\alpha'} = (\beta+i\gamma)\cos\Theta,\\
&\varrho(\beta_1+i\gamma_1)^2\sin2\Theta\frac{\alpha''}{\alpha'}+\varrho_1(\beta_1+i\gamma_1)^2\sin2\Theta_1\frac{\alpha_1}{\alpha'}+\varrho(\beta_1+i\gamma_1)^2\frac{\tilde{\alpha}}{\alpha'}\\
&\qquad +\varrho_1(\beta+i\gamma)\nu_1\frac{\tilde{\alpha}_1}{\alpha'} = \varrho(\beta_1+i\gamma_1)^2\sin2\Theta,\\
&-\varrho(\beta_1+i\gamma_1)^2\cos2\Theta\frac{\alpha''}{\alpha'}+\varrho_1[(\beta_1+i\gamma_1)^2\cos^2\Theta_1-(\beta+i\gamma)^2\sin^2\Theta]\frac{\alpha_1}{\alpha'}\\
&\qquad +\varrho_1\left[\nu_1^2-\frac{1}{2}(\beta+i\gamma)^2\right]\frac{\tilde{\alpha}_1}{\alpha'} = \varrho(\beta_1+i\gamma_1)^2\cos2\Theta,
\end{aligned}
$$

from which the corresponding amplitude indices, i.e. α''/α', α_1/α', $\tilde{\alpha}/\alpha'$, $\tilde{\alpha}_1/\alpha'$ can be calculated by means of appropriate determinants.

Further analysis of wave properties will be restricted to the simpler case of fluids sliding freely at the interface (cf. Sect. 3.1).

4. Phenomena of wave reflection and refraction for fluids sliding freely at the interface

In our further analysis of the wave properties, we can distinguish the following cases: 1) the case of simultaneous reflection and refraction if $\cos^2\Theta_1 > 0$ or $\Theta_1 < \pi/2$, 2) the case of full reflection if $\cos^2\Theta_1 < 0$ or $\Theta_1 > \pi/2$. Let us consider them separately.

4.1. Simultaneous reflection and refraction ($\cos^2\Theta_1 > 0$)

In this case it is useful to present Eqs. (3.12) in the form containing explicit real and imaginary parts of the amplitude indices, viz.

$$(4.1)\qquad \frac{\alpha''}{\alpha'} = \frac{\frac{\varrho^2}{\varrho_1^2}(\beta_1^2+\gamma_1^2)-(\beta^2+\gamma^2)n^2}{\left(\frac{\varrho}{\varrho_1}\beta_1+\beta n\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1+\gamma n\right)} + i\,\frac{2\frac{\varrho}{\varrho_1}(\beta\gamma_1-\beta_1\gamma)n}{\left(\frac{\varrho}{\varrho_1}\beta_1+\beta n\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1+\varrho n\right)^2},$$

and

$$(4.2)\qquad \frac{\alpha_1}{\alpha'} = 2\frac{\varrho}{\varrho_1}\,\frac{\frac{\varrho}{\varrho_1}(\beta_1^2+\gamma_1^2)+(\beta_1\beta+\gamma_1\gamma)n}{\left(\frac{\varrho}{\varrho_1}\beta_1+\beta n\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1+\gamma n\right)^2} + i\,\frac{2\frac{\varrho}{\varrho_1}(\beta\gamma_1-\beta_1\gamma)n}{\left(\frac{\varrho}{\varrho_1}\beta_1+\beta n\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1+\gamma n\right)^2},$$

where we have denoted

$$(4.3)\qquad n = \left(\frac{1}{\cos^2\Theta} - \frac{\beta^2+\gamma^2}{\beta_1^2+\gamma_1^2}\tan^2\Theta\right)^{1/2}, \quad n^2 > 0.$$

The corresponding phase angles take the form

$$(4.4)\qquad \tan\varphi = \frac{2\frac{\varrho}{\varrho_1}(\beta\gamma_1-\beta_1\gamma)n}{\frac{\varrho^2}{\varrho_1^2}(\beta_1^2+\gamma_1^2)-(\beta^2+\gamma^2)n^2}$$

and

$$(4.5)\qquad \tan\varphi_1 = \frac{(\beta\gamma_1-\beta_1\gamma)n}{\frac{\varrho}{\varrho_1}(\beta_1^2+\gamma_1^2)+(\beta\beta+\gamma\gamma)n}.$$

