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Plastic materials with continuous transition between loading and unloading states

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PLASTIC materials with continuous transition between loading and unloading states are proposed. Two different rate type constitutive equations in loading and unloading states are unified into a single equation by introducing a continuous function. The work-hardening is taken into consideration by the method of the internal state variables, and their evolutional equations in both states are also unified into a single set of equations. For a small value of the transition parameter the gradual transition occurs between two states, while for a large value a rapid transition occurs and it is substantially identical with the sudden transition caused by the adoption of two sets of equations.

Zaproponowano teorię materiałów o ciągłym przejściu od stanu obciążania do stanu odciążania. Dwa różne typy równań konstytutywnych dla stanów odciążania i obciążania sprowadzono do jednego równania drogą wprowadzenia ciągłej funkcji przejścia. Wzmocnienie uwzględniono za pomocą metody wewnętrznych zmiennych stanu, a ich równania ewolucyjne w obu stanach zostały również sprowadzone do pojedynczego układu równań. Przy małych wartościach parametru przejścia następuje stopniowe przejście od jednego stanu do drugiego, podczas gdy przy dużych wartościach parametru przejście to ma charakter gwałtowny i jest w zasadzie identyczne z gwałtownym przejściem wynikającym z przyjęcia dwóch układów równań.

Предложена теория материалов с непрерывным переходом от состояния нагружения до состояния разгружения. Два разных типа определяющих уравнений для состояний разгружения и нагружения сведены к одному уравнению путем введения непрерывной функции перехода. Упрочнение учтено при помощи метода внутренних переменных состояния, а их эволюционные уравнения в обоих состояниях тоже сведены к единичной системе уравнений. При малых значениях параметра перехода наступает постепенный переход от одного состояния к другому, в то время когда при больших значениях параметра этот переход имеет внезапный характер и в принципе идентичен с внезапным переходом, следующим из принятия двух систем уравнений.

1. Introduction

FOR THE CLASSICAL theory of plasticity, a plastic material is defined by a set of equations and inequalities, that is, two constitutive equations in loading and unloading states, a yield condition, and a work-hardening rule. These relations reveal the corresponding individual phenomena and there are, in general, no intrinsic interrelations between these relations. For the classical theory of plasticity, refer, for example, to HILL [1] and PAUL [2].

At present there is a trend to reduce the number of these relations in such a way that two or more phenomena can be expressed by a relation. For example, VALANIS [3] proposed the constitutive equations of integral form by means of intrinsic time measures; OWEN [4], WHITE [5] and HOLSAPPLE [6] proposed the functional theories of plasticity, all of which express plasticity of special kinds of simple materials defined by NOLL [7]. LUBLINER [8] and the author [9] attempted to derive a yield condition by the method of the Clausius–Duhem inequality. The author also laid down a statement of hypo-elasticity as basis and from it he derived yield conditions and flow rules [10, 11].

The hypo-elastic material framed by TRUESDELL [12] has a type of constitutive equation

$$\overset{\circ}{\mathbf{T}} = \mathscr{H}(\mathbf{T})[\mathbf{D}], \tag{1.1}$$

where $\mathbf{T}$ is the Cauchy stress,

$$\overset{\circ}{\mathbf{T}} \equiv \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} \tag{1.2}$$

is the co-rotational time rate of stress, and the stretching tensor $\mathbf{D}$ and the spin tensor $\mathbf{W}$ are, respectively, the symmetric and the skew-symmetric parts of the velocity gradient of a material point. Equation (1.1) means a linear and homogeneous relation between the stress rate $\overset{\circ}{\mathbf{T}}$ and the deformation rate $\mathbf{D}$ and then the hypo-elastic material is independent of the time scale. The stress rate (1.2) and the stretching assure that Eq. (1.1) satisfies the principle of objectivity, which denotes the independent property of the material with the observer.

