A hidden variable approach to constrained solids (*)

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A NONLINEAR model of viscous and heat-conducting solids is set up by having recourse to hidden variable thermodynamics and by allowing the solid to be subject to internal constraints. After deriving thermodynamic restrictions on the constitutive equations, an analysis of the acceleration waves propagating in solids inextensible in one direction is developed. Next, a detailed investigation of waves entering the natural state is given; in particular, it is shown that in the absence of thermomechanical coupling the acceleration waves degenerate into a purely mechanical wave and a purely thermal wave. In regard to the purely mechanical wave, the effects of the inextensibility constraint turn out to be the same as those in elastic bodies.

Zbudowano model lepkich i przewodzących ciepło ciał stałych, posługując się termodynamiką zmiennych utajonych i dopuszczając by ciało było poddane więzom wewnętrznym. Po wyprowadzeniu ograniczeń termodynamicznych dla równań stanu przeprowadzono analizę fal przyśpieszenia rozprzestrzeniających się w ciałach nierozciągliwych w jednym kierunku. Zbadano następnie szczegółowo problem fal pojawiających się w stanie naturalnym ciała; w szczególności pokazano, że przy braku sprzężenia termomechanicznego fale przyśpieszenia degenerują się do fali mechanicznej i czysto termicznej. W odniesieniu do fali czysto mechanicznej stwierdzono, że efekt więzów nierozciągliwości jest taki sam jak w przypadku ciał sprężystych.

Построена модель вязких и теплопроводящих твердых тел, послуживаясь термодинамикой неявных переменных и допуская, чтобы тело подвергалось внутренним связям. После вывода термодинамических ограничений для уравнений состояния, проведен анализ волн ускорения, распространяющихся в телах нерастяжимых в одном направлении. Затем подробно исследована проблема волн, появляющихся в натуральном состоянии тела; в частности показано, что при отсутствии термомеханического сопряжения волны ускорения вырождаются в механическую и чисто термическую волны. По отношению к чисто механической волне констатировано, что эффект связей нерастяжимости такой же сам, как в случае упругих тел.

1. Introduction

THERE ARE SEVERAL dissipative mechanisms in solids whereby mechanical energy is transformed into thermal energy. In spite of their complexity, such mechanisms are usually accounted for in phenomenological theories through viscosity and heat conduction only. Various models are applied to describe viscosity effects; among them, those of Maxwell, Kelvin–Voigt, and Zener are well known. Yet many theories based upon these models suffer from the drawback that they are not framed in a consistent thermodynamic approach or that the corresponding systems of partial differential equations are parabolic, rather than hyperbolic, in character.

The recent literature bears evidence of many successful attempts to remedy such deficiencies; let me mention, for example, the hidden variable approach performed by KOSIŃSKI

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and Perzyna [1] and Kosiński [2, 3], the generalized thermoelasticity elaborated by Müller [4] and re-examined by McCarthy [5], the hyperbolic elasticity of dissipative media carried out by Kranyš [6]. While references [1–5] do not account explicitly for viscosity, reference [6] does but only under the assumption of infinitesimal deformations. It seems then worthwhile looking again at dissipative solids, undergoing finite deformations, with the purpose of casting viscosity and heat conduction in a thermodynamic scheme compatible with wave propagation.

In regard to fluids such a scheme has been accomplished by myself via recourse to the hidden variable approach [7–9]; the hyperbolicity of the model has been proved to hold [10]. Subsequent generalizations allowed Bampi and myself to see how flexible the scheme is. For example, nonstationary transport equations investigated by Müller [11], and Israel and Stewart [12] find their strict counterparts in the thermodynamics with hidden variables [13, 14].

Motivated by the encouraging results obtained so far, in this paper I attempt to answer the aforementioned purpose by having recourse to hidden variables. Moreover, so as to arrive at relations applicable to a large extent, the solid is allowed to be subject to internal constraints.

The plan of the paper is as follows. Section 2 delivers general properties of materials with hidden variables and equips the dissipative solid with the structure of a material with hidden variables whose internal constraints involve relations between the deformation and the temperature. Next, in Sect. 3, thermodynamic restrictions are derived by starting from an entropy inequality allowing for the existence of an entropy extra-flux. Then detailed calculations are made in connection with an outstanding free energy function and the internal constraint of inextensibility in one direction: after investigating the jump relations for acceleration waves (Sect. 4) the propagation modes of waves entering the natural state are examined (Sec. 5). As we should expect, and differently from what happens in many theories, it turns out that the acceleration waves need not be homothermal. Meanwhile, and this is similar to McCarthy's result [5], in the absence of thermomechanical coupling the acceleration waves degenerate into a purely mechanical wave and a purely thermal wave. In regard to the purely mechanical wave the effects of the inextensibility constraint are assessed; it is shown that the propagation mode is left unaffected if the direction of propagation is perpendicular to the direction of inextensibility and that a compressive reaction stress lowers the speed of propagation if the amplitude vector is perpendicular to the direction of inextensibility.

2. Dissipative solids with hidden variables

We are dealing with a solid body $\mathscr{B}$ whose particles are labeled by the positions they occupy in a suitable reference configuration $\mathscr{R}$; it is convenient to let $\mathscr{R}$ be in the natural state, namely a stress free configuration of uniform temperature. The motion of $\mathscr{B}$ is described by a function $\mathbf{x}(\mathbf{X}, t)$ specifying the position $\mathbf{x}$ of $\mathbf{X}$ at time t. Direct tensor notation is used throughout; whenever recourse to components is needed we refer to a fixed Cartesian system of axes. Suffixes range over the values 1, 2, 3 and the usual sum-

mation convention is applied; upper case letters represent material suffixes. A superposed dot designates the material time derivative, ∇ and $\nabla \cdot$ stand for the material gradient and divergence operators. The symbols $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{A}$, and $\boldsymbol{\Phi}$ denote finite dimensional real normed vector spaces, subject to the requirement $\dim \mathbf{A} \leqslant \dim \mathbf{Y} + \dim \mathbf{Z}$, while $\mathbf{L}(\cdot, \cdot)$ stands for the normed vector space of all linear maps from a vector space into another; $|\cdot|$ is adopted to denote the usual norm — $|\mathbf{p}| = (\mathbf{p} \cdot \mathbf{p})^{1/2}$ — both in $\mathbf{Y} \times \mathbf{Z}$, $\mathbf{A}$ and in $\mathbf{L}$. The space $\mathrm{Sym} \in \mathbf{L}$ consists of second order symmetric tensors; $\mathscr{V}$ is the vector space associated with the three-dimensional Euclidean point space.