It is seen from Eqs. (4.1) to (4.5) that the amplitude indices always become real quantities and the phase angles are equal to zero if the following proportionality condition is satisfied:

$$(4.6)\qquad \frac{\beta}{\beta_1} = \frac{\gamma}{\gamma_1}.$$

From Eqs. (2.5) and (2.6) it also results that the above condition is equivalent to

$$(4.7)\qquad \frac{\eta''}{\eta'} = \frac{\eta_1''}{\eta_1'} \quad \text{or} \quad \xi = \xi_1 \quad \text{or} \quad \tan\delta = \tan\delta_1.$$

The last condition means the equality of the generalized loss angles in the full range of frequencies. This is the case of Newtonian fluids for which $\beta_1 = \gamma_1$, $\beta = \gamma$, and purely elastic media for which $\beta = \beta_1 = 0$ (cf. Sect. 5).

It also results from Eq. (4.1) that for some value of the incidence angle the amplitude of reflected waves vanishes completely and only refracted waves can be present. This case, called briefly refraction without reflection, requires that

$$\sin^2\Theta = \frac{(\beta_1^2+\gamma_1^2)^2 - \frac{\varrho_1^2}{\varrho^2}(\beta^2+\gamma^2)(\beta_1^2+\gamma_1)^2}{(\beta_1^2+\gamma_1^2)^2 - \frac{\varrho_1^2}{\varrho^2}(\beta^2+\gamma^2)^2} = \frac{1-\frac{\varrho_1^2}{\varrho^2}\frac{c_1^2}{c^2}}{1-\frac{\varrho_1^2}{\varrho^2}\left(\frac{c_1^2}{c^2}\right)^2}. \tag{4.8}$$

The existence of such a critical angle Θ is possible only if c_1/c is contained between 1 and ϱ/ϱ_1.

Taking into account the limit behaviour of viscoelastic fluids discussed in Sect. 5, the formal equation (3.4) can be written in the form

$$\begin{aligned}
\operatorname{Re}\Phi' &= A'\exp(\beta X\sin\Theta+\beta Z\cos\Theta)\cos[\gamma(X\sin\Theta+Z\cos\Theta)+\omega t],\\
\operatorname{Re}\Phi'' &= A''\exp(\beta X\sin\Theta-\beta Z\cos\Theta)\cos[\gamma(X\sin\Theta-Z\cos\Theta)+\omega t+\varphi],\\
\operatorname{Re}\Phi_1 &= A_1\exp(\beta X\sin\Theta+\beta_1 Z\cos\Theta_1)\cos[\gamma X\sin\Theta+\gamma_1 Z\cos\Theta_1+\omega t+\varphi_1],
\end{aligned} \tag{4.9}$$

where A', A'' and A_1 denote the real amplitudes of the primary, reflected and refracted waves, respectively. It turns out that

$$A'' = \frac{1-a}{1+a}A', \quad A_1 = \frac{2}{1+a}A', \tag{4.10}$$

where

$$a = \frac{\varrho_1 c_1}{\varrho c}\left(\frac{1}{\cos^2\Theta} - \frac{c_1^2}{c^2}\tan^2\Theta\right)^{1/2}. \tag{4.11}$$

It is easy to see that under the present notation the case of refraction without reflection corresponds to $a = 1$.

4.2. Full reflection ($\cos^2\Theta_1 < 0$)

The phenomenon of full reflection, well-known in acoustics and optics (cf. [10, 11]), formally corresponds to imaginary values of $\cos\Theta_1$. In terms of the incidence angle Θ, this condition can be expressed as

$$\sin\Theta \geqslant \frac{c}{c_1} = \sqrt{\frac{\beta_1^2+\gamma_1^2}{\beta^2+\gamma^2}}. \tag{4.12}$$

The existence of such a critical angle Θ is possible only if $c/c_1 < 1$, i.e. if the wave velocity in the primary fluid is less than that in the secondary fluid. In other words, the phenomenon of full reflection may occur when the upper fluid is acoustically more dense than the lower one.