When the deformation rate is given, the stress rate is uniquely determined by Eq. (1.1). When the stress rate is given, the deformation rate can be uniquely determined if Eq. (1.1) has one-to-one correspondence and it cannot be done if Eq. (1.1) is singular. The author defined that the singularity relation of Eq. (1.1) denotes the *yield condition* [10, 11]. Thus the yield condition is not a given relation a priori but a reduced one given by the constitutive equation.

However, a single equation of the type (1.1) cannot express both states of *loading* and *unloading*, so the author [13, 14] introduced a set of two constitutive equations, the equation

$$\overset{\circ}{\mathbf{T}} = {}_{\mathrm{E}}\mathscr{H}(\mathbf{T})[\mathbf{D}] + {}_{\mathrm{P}}\mathscr{H}(\mathbf{T})[\mathbf{D}], \tag{1.3}$$

holds in loading state and the other

$$\overset{\circ}{\mathbf{T}} = {}_{\mathrm{E}}\mathscr{H}(\mathbf{T})[\mathbf{D}] \tag{1.4}$$

does in unloading state, where ${}_{\mathrm{E}}\mathscr{H}$ and ${}_{\mathrm{P}}\mathscr{H}$ were chosen appropriately. Also, the author introduced two internal state variables, one is scalar and the other is tensor in order to express the work-hardening. The scalar and tensor internal state variables refer to the *isotropic* and the *translational work-hardenings*, respectively.

In this paper two constitutive equations (1.3) and (1.4) will be unified into a single equations by the method of transition function proposed by the author [15]. The obtained equation holds in both states and gives the continuous transition between them. The evolutional equations which govern the internal state variables will also be unified into a single set of equations.

2. Unification to a single constitutive equation

The *stress power*

$$w^* \equiv \mathrm{tr}(\mathbf{T}^*\mathbf{D}) \tag{2.1}$$

denotes the work rate per unit volume for the deviatoric stress $\mathbf{T}^*$. When $w^* > 0$, the exterior of a body executes work upon the body, and when $w^* < 0$, the body does work upon its exterior. It is then very natural to assume that the loading and the unloading states are defined by the conditions $w^* > 0$ and $w^* < 0$, respectively.

A set of equations (1.3) and (1.4) can be formally put into a single equation by the unit step function

$$U(x) = \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0. \end{cases} \tag{2.2}$$

That is, we have

$$\overset{\circ}{\mathbf{T}} = {}_{\mathrm{E}}\mathscr{H}(\mathbf{T})\,[\mathbf{D}] + {}_{\mathrm{P}}\mathscr{H}(\mathbf{T})\,[\mathbf{D}]\,U(w^*). \tag{2.3}$$

However, we cannot regard this device as a unification to a single equation because the process is very formal and Eq. (2.3) is identically equal to the set of Eqs. (1.3) and (1.4).

Let us try to replace the unit step function $U(x)$ by a monotonically increasing continuous function $V(x)$ which has the following limit properties:

$$\lim_{x\to-\infty} V(x) = 0, \qquad \lim_{x\to+\infty} V(x) = 1. \tag{2.4}$$

We then have the single constitutive equation

$$\overset{\circ}{\mathbf{T}} = {}_{\mathrm{E}}\mathscr{H}(\mathbf{T})\,[\mathbf{D}] + {}_{\mathrm{P}}\mathscr{H}(\mathbf{T})\,[\mathbf{D}]\,V\left(\frac{w^*}{w_0}\right), \tag{2.5}$$

where

$$w_0 \equiv \frac{1}{\varkappa} K|\mathbf{D}| \tag{2.6}$$

is a *characteristic stress power* depending on the material. K (> 0) denotes a material constant with the dimension of stress, $|\mathbf{D}| \equiv \{\mathrm{tr}(\mathbf{D}^2)\}^{1/2}$ is the magnitude of stretching, and $\varkappa$ (> 0) is a material constant called the *transition parameter* which defines the breadth of the *transition region*. Function $V(x)$ is called the *transition function*.

