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Annular punch on a transversely isotropic layer bonded to a half-space

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IN THIS NOTE the author analyzes the axisymmetric indentation of an annular punch into the elastic, transversely isotropic layer perfectly bonded to a dissimilar half-space.

W pracy autor analizuje osiowo-symetryczne wciskanie pierścieniowego stempla w poprzecznie izotropową warstwę sprężystą idealnie połączoną z różną od niej półprzestrzenią.

В работе автор анализирует осесимметричное вдавливание кольцевого штампа в упругий, трансверсально-изотропный слой соединенный с другим полупространством.

1. Introduction

CONTACT problems involving annular contact region have been studied in recent years and several solutions have been given for the case in which the punch face is flat. COOKE [1], COLLINS [2], GUBENKO and MOSSAKOVSKII [3], OLESIAK [4], JAIN and KANWAL [5] and SHIBUYA *et al.* [6, 7] have analyzed the problem for isotropic half-space. The author [8–9] has also analyzed the problem for a transversely isotropic medium. DHALIWAL and SINGH [10] have considered the annular punch problem for the isotropic layer bonded to an isotropic half-space. The same problem for transversely isotropic materials has been considered in this note.

2. Basic equations and their solution

In a cylindrical coordinate system (r, θ, z), the axisymmetrical solutions of the elastic basic equations of equilibrium for a transversely isotropic material are given by the functions $\varphi_1(r, z)$ and $\varphi_2(r, z)$ satisfying the partial differential equations [11]

$$(\partial_r^2 + r^{-1}\partial_r + s_i^{-2}\partial_z^2)\varphi_i(r, z) = 0, \quad i = 1, 2 \text{ (no sum implied)}, \tag{2.1}$$

where ∂_r denotes differentation with respect to r, etc. The nonvanishing components of the displacement and stress due to the functions are [11]

$$u_r = \partial_r(k\varphi_1 + \varphi_2), \quad u_z = \partial_z(\varphi_1 + k\varphi_2); \tag{2.2}$$

$$\begin{gathered}\sigma_{zz} = G_1(k+1)\partial_z^2(s_1^{-2}\varphi_1 + s_2^{-2}\varphi_2),\\ \sigma_{rz} = G_1(k+1)\partial_{rz}^2(\varphi_1 + \varphi_2),\\ \begin{pmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\end{pmatrix} = -G_1(k+1)\partial_z^2(\varphi_1 + \varphi_2) - 2G\begin{pmatrix}r^{-1}\partial_r\\ \partial_r^2\end{pmatrix}(k\varphi_1 + \varphi_2),\end{gathered} \tag{2.3}$$

where G_1 and G denote the shear modulus in the z-direction and in the $r-\theta$ plane respectively, and the nondimensional quantities s_1, s_2, k related to the five elastic constants of the material E, ν, E_1, G_1, ν_1 are given by the following equations [11]:

$$(2.4)\qquad s_{1,2} = \frac{1}{2}(\alpha \pm \beta), \quad \alpha = \varepsilon\sqrt{2(\varrho+1)}, \quad \beta = \varepsilon\sqrt{2(\varrho-1)}, \quad k = q+\sqrt{q^2-1},$$

$$(2.5)\qquad \begin{aligned} &\varepsilon = \left[\frac{1}{1-\nu^2}H(1-\nu_1^2 H)\right]^{1/4}, \quad \varrho = (\Gamma-\nu_1 H)\left[\frac{1-\nu}{1+\nu}H(1-\nu_1^2 H)\right]^{-1/2},\\ &q = \Gamma\left[\nu_1 H+\frac{G_1}{E_1}(1-\nu-2\nu_1^2 H)\right]^{-1}-1, \quad H = \frac{E}{E_1}, \quad \Gamma = \frac{G}{G_1}. \end{aligned}$$

Suitable solutions of Eqs. (2.1) are taken in the form

$$(2.6)\qquad \varphi_i(r,z) = \frac{s_{i\pm 1}}{G_1(k+1)(s_1-s_2)}\int_0^\infty \xi^{-1}[A_i(\xi)e^{-s_i\xi z}+B_i(\xi)e^{s_i\xi z}]J_0(\xi r)d\xi,$$

where $A_i(\xi)$, $B_i(\xi)$ $(i = 1,2)$ are arbitrary functions and $J_n(\xi r)$ is the Bessel function of the first kind of order n.