In this case Eqs. (3.1) can be written in the form

$$(4.13)\quad \frac{\overline{\alpha}''}{\overline{\alpha}'} = \frac{\frac{\varrho^2}{\varrho_1^2}(\beta_1^2+\gamma_1^2)-(\beta^2+\gamma^2)m^2}{\left(\frac{\varrho}{\varrho_1}\beta_1+\gamma m\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1-\beta m\right)^2}+i\frac{2\frac{\varrho}{\varrho_1}(\beta_1\beta+\gamma_1\gamma)m}{\left(\frac{\varrho}{\varrho_1}\beta_1+\gamma m\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1-\beta m\right)^2}$$

and

$$(4.14)\quad \frac{\overline{\alpha}_1}{\overline{\alpha}'} = 2\frac{\varrho}{\varrho_1}\frac{\frac{\varrho}{\varrho_1}(\beta_1^2+\gamma_1^2)+(\beta_1\gamma+\gamma_1\beta)m}{\left(\frac{\varrho}{\varrho_1}\beta_1+\gamma m\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1-\beta m\right)^2}+i\frac{2\frac{\varrho}{\varrho_1}(\beta_1\beta+\gamma_1\gamma)m}{\left(\frac{\varrho}{\varrho_1}\beta_1+\gamma m\right)^2+\left(\frac{\varrho}{\varrho_1}\gamma_1-\beta m\right)^2},$$

where we have denoted

$$(4.15)\quad m = \left(\frac{\beta^2+\gamma^2}{\beta_1^2+\gamma_1^2}\tan^2\Theta-\frac{1}{\cos^2\Theta}\right)^{1/2}, \quad m^2>0.$$

The corresponding phase angles are

$$(4.16)\quad \tan\overline{\varphi} = \frac{2\frac{\varrho}{\varrho_1}(\beta_1\beta+\gamma_1\gamma)m}{\frac{\varrho^2}{\varrho_1^2}(\beta_1^2+\gamma_1^2)-(\beta^2+\gamma^2)m^2}$$

and

$$(4.17)\quad \tan\overline{\varphi}_1 = \frac{(\beta_1\beta+\gamma_1\gamma)m}{\frac{\varrho}{\varrho_1}(\beta_1^2+\gamma_1^2)+(\beta_1\gamma+\gamma_1\beta)m}.$$

Taking into account the limit behaviour of viscoelastic fluids discussed in Sect. 5, we also arrive at

$$(4.18)\quad \begin{aligned} \mathrm{Re}\,\Phi' &= A'\exp(\beta X\sin\Theta+\beta Z\cos\Theta)\cos[\gamma(X\sin\Theta+Z\cos\Theta)+\omega t],\\ \mathrm{Re}\,\Phi'' &= A''\exp(\beta X\sin\Theta-\beta Z\cos\Theta)\cos[\gamma(X\sin\Theta-Z\cos\Theta)+\omega t+\overline{\varphi}],\\ \mathrm{Re}\,\Phi_1 &= A_1\exp\left(\beta X\sin\Theta+\gamma_1 Z\sqrt{d}\right)\cos\left[\gamma X\sin\Theta-\beta_1 Z\sqrt{d}+\omega t+\overline{\varphi}_1\right],\end{aligned}$$

where in the case of $\cos^2\Theta_1<0$ we obtain

$$(4.19)\quad A''=A',\quad A_1=\frac{2}{\sqrt{1+b^2}}A'$$

and

$$(4.20)\quad b=\frac{\varrho_1 c_1}{\varrho c}\left(\frac{c_1^2}{c^2}\tan^2\Theta-\frac{1}{\cos^2\Theta}\right)^{1/2}=\frac{\varrho_1 c_1}{\varrho c}\frac{\sqrt{d}}{\cos\Theta},\quad d=\frac{c_1^2}{c^2}\sin^2\Theta-1.$$

In the case considered Eq. $(4.18)_3$ is only of symbolic character and does not describe a wave in the usual sence. Any disturbances transmitted into the secondary fluid are strongly damped by the exponential term, even if the fluids considered are of purely elastic character ($\beta=\beta_1=0$).