It is clearly seen that Eq. (2.5) is independent of the time scale but it has no linear relation between the stress rate and the deformation rate. Then, it does not belong to the hypo-elasticity in the strict sense, but in a large magnitude of stress, Eq. (2.5) reduces to Eqs. (1.3) and (1.4) according to $w^* > 0$ and $w^* < 0$, respectively, and we can apply the author's definition of the yield condition to them. Equation (2.5) holds in loading and unloading states, and it may fit with the yield phenomenon. However, it cannot show any work-hardening property.

There is a great number of monotonically increasing continuous functions which have the properties (2.4) but here we adopt tentatively the following function:

$$V(x) = \frac{1}{1+e^{-x}}. \tag{2.7}$$

When the transition region is defined by the region which satisfies $0.1 < V(x) < 0.9$, it is given by $|x| < 2.2$, and when it is done by $0.01 < V(x) < 0.99$, it is given by $|x| < 4.6$.

3. Plastic material with combined work-hardening

The theory of internal state variable is one of the most powerful theories to express the internal change of state in a material [16]. PERZYNA [17] and LUBLINER [18] applied it to their plasticity theories. The author also adopted it to his rate type theory of plasticity and he could express well the combined work-hardening [13, 14].

The *constitutive equation* and the *evolutional equations* in rate type form are given by

$$\mathring{\mathbf{T}} = {}_{\mathrm{E}}\mathscr{H}(\mathbf{T})[\mathbf{D}] + {}_{\mathrm{P}}\mathscr{H}(\tilde{\mathbf{T}}, \alpha)[\mathbf{D}], \tag{3.1}$$

$$\begin{aligned} \dot{\alpha} &= \boldsymbol{\Phi}(\tilde{\mathbf{T}}, \alpha)[\mathbf{D}], \\ \mathring{\boldsymbol{\beta}} &= \boldsymbol{\Psi}(\tilde{\mathbf{T}}, \alpha)[\mathbf{D}] \end{aligned} \tag{3.2}$$

in loading state, and

$$\mathring{\mathbf{T}} = {}_{\mathrm{E}}\mathscr{H}(\mathbf{T})[\mathbf{D}], \tag{3.3}$$

$$\begin{aligned} \dot{\alpha} &= 0, \\ \mathring{\boldsymbol{\beta}} &= \mathbf{0} \end{aligned} \tag{3.4}$$

in unloading state, where α and $\boldsymbol{\beta}$ are the scalar and the tensor internal state variables, $\mathring{\boldsymbol{\beta}}$ is the co-rotational time rate of $\boldsymbol{\beta}$ with the similar expression of Eq. (1.2), and

$$\tilde{\mathbf{T}} \equiv \mathbf{T} - \boldsymbol{\beta} \tag{3.5}$$

is the *translated stress*. The internal state variables α and $\boldsymbol{\beta}$ contribute to the *isotropic* and the *translational work-hardenings*, respectively.

We can apply the device which was introduced in the precedent section to Eqs. (3.1)–(3.4). Then we have a single constitutive equation

$$\mathring{\mathbf{T}} = {}_{\mathrm{E}}\mathscr{H}(\mathbf{T})[\mathbf{D}] + {}_{\mathrm{P}}\mathscr{H}(\tilde{\mathbf{T}}, \alpha)[\mathbf{D}]V\left(\frac{\tilde{w}^*}{w_0}\right) \tag{3.6}$$

and a set of evolutional equations

$$\begin{aligned} \dot{\alpha} &= \boldsymbol{\Phi}(\tilde{\mathbf{T}}, \alpha)[\mathbf{D}]V\left(\frac{\tilde{w}^*}{w_0}\right), \\ \mathring{\boldsymbol{\beta}} &= \boldsymbol{\Psi}(\tilde{\mathbf{T}}, \alpha)[\mathbf{D}]V\left(\frac{\tilde{w}^*}{w_0}\right), \end{aligned} \tag{3.7}$$

where

$$\tilde{w}^* \equiv \mathrm{tr}(\tilde{\mathbf{T}}^*\mathbf{D}) \tag{3.8}$$

is the work rate per unit volume for the deviatoric translated stress $\tilde{\mathbf{T}}^*$. Here the loading and the unloading states are defined by $\tilde{w}^* > 0$ and $\tilde{w}^* < 0$, respectively.