A material with hidden variables $\{\mathbf{y}_0, \mathbf{z}_0, \mathbf{a}_0, \mathbf{U}, \mathbf{V}, \boldsymbol{\phi}, \mathbf{f}\}$ on $\mathbf{Y} \times \mathbf{Z} \times \mathbf{A}$ consists of a ground value $(\mathbf{y}_0, \mathbf{z}_0, \mathbf{a}_0)$ of the variables $(\mathbf{y}, \mathbf{z}, \mathbf{a}) \in \mathbf{Y} \times \mathbf{Z} \times \mathbf{A}$, the vector $\mathbf{a} \in \mathbf{A}$ representing the set of hidden variables, together with an open connected neighbourhood $\mathbf{U} \times \mathbf{V}$ of $(\mathbf{y}_0, \mathbf{z}_0)$ and the maps

$$\boldsymbol{\phi} \in \mathbf{C}^2(\mathbf{U} \times \mathbf{A}, \boldsymbol{\Phi}), \quad \mathbf{f} \in \mathbf{C}^1(\mathbf{U} \times \mathbf{V} \times \mathbf{A}, \mathbf{A})$$

representing the response function and the evolution function, respectively. A path is a bounded and piecewise continuously differentiable map $\boldsymbol{\pi} \colon \mathbf{R} \to \mathbf{U} \times \mathbf{V}$; to save writing the symbol $\boldsymbol{\pi}$ will be used also in connection with the values of the path, i.e. the values of the observable variables $(\mathbf{y}, \mathbf{z}) \in \mathbf{U} \times \mathbf{V}$. The hidden variables are time dependent fields on $\mathscr{R} \times \mathbf{R}$; their growth, at a given particle $\mathbf{X} \in \mathscr{R}$, is determined by the path $\boldsymbol{\pi}$ through the evolution function $\mathbf{f}$, namely

$$(2.1) \qquad \dot{\mathbf{a}}(t) = \mathbf{f}\big(\boldsymbol{\pi}(t), \mathbf{a}(t)\big), \quad t \geqslant t_0, \quad \mathbf{a}(t_0) = \mathbf{a}^*.$$

It is an essential requirement on the hidden variables that the following assumption holds.

I. *There is a map* $\boldsymbol{\Gamma} \in \mathbf{L}(\mathbf{A}, \mathbf{A})$ *and a positive constant* δ *such that*

$$(2.2) \qquad |\mathbf{f}(\boldsymbol{\pi}, \mathbf{a} + \mathbf{b}) - \mathbf{f}(\boldsymbol{\pi}, \mathbf{a}) - \boldsymbol{\Gamma}\mathbf{b}| \leqslant \delta|\mathbf{b}|, \quad \boldsymbol{\pi} \in \mathbf{U} \times \mathbf{V}, \quad \mathbf{a}, \mathbf{a} + \mathbf{b} \in \mathbf{A},$$

and each eigenvalue of $\boldsymbol{\Gamma} + \delta\mathbf{I}_A$ *has a negative real part.*

Usually an additional requirement on $\mathbf{f}$, namely a uniform Lipschitz condition in $\boldsymbol{\pi}$, is introduced.

II. *There is a positive constant* ν *such that*

$$(2.3) \qquad |\mathbf{f}(\boldsymbol{\pi} + \boldsymbol{\omega}, \mathbf{a}) - \mathbf{f}(\boldsymbol{\pi}, \mathbf{a})| \leqslant \nu|\boldsymbol{\omega}|, \quad \boldsymbol{\pi}, \boldsymbol{\pi} + \boldsymbol{\omega} \in \mathbf{U} \times \mathbf{V}, \quad \mathbf{a} \in \mathbf{A}.$$

As shown in [15], the properties I, II guarantee the asymptotic stability of the evolution equation (2.1).

Having in mind the aim of accounting for the behaviour of a solid undergoing finite deformations, one way of equipping a dissipative solid with hidden variables is as follows. Look at the material description of the balance laws, namely ([16] $\S$ 43, [17])

$$(2.4) \qquad \varrho_0 \dot{\mathbf{v}} = \nabla \cdot \mathbf{S} + \varrho_0 \mathbf{b},$$

$$(2.5) \qquad \varrho_0 \dot{\varepsilon} = \tfrac{1}{2}\tilde{\mathbf{T}} \cdot \dot{\mathbf{C}} - \nabla \cdot \mathbf{Q} + \varrho_0 r,$$

where ϱ_0 is the mass density (in $\mathscr{R}$), $\mathbf{v}$ the velocity, $\mathbf{b}$ the body force (per unit mass), ε the internal energy, $\mathbf{C}$ the right Cauchy-Green tensor, $\mathbf{Q}$ the heat flux vector (in $\mathscr{R}$), and r the heat supply while $\mathbf{S}$ and $\tilde{\mathbf{T}}$ are the first and second Piola–Kirchhoff stress tensors. If the temperature θ is viewed as an independent variable, the balance equations (2.4)

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and (2.5) suggest that we identify the response functions ϕ with the set $(\psi, \eta, \tilde{T}, Q)$, η being the entropy and $\psi = \varepsilon - \theta\eta$ the free energy (per unit mass). Moreover, we designate y to be the pair (C, θ) and z the pair $(\dot{C}, G)$, where $G = \nabla\theta$. The evolution function f is assumed to be linear in the hidden variables a and the observable variables $\dot{C}, G$. Accordingly, we let a be in fact the pair (Σ, Λ), of $\Sigma \in \mathrm{Sym}(\mathscr{V}, \mathscr{V})$ and $\Lambda \in \mathscr{V}$, and then we write the corresponding evolution equations as

$$(2.6) \qquad \dot{\Sigma} = \sigma_1 \dot{C} - \sigma_2 \Sigma, \quad \Sigma(t_0) = \Sigma^*,$$

$$(2.7) \qquad \dot{\Lambda} = \chi_1 G - \chi_2 \Lambda, \quad \Lambda(t_0) = \Lambda^*,$$

where $\sigma_1, \sigma_2, \chi_1, \chi_2$ are scalar functions on C and θ. Property I requires that σ_2 and χ_2 should be positive; meanwhile, the above evolution equations ascribe to $\sigma_2^{-1}, \chi_2^{-1}$ the meaning of relaxation times. On the other hand, note that we may always set $\sigma_1 > 0$, $\chi_1 > 0$ (should σ_1, χ_1 be negative we could consider $-\Sigma, -\Lambda$ instead of Σ, Λ as hidden variables). These observations complete the hidden variable structure of dissipative solids.