The effect of material properties characterized by c_1/c on the amplitude indices and the phase angles can be illustrated in a graphical form. In Fig. 2 the indices A''/A' and A_1/A' are plotted in function of c_1/c for various indicated values of $\sin\Theta$, assuming that $\varrho_1/\varrho \approx 1$, i.e. the mass densities of both fluids are almost equal. In Figs. 3 and 4 similar graphs are shown for $\varrho_1/\varrho = 0.1$. In the case considered it is seen that the amplitude indices are monotonically decreasing functions of c_1/c. This means that the amplitudes of

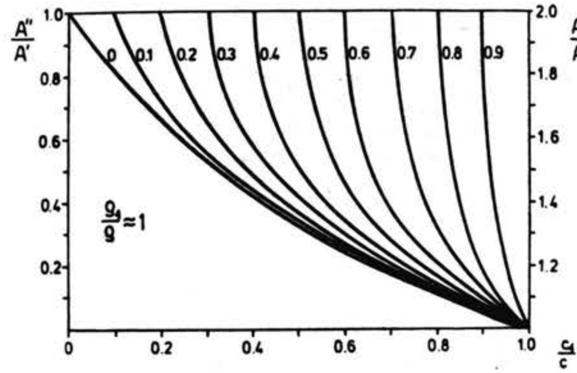

FIG. 2.

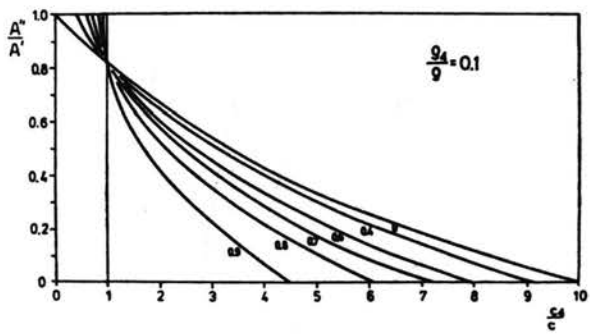

FIG. 3.

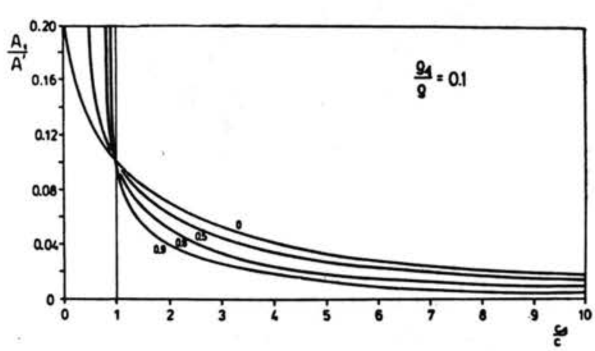

FIG. 4.

reflected and refracted waves (at the interface) are less than the amplitudes of primary waves if the acoustic density of the upper fluid increases. If ϱ_1/ϱ is less than unity, the graphs of A''/A' are shifted to the right, while those of $A_1 A'$ — to the left.

In the case of full reflection the amplitudes of reflected waves are exactly equal to those of primary waves. The dependence of A_1/A' on c/c_1, for $\varrho_1/\varrho \approx 1$, is also shown in Fig. 5. The corresponding phase angles for very high frequencies behave in accordance

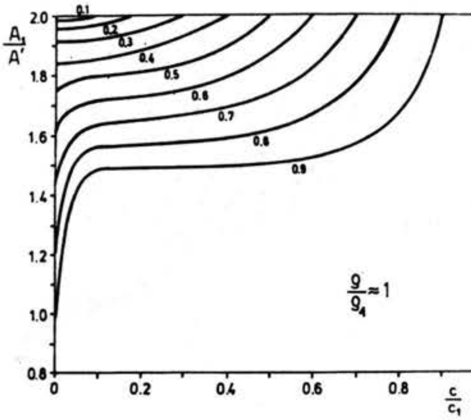

FIG. 5.