Substituting Eq. (2.6) into Eqs. (2.2) and (2.3) the stress, displacement fields are

$$(2.7)\qquad \begin{aligned} u_r &= -\frac{1}{G_1(k+1)\beta}\int_0^\infty [ks_2(A_1 e^{-s_1\xi z}+B_1 e^{s_1\xi z})+s_1(A_2 e^{-s_2\xi z}+B_2 e^{s_2\xi z})]J_1(\xi r)d\xi,\\ u_z &= -\frac{s_1 s_2}{G_1(k+1)\beta}\int_0^\infty [A_1 e^{-s_1\xi z}-B_1 e^{s_1\xi z}+k(A_2 e^{-s_2\xi z}-B_2 e^{s_2\xi z})]J_0(\xi r)d\xi; \end{aligned}$$

$$(2.8)\qquad \begin{aligned} \sigma_{zz} &= \frac{1}{\beta}\int_0^\infty \xi[s_2(A_1 e^{-s_1\xi z}+B_1 e^{s_1\xi z})+s_1(A_2 e^{-s_2\xi z}+B_2 e^{s_2\xi z})]J_0(\xi r)d\xi,\\ \sigma_{rz} &= -\frac{s_1 s_2}{\beta}\int_0^\infty \xi(-A_1 e^{-s_1\xi z}+B_1 e^{s_1\xi z}-A_2 e^{-s_2\xi z}+B_2 e^{s_2\xi z})J_1(\xi r)d\xi. \end{aligned}$$

The other components may be expressed similarly.

3. Statement of the problem and derivation of the integral equation

Consider a layer bounded by a pair of parallel planes $z = 0$ and $z = -h$. The layer is perfectly bonded to a half-space $z \geqslant 0$. The materials of the layer and the half-space are different but homogeneous, transversely isotropic and elastic.

The free surface $z = -h$ is indented over an annular area $r_i \leqslant r \leqslant r_0$ by an annular rigid punch with an arbitrary smooth face. We assume that the punch is indented to a depth δ and the interface between the punch and the layer is fixed, while the remaining part of the surface is free from traction.

The boundary conditions of the present problem are:

$$u_z(r, -h) = \delta - f(r-r_c), \quad r_i \leqslant r \leqslant r_0, \tag{3.1}$$

$$\sigma_{zz}(r, -h) = 0, \quad 0 \leqslant r < r_i, \quad r > r_0, \tag{3.2}$$

$$\sigma_{zr}(r, -h) = 0, \quad r \geqslant 0, \tag{3.3}$$

where δ is the penetration depth of the indentor, $z = f(r-r_c)$ the function describing the indentor profile so that $f(0) = 0$, r_i and r_0 are the inner and outer contact radii respectively, and r_c is a coordinate of the top of the end surface of the punch. The indented load P is given for the static equilibrium of a punch by the equation

$$P = -2\pi \int_{r_i}^{r_0} r\sigma_{zz}(r, -h)dr. \tag{3.4}$$

The displacement and stress fields in the elastic layer are given by Eqs. (2.7) and (2.8). For the half-space they are obtained from the same equations by replacing u_r, u_z, σ_{zz}, σ_{zr}, s_1, s_2, k, G_1, $A_1(\xi)$, $A_2(\xi)$ by u_r', u_z', σ_{zz}', σ_{zr}', s_1', s_2', k', G_1', $D_1(\xi)$, $D_2(\xi)$ respectively, and setting $B_1(\xi) = B_2(\xi) = 0$. $A_i(\xi)$, $B_i(\xi)$, $D_i(\xi)$ $(i = 1, 2)$ are unknown functions of ξ to be determined from the boundary conditions (3.1), (3.2), (3.3) and from the following continuity conditions which must be satisfied at the interface between the two regions $z = 0$

$$u_r'(r, 0+) = u_r(r, 0-), \quad u_z'(r, 0+) = u_z(r, 0-), \quad r \geqslant 0, \tag{3.5}$$

$$\sigma_{zz}'(r, 0+) = \sigma_{zz}(r, 0-), \quad \sigma_{zr}'(r, 0+) = \sigma_{zr}(r, 0-), \quad r \geqslant 0. \tag{3.6}$$