Possible restrictions on the behaviour of the solid are assumed in the form of internal constraints — see, e.g. [16] § 30, [18] — as

$$(2.8) \qquad \lambda^\alpha(C, \theta) = 0, \quad \alpha = 1, \ldots, n;$$

of course, since $C \in \mathrm{Sym}(\mathscr{V}, \mathscr{V})$, n may run from 1 to 6, at the most. Examples of constraints like Eq. (2.8) are provided by the temperature–dependent compressibility [19] and the temperature-dependent extensibility in one direction. Specifically, if e^0 is a unit vector in the reference configuration, inextensibility in the e^0 direction means [16]

$$(2.9) \qquad \lambda(C) = e^0 \cdot C e^0 - 1 = 0.$$

The literature bears evidence of a wide interest in materials meeting the constraint (2.9), namely inextensible materials [20, 21]. Within the context of dissipative solids as described by the scheme outlined above, this paper exhibits a further investigation of wave propagation in inextensible materials.

Sometimes, in connection with fibre-reinforced materials, constraints involving also the temperature gradient G are introduced [22, 23] on the basis of the observation that the fibres are good conductors but the matrix material is not. Precisely, if e^0 is tangent to the fibre, it is assumed that [1]

$$(2.10) \qquad e^0 \cdot G = 0.$$

Here, instead, the possible anisotropy of the heat conductivity is described via an anisotropic conductivity tensor rather than via the constraint (2.10) — see the next section.

3. Thermodynamic restrictions

Restrictions on the constitutive theory described so far may be derived through compatibility with the second law of thermodynamics. While this law is often assumed to be

[1] The analogous constraint on the electric field has been considered by Chen and McCarthy in coniunction with electrically conducting inextensible elastic bodies [21].

expressed by the Clausius–Duhem inequality, here we adopt the viewpoint, which may be traced back to MÜLLER [4, 11], whereby the entropy flux vector $\mathscr{I}$ is a constitutive response function possibly different from $\mathbf{Q}/\theta$; accordingly we set

$$(3.1) \qquad \mathscr{I} = \mathscr{I}(\mathbf{C}, \theta, \mathbf{\Sigma}, \mathbf{\Lambda}).$$

Then, as a statement of the second law of thermodynamics, we say that the inequality

$$(3.2) \qquad \varrho_0 \dot{\eta} + \nabla \cdot \mathscr{I} - \frac{\varrho_0 r}{\theta} \geqslant 0$$

must hold at each particle of the body for each path π. Since $\eta = (\varepsilon - \psi)/\theta$, account of the energy balance (2.5) allows the inequality (3.2) to be written as

$$(3.3) \qquad -\varrho_0(\dot{\psi} + \eta\dot{\theta}) + \tfrac{1}{2}\tilde{\mathbf{T}} \cdot \dot{\mathbf{C}} + \theta\nabla \cdot \mathscr{I} - \nabla \cdot \mathbf{Q} \geqslant 0.$$

Letting $\mathscr{F} = \mathscr{I} - \mathbf{Q}/\theta$, the inequality (3.3) may be given the explicit form

$$(3.4) \qquad -\varrho_0(\psi_\theta + \eta)\dot{\theta} + \tfrac{1}{2}(\tilde{\mathbf{T}} - \varrho_0\psi_c - \varrho_0\sigma_1\psi_\Sigma) \cdot \dot{\mathbf{C}} + \theta\{\mathscr{F}_c \cdot (\nabla\mathbf{C}) + \mathscr{F}_\theta \cdot \mathbf{G} + \mathscr{F}_\Sigma \cdot (\nabla\mathbf{\Sigma})$$

$$+ \mathscr{F}_\Lambda \cdot (\nabla\mathbf{\Lambda})\} - \varrho_0\left(\frac{1}{\theta}\mathbf{Q} + \varrho_0\chi_1\psi_\Lambda\right) \cdot \mathbf{G} + \varrho_0(\chi_2\psi_\Lambda \cdot \mathbf{\Lambda} + \sigma_2\psi_\Sigma \cdot \mathbf{\Sigma}) \geqslant 0,$$

where the suffixes $\mathbf{C}, \theta, \mathbf{\Sigma}, \mathbf{\Lambda}$ denote partial derivatives and, for example, $\mathscr{F}_c \cdot (\nabla\mathbf{C}) = (\partial\mathscr{F}_H/\partial C_{KL})(\partial C_{KL}/\partial X_H)$. As a customary additional restriction on the entropy extra-flux $\mathscr{F}$ we require that it vanishes when the hidden variables vanish. So if $\mathscr{F}_\Sigma = 0$ and $\mathscr{F}_\Lambda = 0$, then $\mathscr{F}$ vanishes identically.

In connection with the inequality (3.4), it is worth emphasizing an essential property of the hidden variables. In fact, the obvious solutions to Eqs. (2.6) and (2.7), namely

$$(3.5) \qquad \mathbf{\Sigma}(t) = \mathbf{\Sigma}^*\exp\left\{-\int_{t_0}^t \sigma_2(\zeta)d\zeta\right\} + \int_{t_0}^t \sigma_1(\xi)\dot{\mathbf{C}}(\xi)\exp\left\{-\int_\xi^t \sigma_2(\zeta)d\zeta\right\}d\xi,$$

$$\mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\epsilon\exp\left\{-\int_{t_0}^t \chi_2(\zeta)d\zeta\right\} + \int_{t_0}^t \chi_1(\xi)\mathbf{G}(\xi)\exp\left\{-\int_\xi^t \chi_2(\zeta)d\zeta\right\}d\xi,$$

show that, although the σ and χ may depend on $\mathbf{C}$ and θ, the hidden variables $\mathbf{\Sigma}(t)$, $\mathbf{\Lambda}(t)$ are independent of the present values $\mathbf{C}(t)$, $\dot{\mathbf{C}}(t)$, $\theta(t)$, $\dot{\theta}(t)$, and $\mathbf{G}(t)$.

The exploitation of the inequality (3.4) proceeds as follows.

a. Owing to Eq. (3.5), the values of $\nabla\mathbf{\Sigma}$ and $\nabla\mathbf{\Lambda}$ may be chosen arbitrarily and independently of the values of $\mathbf{C}$, θ, $\mathbf{\Sigma}$, $\mathbf{\Lambda}$, $\dot{\theta}$, $\dot{\mathbf{C}}$, $\nabla\mathbf{C}$, $\mathbf{G}$. Then the inequality (3.4) holds for each path $\pi(\cdot)$ on $[t_0, t]$ only if

$$\mathscr{F}_\Sigma = 0, \quad \mathscr{F}_\Lambda = 0.$$

Thus $\mathscr{F}$ is a function on $\mathbf{C}$ and θ only; hence the assumption that $\mathscr{F}$ vanishes when $\mathbf{\Sigma} = 0$ and $\mathbf{\Lambda} = 0$ leads us to assert that $\mathscr{F} = 0$ and $\mathscr{I} = \mathbf{Q}/\theta$.