The *Prandtl–Reuss material with combined work-hardening* [13, 14] is a special case of the materials which are defined by Eqs. (3.1)–(3.4). It has special forms of ${}_{\mathrm{E}}\mathscr{H}$, ${}_{\mathrm{P}}\mathscr{H}$, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. Its constitutive equation and evolutional equations in the forms of Eqs. (3.6) and (3.7) are, respectively, given by

$$\mathring{\mathbf{T}} = \lambda\,\mathrm{tr}(\mathbf{D})\mathbf{1} + 2\mu\left[\mathbf{D} - \frac{\tilde{w}^*}{K(\alpha)^2}\tilde{\mathbf{T}}^*\right]V\left(\frac{\tilde{w}^*}{w_0}\right), \tag{3.9}$$

$$\dot{\alpha} = \frac{1}{2\mu}\tilde{w}^* V\left(\frac{\tilde{w}^*}{w_0}\right),$$
(3.10)
$$\mathring{\boldsymbol{\beta}} = \frac{2\mu c\tilde{w}^*}{K(\alpha)^2}\tilde{\mathbf{T}}^* V\left(\frac{\tilde{w}^*}{w_0}\right),$$

where λ and μ are Lamé's modulus, c is a dimensionless material constant and $K(\alpha)$ is a material function. We can assume any appropriate expression of $K(\alpha)$ but here, for simplicity, we adopt

(3.11)
$$K(\alpha) = K_0(1+a\alpha)^n,$$

where K_0, a and n are material constants having positive values.

4. Uniaxial stress extension

In order to estimate the behavior of the material with a continuous transition region, here we will study the responses of the incompressible Prandtl–Reuss material defined by Eqs. (3.9) and (3.10) in the uniaxial stress extension. For numerical calculation the equations are transformed into non-dimensional forms

(4.1)
$$\mathring{\tilde{\mathbf{S}}}^* = \mathbf{D}^* - \frac{\tilde{v}^*}{M(\alpha)^2}\tilde{\mathbf{S}}^* V\left(\frac{\tilde{v}^*}{v_0}\right),$$

$$\dot{\alpha} = \tilde{v}^* V\left(\frac{\tilde{v}^*}{v_0}\right),$$
(4.2)
$$\mathring{\boldsymbol{\gamma}} = \frac{c\tilde{v}^*}{M(\alpha)^2}\tilde{\mathbf{S}}^* V\left(\frac{\tilde{v}^*}{v_0}\right),$$

where we put

$$\mathbf{S} \equiv \frac{T}{2\mu}, \quad \boldsymbol{\gamma} \equiv \frac{\boldsymbol{\beta}}{2\mu}, \quad \tilde{v}^* \equiv \operatorname{tr}(\tilde{\mathbf{S}}^*\mathbf{D}),$$
(4.3)
$$M(\alpha) \equiv M_0(1+a\alpha)^n, \quad M_0 \equiv \frac{K_0}{2\mu(1+c)^{1/2}},$$

and we assume

(4.4)
$$v_0 \equiv \frac{1}{\varkappa}\left(\frac{2}{3}\right)^{1/2} M_0|\mathbf{D}|.$$

Then, a material constant K in Eq. (2.6) is equated to $(2/3)^{1/2} K_0/(1+c)^{1/2}$.