Solving Eqs. (3.3), (3.5) and (3.6) for $A_i(\xi)$, $B_i(\xi)$ and $D_i(\xi)$, we obtain

$$\begin{aligned}
A_1(\xi) &= -(a_5 e^{-s_1 x} - a_1 e^{s_2 x} + a_3 e^{-s_2 x}) G(\xi)[X(x)]^{-1},\\
A_2(\xi) &= -(a_1 e^{s_1 x} + a_4 e^{-s_1 x} + a_6 e^{-s_2 x}) G(\xi)[X(x)]^{-1},\\
B_1(\xi) &= -(a_5 e^{s_1 x} + a_4 e^{s_2 x} - a_2 e^{-s_2 x}) G(\xi)[X(x)]^{-1},\\
B_2(\xi) &= -(a_3 e^{s_1 x} + a_2 e^{-s_1 x} + a_6 e^{s_2 x}) G(\xi)[X(x)]^{-1},\\
D_1(\xi) &= -(-b_1 e^{s_1 x} + b_2 e^{-s_1 x} - b_3 e^{s_2 x} - b_4 e^{-s_2 x}) G(\xi)[X(x)]^{-1},\\
D_2(\xi) &= -(b_5 e^{s_1 x} - b_6 e^{-s_1 x} + b_7 e^{s_2 x} + b_8 e^{-s_2 x}) G(\xi)[X(x)]^{-1}, \quad (x = \xi h),
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
b_{1,2} &= c_1[s_1(g-1) \mp s_2'(gk'-k)],\\
b_{3,4} &= c_1[s_2'(gk'-1) \mp s_2(g-k)],\\
b_{5,6} &= c_2[s_1(gk'-1) \mp s_1'(g-k)],\\
b_{7,8} &= c_2[s_1'(g-1) \mp s_2(gk'-k)],
\end{aligned} \tag{3.8}$$

$$g = \frac{G_1(k+1)}{G_1'(k'+1)}, \quad c_{1,2} = \frac{\beta s_{1,2}'}{2s_1 s_2 \beta'(k-1)},$$

$$a_1 = b_3 b_5 - b_1 b_7, \quad a_2 = b_4 b_6 - b_2 b_8,$$

$$a_3 = b_1 b_8 - b_4 b_5, \quad a_4 = b_2 b_7 - b_3 b_6,$$

$$a_5 = b_2 b_5 - b_1 b_6, \quad s_1 a_6 = s_2 a_5.$$

$G(\xi)$ is an unknown function to be determined from the boundary conditions (3.1) and (3.2) and the function $X(x)$ ($x = \xi h$) is defined by the equation

$$X(x) = a_1 e^{\alpha x} + a_2 e^{-\alpha x} + a_3 \alpha\beta^{-1} e^{\beta x} + a_4 \alpha\beta^{-1} e^{-\beta x} + 4s_2 a_5 \beta^{-1}; \tag{3.9}$$
$$\alpha = s_1 + s_2, \quad \beta = s_1 - s_2.$$

The substitution of $A_i(\xi)$ and $B_i(\xi)$ from Eq. (3.7) into Eqs. (2.7) and (2.8) leads to the following expressions for normal displacement $u_z(r, -h)$ and stress $\sigma_{zz}(r, -h)$:

$$u_z(r, -h) = C^{-1} \int_0^\infty [1 + M(\xi h)] G(\xi) J_0(\xi r) d\xi, \tag{3.10}$$

$$\sigma_{zz}(r, -h) = -\int_0^\infty \xi G(\xi) J_0(\xi r) d\xi, \tag{3.11}$$

where

$$C = \frac{G_1(k+1)(s_1 - s_2)}{s_1 s_2 (k-1)} \tag{3.12}$$

is a parameter depending only on elastic constants of the material. The function $M(x)$ ($x = \xi h$) is defined by the equation

$$M(x) = -2(a_2 e^{-\alpha x} + s_1 a_3 \beta^{-1} e^{\beta x} + s_2 a_4 \beta^{-1} e^{-\beta x} + 2s_2 a_5 \beta^{-1})[X(x)]^{-1}, \tag{3.13}$$

where $X(x)$ is given by the expression (3.9).

When the medium is homogeneous ($a_2 = a_3 = a_4 = a_5 = 0$, $a_1 = 1$) or $h \to \infty$ (the cases of the half-space), the function $M(\xi h)$ is identically zero.

Substituting Eqs. (3.10) and (3.11) in the boundary conditions (3.1) and (3.2), we find that they are satisfied if $G(\xi)$ is the solution of the triple integral equations:

$$\int_0^\infty [1 + M(\xi h)] G(\xi) J_0(\xi r) d\xi = C[\delta - f(r - r_c)], \quad r_i \leqslant r \leqslant r_0, \tag{3.14}$$

$$\int_0^\infty \xi G(\xi) J_0(\xi r) d\xi = 0, \quad 0 \leqslant r < r_i, \quad r > r_0. \tag{3.15}$$

To solve the triple integral equations (3.14) and (3.15) we assume the following integral representation of the function $G(\xi)$ as

$$G(\xi) = \delta \int_{r_i}^{r_0} g(u) J_0(\xi u) du, \tag{3.16}$$

where $g(u)$ is an arbitrary continuous function.