It is worthy of note that a similar analysis on the entropy extra-flux has already been accomplished by LEBON [24] [2]; his different scheme, however, allowed him to find that the entropy extra-flux vanishes to within the entropy-production-free term $\mathbf{\Omega}\mathbf{G}$, $\mathbf{\Omega} = \mathbf{\Omega}(\theta)$ being a skew-symmetric tensor.

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[2] See also ref. [25].

b. Since $\mathscr{F} = 0$, the inequality (3.4) reads

(3.6) $-\left\{\varrho_0(\psi_\theta+\eta)\dot\theta+(\tfrac{1}{2}\tilde{\mathbf{T}}-\varrho_0\psi_\mathbf{c}-\varrho_0\sigma_1\psi_\Sigma)\cdot\dot{\mathbf{C}}-\left(\dfrac{1}{\theta}\mathbf{Q}+\varrho_0\chi_1\psi_\Lambda\right)\cdot\mathbf{G}\right.$

$$+\varrho_0(\chi_2\psi_\Lambda\cdot\mathbf{\Lambda}+\sigma_2\psi_\Sigma\cdot\mathbf{\Sigma})\geqslant 0.$$

The values of $\dot\theta$ and $\dot{\mathbf{C}}$ are not independent of each other; indeed, in view of Eqs. (2.8) they are related by

$$\lambda_\mathbf{c}^\alpha\cdot\dot{\mathbf{C}}+\lambda_\theta^\alpha\dot\theta = 0, \quad \alpha = 1, ..., n.$$

Then, on introducing n Lagrange multipliers p^α, we may write the inequality (3.6) as

(3.7) $-\{\varrho_0(\psi_\theta+\eta)+p^\alpha\lambda_\theta^\alpha\}\dot\theta+(\tfrac{1}{2}\tilde{\mathbf{T}}-\varrho_0\psi_\mathbf{c}-\varrho_0\sigma_1\psi_\Sigma-p^\alpha\lambda_\mathbf{c}^\alpha)\cdot\dot{\mathbf{C}}-\left(\dfrac{1}{\theta}\mathbf{Q}+\varrho_0\chi_1\psi_\Lambda\right)\cdot\mathbf{G}$

$$+\varrho_0(\sigma_2\psi_\Sigma\cdot\mathbf{\Sigma}+\chi_2\psi_\Lambda\cdot\mathbf{\Lambda})\geqslant 0$$

and assert that Eq. (3.7) must hold for each path $\pi(\cdot)$ on $[t_0, t]$. As $\mathbf{\Sigma}(t)$ and $\mathbf{\Lambda}(t)$ are independent of $\dot\theta(t)$, $\dot{\mathbf{C}}(t)$, and $\mathbf{G}(t)$, the inequality (3.7) holds if and only if

(3.8) $$\eta = -\psi_\theta-\frac{1}{\varrho_0}p^\alpha\lambda_\theta^\alpha,$$

(3.9) $$\tilde{\mathbf{T}} = 2\varrho_0\psi_\mathbf{c}+2\varrho_0\sigma_1\psi_\Sigma+p^\alpha\lambda_\mathbf{c}^\alpha,$$

(3.10) $$\mathbf{Q} = -\varrho_0\theta\chi_1\psi_\Lambda,$$

(3.11) $$\sigma_2\psi_\Sigma\cdot\mathbf{\Sigma}+\chi_2\psi_\Lambda\cdot\mathbf{\Lambda}\geqslant 0.$$

In view of Eqs. (3.8)–(3.10), the response functions η, $\tilde{\mathbf{T}}$, $\mathbf{Q}$ are determined by the free energy function ψ only to within a constraint entropy $-p^\alpha\lambda_\theta^\alpha/\varrho_0$ and a constraint stress $p^\alpha\lambda_\mathbf{c}^\alpha$. Accordingly, once ψ meets the requirement (3.11), the response functions η, $\tilde{\mathbf{T}}$, $\mathbf{Q}$, as determined by Eqs. (3.8)–(3.10), are automatically consistent with thermodynamics. Among the free energy functions satisfying the condition (3.11), it is worth considering

$$\psi = \Psi(\mathbf{C}, \theta)+\frac{\sigma_2}{4\varrho_0\sigma_1^2}\,\mathbf{\Sigma}\cdot\mathbf{M\Sigma}+\frac{\chi_2}{2\varrho_0\theta\chi_1^2}\,\mathbf{\Lambda}\cdot\mathbf{K\Lambda},$$

the temperature-dependent tensors $\mathbf{M}$ and $\mathbf{K}$, $\mathbf{K}\in\mathrm{Sym}(\mathscr{V},\mathscr{V})$, being positive definite. The mechanical anisotropies of the solid are accounted for through the constraints (2.8). It seems then natural to assume that $\mathbf{M}$ is an isotropic tensor, as such being singled out by two scalars only, say μ_1, μ_2. So we arrive at

(3.12) $$\psi = \Psi(\mathbf{C}, \theta)+\frac{\sigma_2}{2\varrho_0\sigma_1^2}\,\{\mu_1\mathbf{\Sigma}\cdot\mathbf{\Sigma}+\tfrac{1}{2}\mu_2(\mathrm{tr}\,\mathbf{\Sigma})^2\}+\frac{\chi_2}{2\varrho_0\theta\chi_1^2}\,\mathbf{\Lambda}\cdot\mathbf{K\Lambda};$$

hence the requirement (3.11) is satisfied if and only if

(3.13) $$\mu_1 \geqslant 0, \quad 2\mu_1+3\mu_2 \geqslant 0,$$

(3.14) $$\mathbf{w}\cdot\mathbf{Kw} \geqslant 0, \quad \forall\mathbf{w}\in\mathscr{V}.$$

Moreover, to get a handier model, henceforth we let the quantities σ and χ, besides μ_1, μ_2, and $\mathbf{K}$, depend on the temperature θ only.