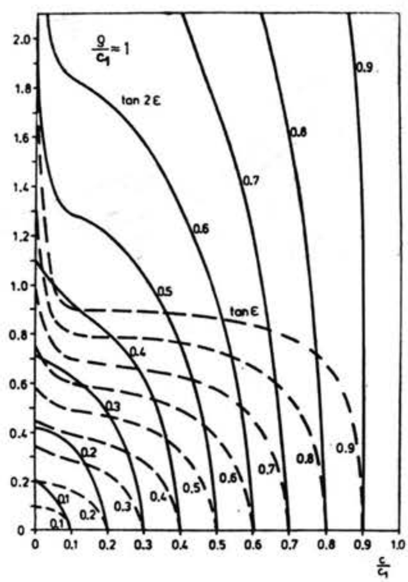

FIG. 6.

with Fig. 6, where $\tan\bar{\varphi} \approx \tan 2\varepsilon$ and $\tan\bar{\varphi}_1 \approx \tan\varepsilon$ are plotted in function of c/c_1 for various $\sin\Theta$ and $\varrho_1/\varrho \approx 1$. The phase angles of reflected waves take the highest values for c/c_1 tending to zero; if $\varrho_1/\varrho < 1$, the graphs are shifted to the left.

It is worthwhile to emphasize that the results obtained above are very similar to those known from the analysis of waves in purely elastic media. A discussion of simultaneous effects of high frequencies and material properties will be presented in Sect. 6.

5. Limit behaviour for very low and very high frequencies

We assume that according to Eqs. (2.10) to (2.14) the fluids considered behave like Newtonian ones at very low frequencies, and viscoelastic of the Maxwell-like or Kelvin-like type at very high frequencies. We shall discuss these two cases separately.

5.1. Viscous fluids (very low frequencies)

This case corresponds to the assumption that either $\omega \to 0$ or $\beta = \gamma$ and $\beta_1 = \gamma_1$. Then, for the case of simultaneously reflected and refracted waves ($\cos^2\Theta_1 > 0$), Eqs. (4.1) and (4.2) lead to

$$\lim_{\omega\to 0}\frac{\alpha''}{\alpha'} = \lim_{\omega\to 0}\frac{A''}{A'} = \lim_{\omega\to 0}\frac{1-a}{1+a}, \quad \lim_{\omega\to 0}\frac{\alpha_1}{\alpha'} = \lim_{\omega\to 0}\frac{A_1}{A'} = \lim_{\omega\to 0}\frac{2}{1+a}, \tag{5.1}$$

where a has been defined by Eq. (4.11). From Eqs. (4.4) and (4.5) it results that

$$\lim_{\omega\to 0}\tan\varphi = \lim_{\omega\to 0}\tan\varphi_1 = 0. \tag{5.2}$$

For the case of full reflection ($\cos^2\Theta_1 < 0$), we arrive at

$$\lim_{\omega\to 0}\frac{A''}{A'} = 1, \quad \lim_{\omega\to 0}\frac{A_1}{A'} = \lim_{\omega\to 0}\frac{2}{\sqrt{1+b^2}} \tag{5.3}$$

and

$$\lim_{\omega\to 0}\tan\bar{\varphi} = \lim_{\omega\to 0}\frac{2b}{1-b^2}, \quad \lim_{\omega\to 0}\tan\bar{\varphi}_1 = \lim_{\omega\to 0}\frac{b}{1+b}, \tag{5.4}$$

where b has been defined by Eq. (4.20). Since in this case the phase angles are not equal to zero, we also have

$$\begin{aligned} &\lim_{\omega\to 0}\mathrm{Re}\frac{\alpha''}{\alpha'} = \lim_{\omega\to 0}\frac{1-b^2}{1+b^2}, \quad \lim_{\omega\to 0}\mathrm{Im}\frac{\alpha''}{\alpha'} = \lim_{\omega\to 0}\frac{2b}{1+b^2}, \\ &\lim_{\omega\to 0}\mathrm{Re}\frac{\alpha_1}{\alpha'} = \lim_{\omega\to 0}\frac{2(1+b)}{1+b^2}, \quad \lim_{\omega\to 0}\mathrm{Im}\frac{\alpha_1}{\alpha'} = \lim_{\omega\to 0}\frac{2b}{1+b^2}. \end{aligned} \tag{5.5}$$

For purely viscous fluids β and γ are defined as follows (cf. [3, 4]):

$$\beta^2 = \gamma^2 = \frac{\varrho\omega}{2\eta'(0)}. \tag{5.6}$$

Then, the ratios β/β_1 and γ/γ_1 are constants idependent of ω, and so are a and b defined by Eqs. (4.11) and (4.20). Thus the limits described in Eqs. (5.1) to (5.5) are independent of frequency ω.