In incompressible uniaxial stress extension along the x_1-axis, we may assume

$$S_1 = S, \quad S_2 = S_3 = 0; \quad D_1 = D, \quad D_2 = D_3 = -\frac{1}{2}D;$$
(4.5)
$$\gamma_1 = \frac{2}{3}\gamma, \quad \gamma_2 = \gamma_3 = -\frac{1}{3}\gamma.$$

Then, the spin tensor vanishes identically, and we have

$$\tilde{v}^* = \tilde{S}D, \quad v_0 = \frac{1}{\varkappa} M_0|\mathbf{D}|, \quad \tilde{S} \equiv S-\gamma. \tag{4.6}$$

Therefore the constitutive equation (3.9) and the evolutional equations (3.10) are reduced to

$$\frac{d\bar{S}}{dE} = \frac{3}{2} - \frac{\tilde{S}^2}{M_0^2(1+a\alpha)^{2n}} V\left(\frac{\tilde{v}^*}{v_0}\right), \tag{4.7}$$

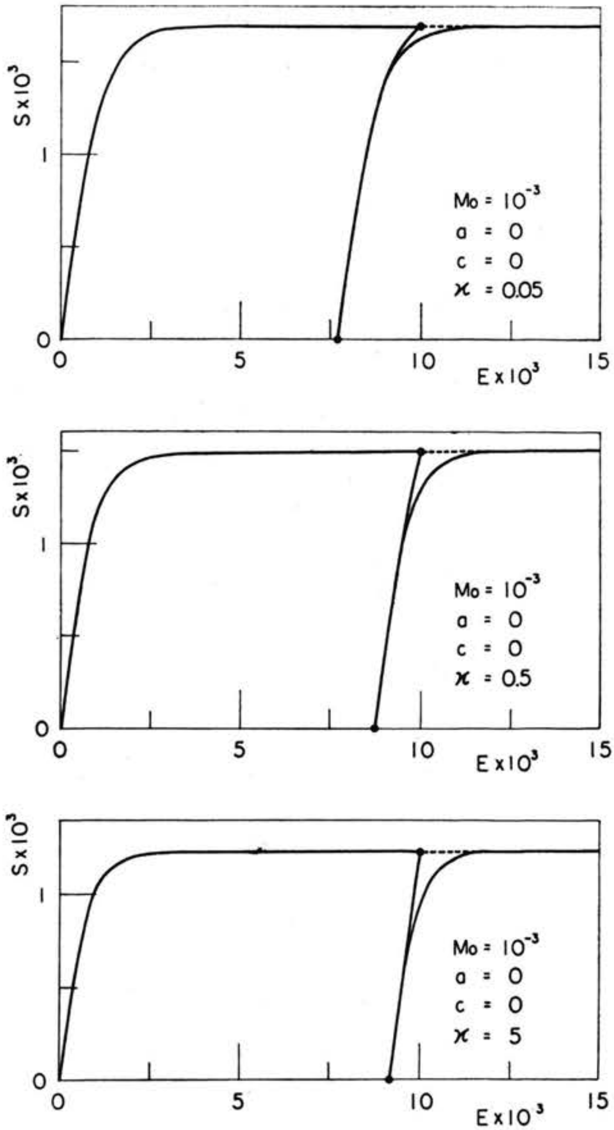
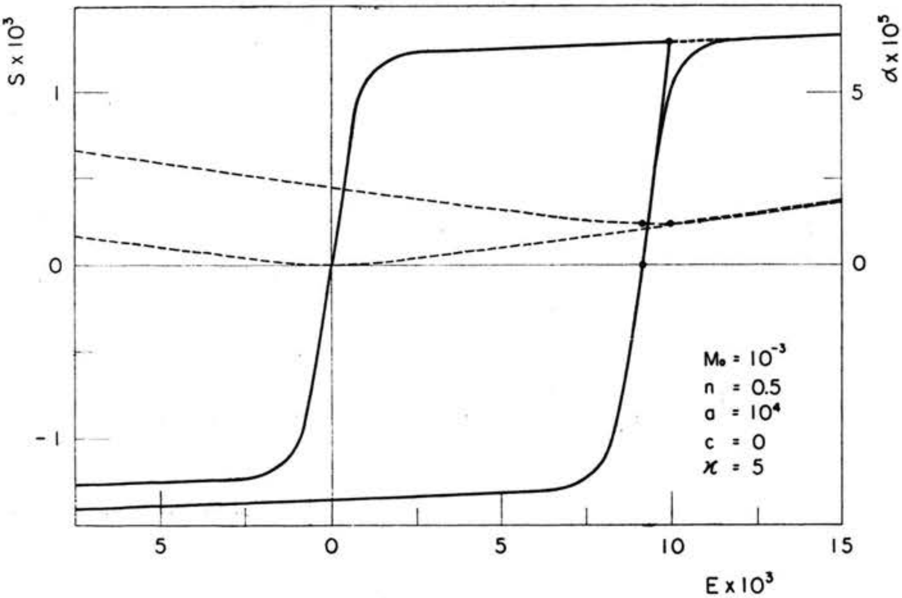
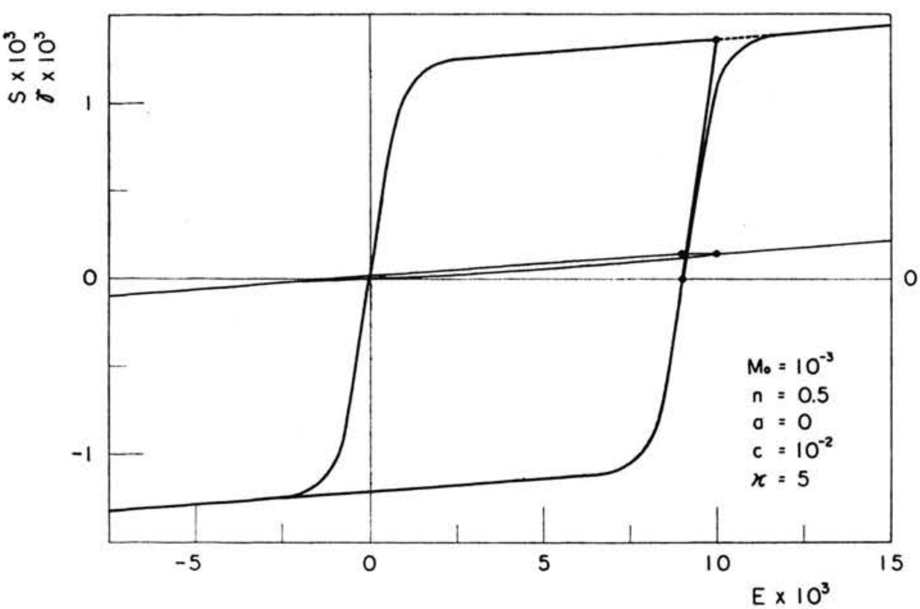