Substituting Eq. (3.16) into $\sigma_{zz}(r, -h)$ of Eq. (3.11), we obtain

$$\sigma_{zz}(r, -h) = -\delta \int_{r_i}^{r_0} g(u) du \int_0^\infty \xi J_0(u\xi) J_0(r\xi) d\xi = -\delta \int_{r_i}^{r_0} g(u) \frac{\delta(u - r)}{\sqrt{ur}} du, \tag{3.17}$$

where $\delta(u-r)$ is a Dirac's delta function and the following formula is used:

$$\int_0^\infty \xi J_0(u\xi)J_0(r\xi)d\xi = \frac{\delta(u-r)}{\sqrt{ur}}. \tag{3.18}$$

Since the argument in the delta function is $u-r \neq 0$ in $0 \leqslant r < r_i$ or $r_0 < r$ because of $r_i \leqslant u \leqslant r_0$, the stress $\sigma_{zz}(r, -h)$ is always zero in the free surface independent of the function $g(u)$. Then Eq. (3.15) are identically satisfied. The stress in the contact region is from Eq. (3.17):

$$\sigma_{zz}(r, -h) = -\delta r^{-1}g(r), \quad r_i < r < r_0. \tag{3.19}$$

Substituting Eq. (3.16) into Eq. (3.14), we reduce the problem to the solution of a Fredholm integral equation of the first kind for the determination of an unknown function $g(r)$:

$$\int_{r_i}^{r_0} K(u, r)g(u)du = C[1-\delta^{-1}f(r-r_c)], \quad r_i < r < r_0. \tag{3.20}$$

The symmetric kernel $K(u, r)$ is in the form

$$K(u, r) = \int_0^\infty [1+M(\xi h)]J_0(\xi u)J_0(\xi r)d\xi. \tag{3.21}$$

From Eqs. (3.4) and (3.19) we have the expression for the total load P which must be applied to the punch to maintain the penetration δ:

$$P = 2\pi\delta \int_{r_i}^{r_0} g(r)dr. \tag{3.22}$$

The inner and outer contact radii r_i and r_0 are not known *a priori*. Since the punch renders contact smooth at the edges $r = r_i$ and $r = r_0$ with the layer, $\sigma_{zz}(r, -h)$ must be finite as $r \to r_i+0$ or $r \to r_0-0$.

This is equivalent to the conditions where the function $g(r)$ is zero at $r = r_i$ and $r = r_0$. The inner and outer contact radii can be expressed by the following equations:

$$g(r_i) = 0, \quad g(r_0) = 0. \tag{3.23}$$

In the special case of flat-ended annular punch, the contact radii are given by the inner and outer radii of the annulus.

4. Solution of the integral equation

The integral equation (3.20) can be solved approximately by the GUBENKO–MOSSAKOVSKII technique [3], DHALIWAL and SINGH [10], JAIN and KANWAL [5] have discussed equations of this type.

By following their analysis, we derive from Eq. (3.20) a system of four Fredholm integral equations of the second kind with four unknown functions [9]. The solutions of these integral equations are obtained approximately by an iterative process when the parameters $\lambda = r_i/r_0$ and $\varepsilon = r_0/h$ are small. For the case of a flat-ended annular punch, i.e. for $f(r-r_c) = 0$ and $r_i = a$, $r_0 = b$ (a, b are the inner and outer radii of the annulus), the following values of the contact stress and penetration depth are obtained correct to $O(\lambda^4)$ or $O(\varepsilon^5)$ (setting $r = br'$ and dropping the dashes)

$$\sigma_{zz}(rb, -h) = -\frac{P}{\pi^2 b^2}\frac{g(r, \lambda, \varepsilon)}{g_1(\lambda, \varepsilon)}, \quad \lambda < r < 1, \tag{4.1}$$