5.2. Viscoelastic fluids (very high frequencies)

This case corresponds to the assumption that for $\omega \to \infty$, the parameters β and γ reach their limits described by Eqs. (2.13) and (2.14) for the Kelvin-like behaviour and the Maxwell-like behaviour, respectively. Since in the problems considered we deal with two different fluids (the upper and the lower one), there exist at least four possibilities of material behaviour at very high frequencies. The situations in which the lower fluid is of the Maxwell-like type, while the upper one is of the Kelvin-like type and vice versa seem especially interesting.

A thorough inspection of the above possibilities proves that for simultaneous reflection and refraction the following relations are valid:

$$(5.7)\qquad \lim_{\omega\to\infty}\frac{\alpha''}{\alpha'} = \lim_{\omega\to\infty}\frac{A''}{A'} = \lim_{\omega\to\infty}\frac{1-a}{1+a}, \quad \lim_{\omega\to\infty}\frac{\alpha_1}{\alpha'} = \lim_{\omega\to\infty}\frac{A_1}{A'} = \lim_{\omega\to\infty}\frac{2}{1+a}$$

and

$$(5.8)\qquad \lim_{\omega\to\infty}\tan\varphi = \lim_{\omega\to\infty}\tan\varphi_1 = 0,$$

where a has been defined by Eq. (4.11).

The different relations result for the case of full reflection, viz.

$$(5.9)\qquad \lim_{\omega\to\infty}\frac{A''}{A'} = 1, \quad \lim_{\omega\to\infty}\frac{A_1}{A'} = \lim_{\omega\to\infty}\frac{2}{\sqrt{1+b^2}},$$

$$(5.10)\qquad \begin{aligned} \lim_{\omega\to\infty}\tan\bar{\varphi} &= \lim_{\omega\to\infty}\frac{2b}{1-b^2} = \tan 2\varepsilon,\\ \lim_{\omega\to\infty}\tan\bar{\varphi}_1 &= \lim_{\omega\to\infty} b = \tan\varepsilon, \end{aligned}$$

and also

$$(5.11)\qquad \begin{aligned} \lim_{\omega\to\infty}\mathrm{Re}\frac{\alpha''}{\alpha'} &= \lim_{\omega\to\infty}\frac{1-b^2}{1+b^2}, \quad \lim_{\omega\to\infty}\mathrm{Im}\frac{\alpha''}{\alpha'} = \lim_{\omega\to\infty}\frac{2b}{1+b^2},\\ \lim_{\omega\to\infty}\mathrm{Re}\frac{\alpha_1}{\alpha'} &= \lim_{\omega\to\infty}\frac{2}{1+b^2}, \quad \lim_{\omega\to\infty}\mathrm{Im}\frac{\alpha_1}{\alpha'} = \lim_{\omega\to\infty}\frac{2b}{1+b^2}, \end{aligned}$$

where b has been defined by Eq. (4.20).

The most interesting cases occur when for $\omega \to \infty$ the upper fluid is of the Maxwell-like type, while the lower one is of the Kelvin-like type and vice versa. In the first case, $c/c_1 \to 0$ and, according to Eq. (4.12), the waves are always fully reflected for any angle Θ. In the second case, $c/c_1 \to \infty$ and, depending on the value of ϱ/ϱ_1, the waves may be refracted without being reflected. Of course, the case of a homogeneous fluid corresponds to: $\varrho/\varrho_1 = 1$, $c/c_1 = 1$.

6. Final discussion of the results

The results presented in Sect. 4 and 5 allow for some discussion of the wave properties. In general, such a discussion is very difficult since for viscoelastic fluids we have to deal with a simultaneous effect of four parameters: the incidence angle Θ, the frequency ω, the ratio of mass densities ϱ_1/ϱ and the ratio c_1/c which itself depends on the frequency ω. Certain remarks on the effect of Θ and c_1/c on the amplitude indices and phase angles have been expressed in Sect. 4. Now we shall briefly discuss the simultaneous influence of ω and c_1/c on the above mentioned quantities.