That the ansatz (3.12), subject to the conditions (3.13) and (3.14), is noteworthy and may be shown as follows. The function (3.12), in addition to fulfilling (3.11), provides

$$(3.15) \quad \eta = -\Psi_\theta - \frac{1}{2\varrho_0}\left\{\left(\frac{\mu_1\sigma_2}{\sigma_1^2}\right)_\theta \mathbf{\Sigma}\cdot\mathbf{\Sigma} + \frac{1}{2}\left(\frac{\mu_2\sigma_2}{\sigma_1^2}\right)_\theta (\mathrm{tr}\mathbf{\Sigma})^2 + \mathbf{\Lambda}\cdot\left(\frac{\chi_2\mathbf{K}}{\theta\chi_1^2}\right)_\theta \mathbf{\Lambda}\right\} - \frac{1}{\varrho_0}p^\alpha\lambda_\theta^\alpha,$$

$$(3.16) \quad \tilde{\mathbf{T}} = 2\varrho_0\Psi_c + 2\mu_1\frac{\sigma_2}{\sigma_1}\mathbf{\Sigma} + \mu_2\frac{\sigma_2}{\sigma_1}(\mathrm{tr}\mathbf{\Sigma})\mathbf{I} + p^\alpha\lambda_c^\alpha,$$

$$(3.17) \quad \mathbf{Q} = -\frac{\chi_2}{\chi_1}\mathbf{K}\mathbf{\Lambda}.$$

Now one glance at the evolution equations (2.6) and (2.7) tells us that when $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ are constant in time we have $\mathbf{\Sigma} = \sigma_1\sigma_2^{-1}\dot{\mathbf{C}}$, $\mathbf{\Lambda} = \chi_1\chi_2^{-1}\mathbf{G}$. Hence, on account of Eqs. (3.13) and (3.14), the functions $\mu_1(\theta)$, $\mu_2(\theta)$ may be regarded as the usual coefficients of viscosity and $\mathbf{K}(\theta)$ as the conductivity tensor [4, 5]. Meanwhile, the contribution $2\mu_1\sigma_2\sigma_2^{-1}\mathbf{\Sigma} + \mu_2\sigma_2\sigma_2^{-1}(\mathrm{tr}\mathbf{\Sigma})\mathbf{I}$ to the stress closely resembles the description of the stress in viscoelastic isotropic materials [3]. Indeed, in view of Eq. (3.5)$_1$,

$$\frac{\mu_1\sigma_2}{\sigma_1}(t)\sigma_1(\xi)\exp\left\{-\int_\xi^t \sigma_2(\zeta)d\zeta\right\}, \quad \frac{(2\mu_1+3\mu_2)\sigma_2}{3\sigma_1}(t)\sigma_1(\xi)\exp\left\{-\int_\xi^t \sigma_2(\zeta)d\zeta\right\},$$

where $\xi \in [t_0, t]$, play the role of shear relaxation function and bulk relaxation function, respectively.

Within the scheme elaborated so far we could admit a further restriction on the constitutive functions σ_1, σ_2, χ_1, χ_2, and μ_1, μ_2, $\mathbf{K}$. Confine our attention to unconstrained bodies — and then $\lambda_\theta^\alpha = 0$ — and look at the specific heat $\varepsilon_\theta = \theta\eta_\theta$. By Eq. (3.15) we get the relation

$$\varepsilon_\theta = -\theta\Psi_{\theta\theta} - \frac{\theta}{2\varrho_0}\left\{\left(\frac{\mu_1\sigma_2}{\sigma_1^2}\right)_{\theta\theta}\mathbf{\Sigma}\cdot\mathbf{\Sigma} + \frac{1}{2}\left(\frac{\mu_2\sigma_2}{\sigma_1^2}\right)_{\theta\theta}(\mathrm{tr}\mathbf{\Sigma})^2 + \mathbf{\Lambda}\cdot\left(\frac{\chi_2\mathbf{K}}{\theta\chi_1^2}\right)_{\theta\theta}\mathbf{\Lambda}\right\}$$

emphasizing the additive contribution of the hidden variables. Since we are used to let $\Psi_{\theta\theta} \leqslant 0$, it follows that $\varepsilon_\theta \geqslant 0$ provided the inequalities

$$\left(\frac{\mu_1\sigma_2}{\sigma_1^2}\right)_{\theta\theta} \leqslant 0; \quad \left(\frac{(2\mu_1+3\mu_2)\sigma_2}{\sigma_1^2}\right)_{\theta\theta} \leqslant 0; \quad \mathbf{w}\cdot\left(\frac{\chi_2\mathbf{K}}{\theta\chi_1^2}\right)_{\theta\theta}\mathbf{w} \leqslant 0, \quad \forall\mathbf{w}\in\mathscr{V};$$

hold. In particular, if $\mathbf{K}$ is isotropic, namely $\mathbf{K} = \varkappa\mathbf{I}$, then the last inequality gives

$$(3.18) \qquad\qquad \left(\frac{\chi_2\varkappa}{\theta\chi_1^2}\right)_{\theta\theta} \leqslant 0.$$

So as to test the validity of the condition (3.18) look now at heat conduction in metals. According to the Wiedemann–Franz law the relaxation time χ_2^{-1} and the heat conductivity $\varkappa$ are related by

$$\varkappa\chi_2 = \beta n_e\theta,$$

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[3] See, e. g. [26], Sects. 4.2, 6.1.

where β is a constant and n_e is the electron density. So, if we let the undetermined function $\chi_1(\theta)$ be proportional to $n_e^{1/2}$, then the inequality (3.18) is true identically. In this regard observe that if $\varkappa$, χ_1, and χ_2 are assumed to be constant, the inequality (3.18) is not fulfilled.

4. Propagation of acceleration waves

Henceforth we investigate the propagation of acceleration waves in dissipative solids with internal constraints as described by Eqs. (3.12)–(3.17); meanwhile, in order not to make the theory overly cumbersome, we content ourselves with looking at solids subject to the constraint (2.9) of inextensibility in the e^0 direction. In such a case the relations (3.15) and (3.16) simplify to

$$(4.1) \qquad \eta = -\Psi_\theta - \frac{1}{2\varrho_0}\left\{\left(\frac{\mu_1\sigma_2}{\sigma_1^2}\right)_\theta \mathbf{\Sigma}\cdot\mathbf{\Sigma} + \frac{1}{2}\left(\frac{\mu_2\sigma_2}{\sigma_1^2}\right)_\theta (\mathrm{tr}\mathbf{\Sigma})^2 + \mathbf{\Lambda}\cdot\left(\frac{\chi_2\mathbf{K}}{\theta\chi_1^2}\right)_\theta \mathbf{\Lambda}\right\},$$

$$(4.2) \qquad \tilde{\mathbf{T}} = 2\varrho_0\Psi_c + 2\mu_1\frac{\sigma_2}{\sigma_1}\mathbf{\Sigma} + \mu_2\frac{\sigma_2}{\sigma_1}(\mathrm{tr}\mathbf{\Sigma})\mathbf{I} + pe^0\otimes e^0.$$

Moreover, owing to the presence of the fibres, it seems natural to set

$$(4.3) \qquad\qquad\qquad \mathbf{K}(\theta) = \varkappa(\theta)\mathbf{I} + \varkappa^0(\theta)e^0\otimes e^0.$$

Then the condition (3.14) requires that $\varkappa \geqslant 0$, $\varkappa^0 \geqslant -\varkappa$. If, however, the fibres are good conductors whereas the matrix material is a poor conductor, the condition $\varkappa^0 \geqslant -\varkappa$ should be replaced with $\varkappa^0 \geqslant 0$.