FIG. 1. Stress-strain diagrams for the loading-unloading-reloading uniaxial stress extension, where the transition parameter takes three values.

$$(4.8)\qquad \begin{aligned} \frac{d\alpha}{dE} &= \tilde{S}V\left(\frac{\tilde{v}^*}{v_0}\right), \\ \frac{d\gamma}{dE} &= \frac{c\tilde{S}^2}{M_0^2(1+a\alpha)^{2n}} V\left(\frac{\tilde{v}^*}{v_0}\right), \end{aligned}$$





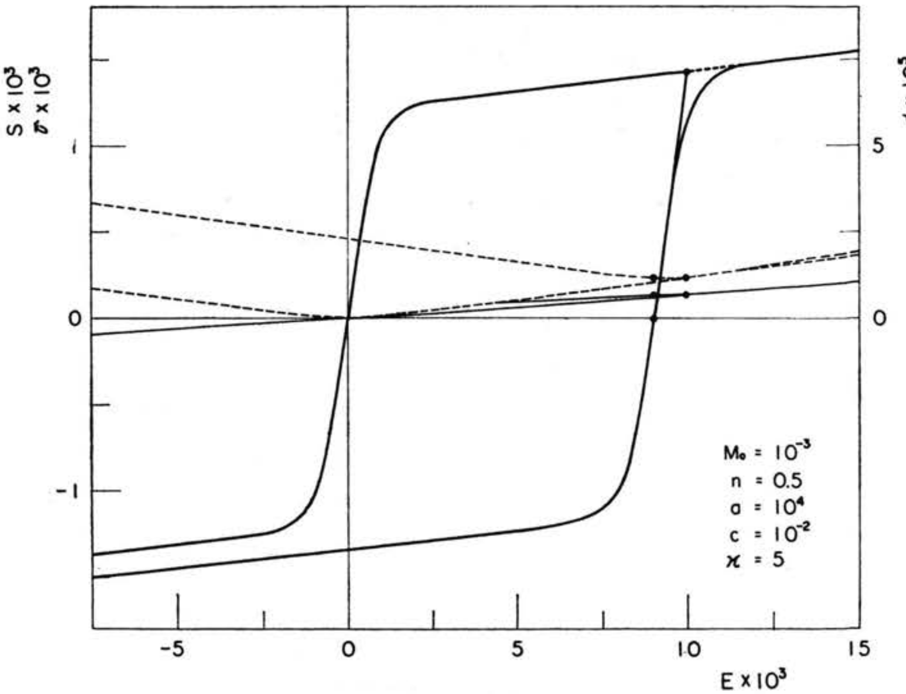


FIG. 2. Stress-strain and internal state variables-strain diagrams for the loading-unloading-reloading uniaxial stress extension.

where E is the logarithmic strain defined by $dE = D dt$,

$$V\left(\frac{\tilde{v}^*}{v_0}\right) = \frac{1}{1+\exp(-\varepsilon\varkappa\tilde{S}/M_0)}, \tag{4.9}$$

and $\varepsilon = D/|D|$ equals 1 for elongation and -1 for compression. The transition regions mentioned in the last part of Section 2 are given by

$$|\tilde{S}| < 2.2\frac{M_0}{\varkappa}, \quad |\tilde{S}| < 4.6\frac{M_0}{\varkappa} \tag{4.10}$$

for $0.1 < V(\tilde{v}^*/v_0) < 0.9$ and $0.01 < V(\tilde{v}^*/v_0) < 0.99$, respectively. Here and henceforth, $M_0 = 10^{-3}$ and $n = 0.5$ are assumed.

Figure 1 shows the diagrams of stress-strain curves for a perfectly plastic material, which has no work-hardening. The material starts at $S = E = 0$ and it is extended (loading) to $E = 10^{-2}$, then compressed (unloading) to $S = 0$. It is again extended to $E = 1.5\times10^{-2}$. The diagrams of stress-strain curves are depicted for the cases $\varkappa = 5\times10^{-2}$, 5×10^{-1} and 5. For the case $\varkappa = 5$ the responses are substantially identical with those for the two-constitutive equation systems which correspond to the case $\varkappa = \infty$. Here and henceforth a black circle denotes a turning point from extension to compression or vice versa.

Figure 2 shows the diagrams of stress-strain and internal state variable-strain curves for plastic materials with a) isotropic, b) translational, and c) combined work-hardening, where the transition parameter is assumed to be 5. The results are very similar to those for the two constitutive equation systems which were reported in [19]. The solid bold curves, the solid fine curves, and the dashed fine curves refer, respectively, to the diagrams for the stress, the tensor internal state variable, and the scalar internal state variable, respectively.

Figure 3 shows the diagram of stress-strain curve for repeated loading-unloading cycles in the limit of strain $-5\times10^{-3} \leqq E \leqq 5\times10^{3}$.