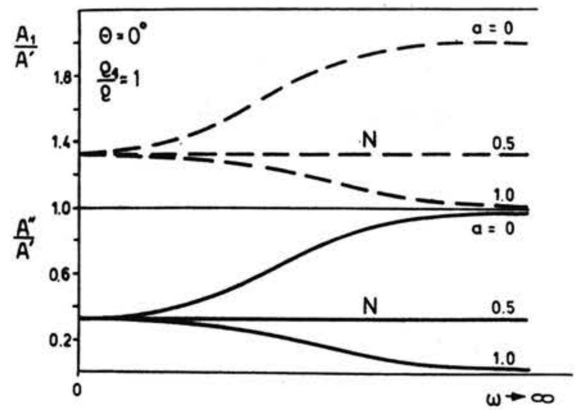

FIG. 7.

In Fig. 7 the dependence of the amplitude indices A''/A' and A_1/A' on ω is schematically plotted for the case of simultaneous reflection and refraction ($\cos^2\Theta_1 > 0$) under the assumption that $\varrho_1/\varrho \approx 1, \Theta \approx 0$. The numbers at the graphs denote the values of a (cf. Eq. (4.11), the Newtonian behaviour is described by straight lines marked with N. It is seen that the amplitudes of reflected and refracted waves increase with frequency ω, only for decreasing values of c_1/c, i.e. for increasing acoustical density of the lower fluid. The maximum amplification effects, irrespective of damping effects caused by viscous properties, are achieved for $c_1/c \to 0$, i.e. for the case when the lower fluid becomes the Maxwell-like type and the upper one — the Kelvin-like type. On the other hand, the maximum attenuation effect occurs at $a = 1$ (or $c_1/c = 1$), i.e. when both fluids become identical; then the case of refraction without reflection takes place.

Figures 8 and 9 illustrate schematically the dependence of the phase angles φ, φ_1 and $\bar{\varphi}$, $\bar{\varphi}_1$ on frequency ω for the case of $\cos^2\Theta_1 > 0$ and $\cos^2\Theta_1 < 0$, respectively. For

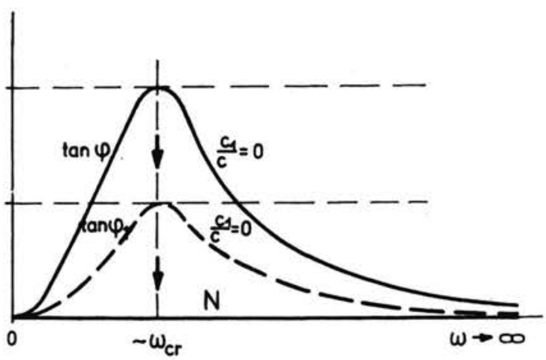

FIG. 8.

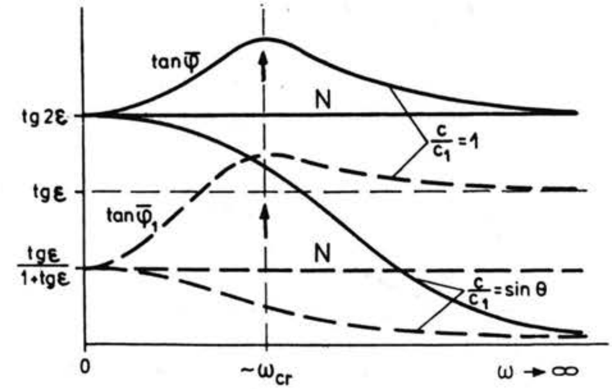

FIG. 9.

simultaneous reflection and refraction, the phase angles φ and φ_1 reach maximum values at some frequencies close to those at which damping effects are the most pronounced. For increasing values of c_1/c (marked with arrows), these maxima become weaker, reaching the zero limit for purely viscous fluids. The same effect can be observed when the generalized loss angles are the same for both fluids (cf. Eq. (4.7)). For the case of full reflection the phase angles $\overline{\varphi}$ and $\overline{\varphi}_1$ may also reach similar maxima which entirely disappear for Newtonian fluids. The limit values at $\omega \to 0$ and $\omega \to \infty$ essentially depend on the incidence angle Θ; for $\Theta \approx 0$ they are equal to zero. The smallest values of phase angles are expected for very high frequencies and c/c_1 close to the value of $\sin\Theta$.