Usually the velocity $\mathbf{v}$, the deformation gradient $\mathbf{F}$ and the temperature θ are assumed to be continuous across an acceleration wave front but $\dot{\mathbf{v}}$, $\dot{\mathbf{F}}$, and $\mathbf{G}$ are not. On the other hand, on account of Eq. (3.5), it is quite legitimate to assume that $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ are continuous across the wave front. Thus we are led to assert that the acceleration waves are characterized by the following

DEFINITION. *A wave $\mathscr{S}(t)$ is said to be an acceleration wave if the functions* $\mathbf{v}$, $\mathbf{F}$, θ, $\mathbf{\Sigma}$, $\mathbf{\Lambda}$, $\mathbf{b}$, r *are continuous on $\mathscr{R}\times\mathbf{R}$ while their time and spatial derivatives of any order suffer jump discontinuities across $\mathscr{S}(t)$ but are continuous functions on $\mathscr{R}\times\mathbf{R}\setminus\mathscr{S}(t)$.*

For later reference we summarize now some well-known results about waves. Following standard notations [27], let U_N and $\mathbf{N}$ be the speed of propagation of and the unit normal to $\mathscr{S}(t)$ with respect to the reference configuration $\mathscr{R}$. Moreover, for any field $\omega(\cdot, t)$ on $\mathscr{R}$ at time t, let $[\omega](t)$ be the jump of ω across $\mathscr{S}(t)$. If $[\omega] = 0$, then Maxwell's theorem asserts that

$$(4.4) \qquad\qquad\qquad [\nabla\omega] = ([\nabla\omega]\cdot\mathbf{N})\mathbf{N}$$

while $[\nabla\omega]$ is related to $[\dot\omega]$ through the compatibility condition

$$(4.5) \qquad\qquad\qquad [\dot\omega] + U_N\mathbf{N}\cdot[\nabla\omega] = 0.$$

Letting $\mathbf{a} = [\dot{\mathbf{v}}]$, $\mathscr{G} = [\mathbf{G}]\cdot\mathbf{N}$, Eqs. (4.4) and (4.5) enable us to find that

$$(4.6) \qquad\qquad\qquad [\dot{\mathbf{F}}] = -U_N^{-1}\mathbf{a}\otimes\mathbf{N},$$

$$(4.7) \qquad\qquad\qquad [\dot{\mathbf{C}}] = -2U_N^{-1}\mathrm{sym}\{(\mathbf{a}\mathbf{F})\otimes\mathbf{N}\},$$

$$(4.8) \qquad\qquad\qquad [\mathbf{G}] = \mathscr{G}\mathbf{N} = -U_N^{-1}[\dot\theta]\mathbf{N}.$$

On the other hand, associated with an integral balance equation of the form

$$\frac{d}{dt}\int_{\Omega}\varphi d\Omega + \int_{\partial\Omega}\mathbf{J}\cdot\mathbf{v}dS + \int_{\Omega}md\Omega = 0,$$

where $\Omega\subset\mathscr{R}$ and $\mathbf{v}$ is the unit outward normal to $\partial\Omega$, there is the jump relation

(4.9) $[\varphi]U_N - [\mathbf{J}]\cdot\mathbf{N} = 0.$

Accordingly, in connection with the integral counterparts of Eqs. (2.4) and (2.5) we find that

(4.10) $[\mathbf{S}]\mathbf{N} = \mathbf{0}$

and

(4.11) $\varrho_0[\varepsilon]U_N + \mathbf{v}\cdot[\mathbf{S}]\mathbf{N} - [\mathbf{Q}]\cdot\mathbf{N} = 0$

must hold at an acceleration wave front. Since $\mathbf{S} = \mathbf{F}\tilde{\mathbf{T}}$, the response function (4.2) meets the condition (4.10) provided that

(1.12) $[p]\mathbf{e}^0\cdot\mathbf{N} = 0$

whereby $p\mathbf{e}^0\cdot\mathbf{N}$ must be continuous across the wave while p must be so unless $\mathbf{e}^0\cdot\mathbf{N} = 0$ [4]. Moreover, as $\varepsilon = \psi + \theta\eta$, in view of Eqs. (4.1), (3.17), and the definition of acceleration wave, the condition (4.11) turns out to be equivalent to Eq. (4.10).

The constraint (2.9) results in a restriction on the vector $\mathbf{a}$. Observe that $\mathbf{e} = \mathbf{F}\mathbf{e}^0$ is a vector tangent to the fibre in the actual configuration and that Eq. (2.9) is equivalent to

$$|\mathbf{e}| = |\mathbf{F}\mathbf{e}^0| = 1.$$

Then, on differentiating with respect to time and evaluating the jump across $\mathscr{S}$, we find that [21]

(4.13) $(\mathbf{a}\cdot\mathbf{e})(\mathbf{N}\cdot\mathbf{e}^0) = 0.$

On the other hand, the forces maintaining the constraint (2.9) do no work, namely $\lambda_c\cdot\dot{\mathbf{C}} = 0$, and then $[p(\mathbf{e}^0\otimes\mathbf{e}^0)\cdot\dot{\mathbf{C}}] = 0$. Hence application of Eq. (4.5) to Eqs. (2.4) and (2.5) allows us to obtain

$$\varrho_0\mathbf{a} = -U_N^{-1}[\dot{\mathbf{S}}]\mathbf{N},$$

$$\varrho_0[\dot{\varepsilon}] = -U_N^{-1}(\tilde{\mathbf{T}} - p^0\mathbf{e}\otimes\mathbf{e}^0)\cdot((\mathbf{a}\mathbf{F})\otimes\mathbf{N}) + U_N^{-1}[\dot{\mathbf{Q}}]\cdot\mathbf{N}.$$