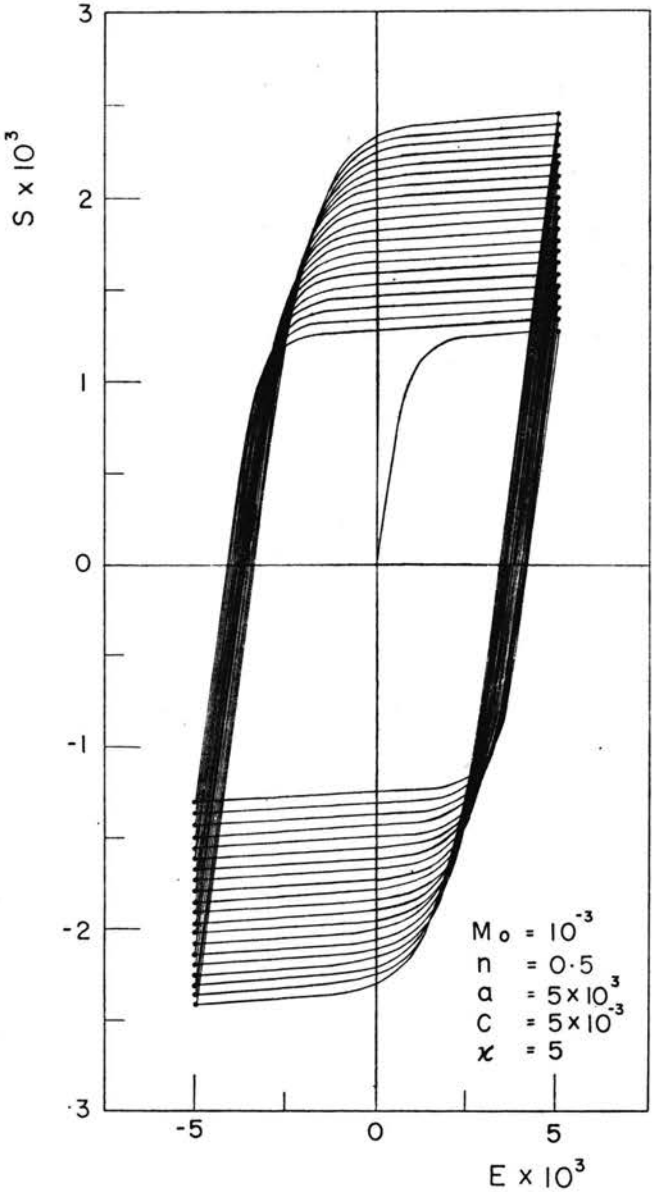


FIG. 3. Stress-strain diagram for repeated loading-unloading cycles.

5. Concluding remarks

1. A single constitutive equation unified by a transition function is in conformity with both states, that is, the loading and the unloading states. The combined work-hardening can be expressed by the scalar and the tensor internal state variables and their evolutional equations in both states are also unified by the transition function.

2. When the value of the transition parameter is small, the unified equations reduce to the equations which hold in the loading state, and when it is large, their stress-strain relations are substantially equal to those of the set of equations in both states.

3. We can say that there are two main advantages in our unification of plastic constitutive equations. One is the theoretical simplicity and the other is the practical simplicity. The numerical calculations of any elasto-plastic deformation can be considerably reduced by using our unified equations.

4. The proposed constitutive and evolutional equations can be applied to any large deformation with large rotation. Refer to [20, 21].

5. The stress-strain curves depend upon the forms of ${}_{E}\mathscr{H}(\mathbf{T})$, ${}_{P}\mathscr{H}(\tilde{\mathbf{T}}, \alpha)$, $\mathbf{\Phi}(\tilde{\mathbf{T}}, \alpha)$ and $\mathbf{\Psi}(\tilde{\mathbf{T}}, \alpha)$. Equation (3.9) corresponds to an isotropic elastoplastic material with the modified von Mises type yield condition. The constitutive equation which corresponds to the material with the Tresca type yield condition was also proposed by the author [13, 14].

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