It is seen from the above discussion that the behaviour of reflected and refracted waves essentially depends on viscoelastic properties of the fluids considered, especially at very high frequencies. It is also worthwhile to notice that any disturbances propagating in viscoelastic fluids in the form of harmonic shear waves are less damped and may be transmitted at longer distances than in purely viscous fluids.

Appendix A

We can consider Eq. (3.4) written in the more general form for which

(A.1) $$\Phi_1 = \alpha_1(\omega)\exp[(\beta_1+i\gamma_1)Z\cos\Theta_1+(\beta_1+i\gamma_1)X\sin\Theta_1+i\omega t]$$

and the notations are similar to those used in Eqs. (3.4). The boundary conditions (3.6) and (3.7), after taking into account Eqs. (2.4) and (3.2), lead to

(A.2) $$(\alpha'+\alpha'')(\beta+i\gamma)\sin\Theta = \alpha_1(\beta_1+i\gamma_1)\sin\Theta_1\exp[(\beta_1+i\gamma_1)X\sin\Theta_1-(\beta+i\gamma)X\sin\Theta]$$

and

(A.3) $$(\alpha'-\alpha'')\cos\Theta\sin\Theta = \frac{\varrho_1}{\varrho}\alpha_1\cos\Theta_1\sin\Theta_1\exp[(\beta_1+i\gamma_1)X\sin\Theta_1-(\beta+i\gamma)X\sin\Theta],$$

respectively. Since the left-hand sides of the above equations are independent of X, the right-hand sides of both equations require that

(A.4) $$(\beta_1+i\gamma_1)\sin\Theta_1-(\beta+i\gamma)\sin\Theta = 0.$$

Thus we obtain the refraction law presented in Eq. (3.5) and the boundary conditions in the forms (3.9) and (3.10).

It can also be shown that the boundary conditions (3.17) lead to the refraction law in the form (A.4).

Appendix B

Let us consider the auxiliary case in which an incompressible fluid with a free surface occupies the lower half-space $Z \leqslant 0$. The harmonic wave equation in the plane XZ, viz.

(B.1) $$\left(\nabla^2 - \frac{i\varrho\omega}{\eta^*}\right)\Phi = 0$$

is identically satisfied if

(B.2) $$\Phi = \tilde{\alpha}(\omega)\exp(\nu Z + \mu X), \quad \nu^2 + \mu^2 = \frac{i\varrho\omega}{\eta^*}.$$

The stress components are determined by Eq. (3.8) and

(B.3) $$T^{13} = \eta^*\left(\frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X}\right) = \eta^*\left(\frac{\partial^2 \Phi}{\partial Z^2} - \frac{\partial^2 \Phi}{\partial X^2}\right),$$

(B.4) $$p = p_0(\omega)\exp(\nu Z + \mu X),$$

where p denotes a hydrostatic pressure. When surface tension effects can be neglected, the boundary conditions at the free surface, viz.

(B.5) $$T^{33} = 0, \quad T^{13} = 0,$$

lead to

(B.6) $$2\eta^*\nu\mu\tilde{\alpha} + p_0 = 0,$$

(B.7) $$\eta^*(\nu^2 - \mu^2)\tilde{\alpha} = 0.$$

The above conditions can be satisfied if and only if

(B.8) $$\nu^2 = \mu^2 = \frac{i\varrho\omega}{2\eta^*} = \frac{1}{2}(\beta + i\gamma)^2.$$

Thus the solution of the surface-wave type takes the following form:

(B.9) $$\Phi = \tilde{\alpha}(\omega)\exp\left[\frac{1}{\sqrt{2}}(\beta + i\gamma)Z + \frac{1}{\sqrt{2}}(\beta + i\gamma)X + i\omega t\right].$$

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