On account of Eqs. (4.1), (4.2), (3.17), (4.6)–(4.8) we can derive explicit expressions for the jumps $[\dot{\mathbf{S}}]$, $[\dot{\varepsilon}]$, $[\dot{\mathbf{Q}}]$ in terms of $\mathbf{a}$ and $\mathscr{G}$. So, upon substitution we arrive at the system of equations

(4.14) $\{Q + p(\mathbf{e}^0\cdot\mathbf{N})^2\mathbf{I} - \varrho_0 U_N^2\mathbf{I}\}\mathbf{a} + 2\mu_1\sigma_2\mathbf{F}\mathbf{F}^\mathsf{T}\mathbf{a} + 2(\mu_1 + \mu_2)\sigma_2\mathbf{F}\mathbf{N}(\mathbf{N}\mathbf{F}^\mathsf{T})\cdot\mathbf{a}$

$$+ \left(2\mu_1\frac{\sigma_2}{\sigma_1}\mathbf{N}\cdot\mathbf{\Sigma}\mathbf{N} + \mu_2\frac{\sigma_2}{\sigma_1}\operatorname{tr}\mathbf{\Sigma}\right)\mathbf{a} + U_N^2\left\{2\varrho_0\mathbf{F}\Psi_{\theta c}\mathbf{N} + 2\left(\frac{\mu_1\sigma_2}{\sigma_1}\right)_\theta\mathbf{F}\mathbf{\Sigma}\mathbf{N} + \left(\frac{\mu_2\sigma_2}{\sigma_1}\right)_\theta(\operatorname{tr}\mathbf{\Sigma})\mathbf{F}\mathbf{N}\right\}\mathscr{G}$$

$$- U_N\mathbf{e}(\mathbf{e}^0\cdot\mathbf{N})[\dot{p}] = \mathbf{0},$$

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[4] The exceptional nature of the case $\mathbf{e}^0\cdot\mathbf{N} = 0$ has already been noticed by CHEN and GURTIN [20].

(4.15) $\mathbf{N}\left\{2\theta\varrho_0\,\Psi_{\theta c}-2\varrho_0\sigma_1\,\varepsilon_\Sigma+2\mu_1\dfrac{\sigma_2}{\sigma_1}\,\Sigma+\mu_2\dfrac{\sigma_2}{\sigma_1}\,(\mathrm{tr}\Sigma)\mathbf{I}\right\}\mathbf{F}^{\mathsf{T}}\cdot\mathbf{a}$

$$-\left\{\varrho_0cU_N^2-\varrho_0\chi_1\mathbf{N}\cdot\varepsilon_\Lambda U_N+\mathbf{N}\cdot\left(\dfrac{\chi_2\mathbf{K}}{\chi_1}\right)_\theta\Lambda U_N-\chi_2\mathbf{N}\cdot\mathbf{KN}\right\}\mathscr{G}=0,$$

where $\mathbf{Q}=4\varrho_0\mathbf{F}\otimes\mathbf{N}\Psi_{cc}\mathbf{N}\otimes\mathbf{F}^{\mathsf{T}}+2\varrho_0(\mathbf{N}\cdot\Psi_c\mathbf{N})\mathbf{I}$ is the acoustic tensor,

$$\varepsilon_\Sigma=\dfrac{\sigma_2}{\varrho_0\sigma_1^2}\left(\mu_1\Sigma+\tfrac{1}{2}\mu_2(\mathrm{tr}\,\Sigma)\mathbf{I}\right)-\dfrac{1}{\varrho_0}\left\{\left(\dfrac{\mu_1\sigma_2}{\sigma_1^2}\right)_\theta\Sigma+\dfrac{1}{2}\left(\dfrac{\mu_2\sigma_2}{\sigma_1^2}\right)_\theta(\mathrm{tr}\Sigma)\mathbf{I}\right\},$$

$$c=\varepsilon_\theta=-\theta\Psi_{\theta\theta}-\dfrac{\theta}{2\varrho_0}\left\{\left(\dfrac{\mu_1\sigma_2}{\sigma_1^2}\right)_{\theta\theta}\Sigma\cdot\Sigma+\dfrac{1}{2}\left(\dfrac{\mu_2\sigma_2}{\sigma_1^2}\right)_{\theta\theta}(\mathrm{tr}\Sigma)^2+\Lambda\cdot\left(\dfrac{\chi_2\mathbf{K}}{\theta\chi_1^2}\right)_{\theta\theta}\Lambda\right\}$$

is the specific heat, and

$$\varepsilon_\Lambda=\dfrac{\chi_2}{\varrho_0\theta\chi_1^2}\mathbf{K}\Lambda-\dfrac{\theta}{\varrho_0}\left(\dfrac{\chi_2\mathbf{K}}{\theta\chi_1^2}\right)_\theta\Lambda.$$

On account of the system of equations (4.13)–(4.15) we are now in a position to derive the propagation speed U_N and the corresponding jump discontinuities $\mathbf{a}$, $\mathscr{G}$, $[\dot{p}]$. In the next section this is accomplished in connection with a simple but remarkable case.

5. Acceleration waves entering the natural state

Suppose now that the region ahead has been kept in the natural state until the arrival of the wave. In such a case Σ and Λ vanish in the region ahead and then, owing to their continuity across the wave, they vanish at the wave front. Moreover, we have $\mathbf{F}=\mathbf{I}$, $\mathbf{C}=\mathbf{I}$ in the region ahead and at the wave front. Hence, in conjunction with the wave discontinuities, we may set $\varrho=\varrho_0$, $\mathbf{e}=\mathbf{e}^0$, $\mathbf{n}:=\mathbf{FN}=\mathbf{N}$ and identify the speed of propagation U_N with the local speed of propagation U [27]. So the system of equations (4.13)–(4.15) simplifies to

(5.1) $(\mathbf{a}\cdot\mathbf{e})\,(\mathbf{n}\cdot\mathbf{e})=0,$

(5.2) $(\mathfrak{Q}+p(\mathbf{e}\cdot\mathbf{n})^2\mathbf{I}-\varrho U^2\mathbf{I})\mathbf{a}+U^2\mathfrak{z}\mathscr{G}-U(\mathbf{e}\cdot\mathbf{n})\mathbf{e}[\dot{p}]=\mathbf{0},$

(5.3) $-\theta\mathfrak{z}\cdot\mathbf{a}+(\varrho cU^2-\chi_2\mathbf{n}\cdot\mathbf{Kn})\mathscr{G}=0,$

where $\mathfrak{Q}(\mathbf{n})$ denotes the effective acoustic tensor

(5.4) $\mathfrak{Q}=\mathbf{Q}+2\mu_1\sigma_2\mathbf{I}+2(\mu_1+\mu_2)\sigma_2\mathbf{n}\otimes\mathbf{n}$

and $\mathfrak{z}(\mathbf{n})$ the vector

$$\mathfrak{z}=2\varrho\Psi_{\theta c}\,\mathbf{n}.$$

The relation (5.4) tells us that the effect of viscosity on the effective acoustic tensor results in an additive contribution given by the symmetric tensor $2\mu_1\sigma_2\mathbf{I}+2(\mu_1+\mu_2)\sigma_2\mathbf{n}\otimes\mathbf{n}$ which, in view of the inequalities (3.13), is positive definite. Hence the eigenvalues of $\mathfrak{Q}$ are greater than the corresponding ones of $\mathbf{Q}$.

Because of Eq. (5.1), nontrivial wave discontinuities pertain to one of the following two cases.

Direction of propagation perpendicular to the direction of inextensibility

Let $\mathbf{n} \cdot \mathbf{e} = 0$. As a consequence $\mathbf{n} \cdot \mathbf{Kn} = \varkappa$ and the system (5.1)–(5.3) becomes

$$\mathbf{n} \cdot \mathbf{e} = 0,$$
$$(\mathfrak{Q} - \varrho U^2 \mathbf{I})\mathbf{a} + U^2 \mathfrak{z} \mathcal{G} = \mathbf{0},$$
$$-\theta \mathfrak{z} \cdot \mathbf{a} + (\varrho c U^2 - \chi_2 \varkappa) \mathcal{G} = 0.$$

The corresponding determinantal equation, which may be written as

(5.5) $$\det\{(\varrho c U^2 - \chi_2 \varkappa)(\mathfrak{Q} - \varrho U^2 \mathbf{I}) + \theta U^2 \mathfrak{z} \otimes \mathfrak{z}\} = 0,$$

is a quadratic equation in U^2. Accordingly, in general we find two values of U^2; if these two values are positive, then they correspond to two pairs of waves propagating with equal speeds but in opposite directions.

Whether or not all four solutions U to Eq. (5.5) are real depends on the thermomechanical properties of the body. In this regard a simple case occurs in the absence of thermomechanical coupling, namely when $\Psi_{\theta c}$ is such that $\theta \mathfrak{z} \otimes \mathfrak{z}$ is negligible with respect to $\varrho c \mathfrak{Q}$ or $\varrho \chi_2 \varkappa \mathbf{I}$. If this is so, Eq. (5.5) splits into the two conditions

$$\det(\mathfrak{Q} - \varrho U^2 \mathbf{I}) = 0,$$
$$\varrho c U^2 - \chi_2 \varkappa = 0,$$

which corresponds to purely mechanical (acceleration) waves and to purely thermal waves with speed $U = (\chi_2 \varkappa / \varrho c)^{1/2}$. Hence account of viscosity and heat conduction results in the increase of the speed of the mechanical waves and in the existence of second sound.

Amplitude vector perpendicular to the direction of inextensibility

Let $\mathbf{a} \cdot \mathbf{e} = 0$. The unknowns of the system (5.1)–(5.3) are $\mathbf{a}$, $\mathcal{G}$, and $[\dot{p}]$. In the present case, on taking the inner product of Eq. (5.2) with $\mathbf{e}$, we find that $[\dot{p}]$ is determined by $\mathbf{a}$ and $\mathcal{G}$ through the relation

$$U \mathbf{e} \cdot \mathbf{n}[\dot{p}] = \mathbf{e} \cdot \mathfrak{Q} \mathbf{a} + U^2 \mathfrak{z} \cdot \mathbf{e} \mathcal{G}.$$

Meanwhile, letting $\mathbf{P} = \mathbf{I} - \mathbf{e} \otimes \mathbf{e}$, we achieve the system of equations

(5.6) $$\mathbf{a} \cdot \mathbf{e} = 0,$$
$$(\mathbf{P}\mathfrak{Q} + p(\mathbf{e} \cdot \mathbf{n})^2 \mathbf{I} - \varrho U^2 \mathbf{I})\mathbf{a} + U^2 \mathbf{P} \mathfrak{z} \mathcal{G} = \mathbf{0};$$
(5.7) $$-\theta \mathfrak{z} \cdot \mathbf{a} + (\varrho c U^2 - \chi_2 \mathbf{n} \cdot \mathbf{Kn}) \mathcal{G} = 0,$$

in the unknowns $\mathbf{a}$ and $\mathcal{G}$ only. The associated determinantal equation

(5.8) $$\det\{(\varrho c U^2 - \chi_2 \mathbf{n} \cdot \mathbf{Kn})(\mathbf{P}\mathfrak{Q} + p(\mathbf{e} \cdot \mathbf{n})^2 \mathbf{I} - \varrho U^2 \mathbf{I}) + \theta U^2 (\mathbf{P}\mathfrak{z}) \otimes \mathfrak{z}\} = 0,$$

is again a quadratic equation in U^2. Likewise to Eq. (5.5), in the absence of thermomechanical coupling the determinantal equation (5.8) results in two conditions:

(5.9) $$\det\{\mathbf{P}\mathfrak{Q} + p(\mathbf{e} \cdot \mathbf{n})^2 \mathbf{I} - \varrho U^2 \mathbf{I}\} = 0,$$
(5.10) $$\varrho c U^2 - \chi_2 \mathbf{n} \cdot \mathbf{Kn} = 0.$$

The condition (5.9) corresponds to purely mechanical waves, the condition (5.10) to purely thermal waves travelling at the speed $U = (\chi_2 \mathbf{n} \cdot \mathbf{Kn}/\varrho c)^{1/2}$, usually greater than $(\chi_2 \varkappa / \varrho c)^{1/2}$.

As to the purely mechanical waves it is of interest to look at the effect of the constraint. On denoting by $\hat{\mathbf{a}}$ and $\hat{U}$ the values of $\mathbf{a}$ and U corresponding to the vanishing of the reaction stress, in view of the conditions (5.6) — with $\mathfrak{z} = \mathbf{0}$ — and (5.9), it follows that

$$\mathbf{a} = \hat{\mathbf{a}},$$

$$\varrho U^2 = \varrho \hat{U}^2 + p(\mathbf{e} \cdot \mathbf{n})^2,$$

whereby a compressive reaction stress lowers the speed of propagation. This conclusion has been arrived at also in connection with inextensible elastic bodies [20, 28].

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