$$\delta = \frac{P}{4bCg_1(\lambda, \varepsilon)}, \tag{4.2}$$

where

$$\begin{aligned} g(r, \lambda, \varepsilon) = &\left(1-\frac{2\varepsilon}{\pi}I_0+\frac{4\varepsilon^2}{\pi^2}I_0^2\right)\left[\frac{\pi}{2\sqrt{1-r^2}}+\frac{\lambda}{\sqrt{r^2-\lambda^2}}-\arcsin\left(\frac{\lambda}{r}\right)\right] \\ &-\frac{\lambda^2 I_2}{6}\left[3\left(\frac{r}{\lambda}\right)^2\arcsin\left(\frac{\lambda}{r}\right)-\frac{3r^2-\lambda^2}{\lambda\sqrt{r^2-\lambda^2}}\right]-\frac{\lambda^3}{3r^3}\left[1-\frac{2}{\pi}\arccos(r)-\frac{2r}{\pi\sqrt{1-r^2}}\right] \\ &-\frac{\varepsilon^3}{3\sqrt{1-r^2}}\left[I_2(1-3r^2)+\frac{12}{\pi^2}I_0^3\right]+O(\lambda^4), \quad \lambda < r < 1, \end{aligned} \tag{4.3}$$

$$\begin{aligned} g_1(\lambda, \varepsilon) = 1-\frac{4\lambda^3}{3\pi^2}-\frac{2\varepsilon}{\pi}I_0\left(1-\frac{8\lambda^3}{3\pi^2}-\frac{2\varepsilon}{\pi}I_0+\frac{4\varepsilon^2}{\pi^2}I_0^2-\frac{8\varepsilon^3}{\pi^3}I_0^3\right) \\ +\frac{2\varepsilon^3}{3\pi}I_2\left(1-\frac{4\varepsilon}{\pi}I_0\right)+O(\varepsilon^5), \end{aligned} \tag{4.4}$$

$$\lambda = \frac{a}{b} \ll 1, \quad \varepsilon = \frac{b}{h} \ll 1, \quad \varepsilon = O(\lambda)$$

and

$$I_n = \frac{1}{(2n)!}\int_0^\infty x^{2n}M(x)dx, \quad n = 0, 1, 2, \ldots. \tag{4.5}$$

The integrals I_n are convergent because the function $M(x)$ given by Eqs. (3.13) and (3.9) satisfies

$$\lim_{x\to 0} x^{2n}M(x) = \begin{cases} g < \infty, & \text{for } n = 0, \\ 0 \quad , & \text{for } n = 1, 2, \ldots, \end{cases} \tag{4.6}$$

for arbitrary values of the elastic constants;

$$\lim_{x\to 0} M(x)\frac{e^{-\beta x}}{e^{-\alpha x}} = \begin{cases} \text{const}, & \text{when} \quad \alpha \in R_+ \\ -2\dfrac{a_3 s_1}{a_1 \beta}, & \text{when} \quad \alpha \in R_+ \quad \text{and} \quad \beta \in R_+; \quad (\alpha > \beta). \end{cases} \tag{4.7}$$

$M(x)$ is continuous for $x \in \langle 0, \infty)$ when $\alpha \in R_+$.

The parameter α is real when the five elastic constants of the layer satisfy the condition (see Eqs. (2.4) and (2.5))

$$\frac{G}{G_1}-\nu_1\frac{E}{E_1} > -\left[\frac{1-\nu}{1+\nu}\frac{E}{E_1}\left(1-\nu_1^2\frac{E}{E_1}\right)\right]^{1/2}. \tag{4.8}$$

The elastic constants for some practical materials such as magnesium, cadmium and laminates, fiber composites and other transversely isotropic materials satisfy the condition (4.8).

In special cases when $\varepsilon = 0$ ($I_n = 0$) or $\lambda = 0$, $I_n \neq 0$ or $\lambda = 0$, $\varepsilon = 0$ ($I_n = 0$), we obtained from Eqs. (4.3) and (4.4) the corresponding functions $g(r, \lambda, 0)$ and $g_1(\lambda, 0)$ for annular punch and half-space, $g(r, 0, \varepsilon)$ and $g_1(0, \varepsilon)$ for cylindrical punch and layered half-space, $g(r, 0, 0) = \pi/2\sqrt{1-r^2}$ and $g_1(0, 0) = 1$ for cylindrical punch and half-space. In the special case of isotropic ($C = G/(1-\nu)$) half-space the penetration depth is

$$\delta_1 = \frac{P(1-\nu)}{4bG}\left(1 - \frac{4\lambda^3}{3\pi^2}\right)^{-1}, \quad \lambda = \frac{a}{b}. \tag{4.9}$$

5. Conclusions

The stress in the contact region and penetration depth of the punch depend on the geometric parameters $\lambda = a/b$ and $\varepsilon = b/h$ and on the elastic constants of the materials.

The influence of the transverse anisotropy for a half-space is considered by the constant C and for the case of the half-space with a surface layer also by the function $M(x)$, defining the integrals I_n. The composite materials are strong transversely anisotropic, i.e. $H = E/E_1 \gg 1$ and $\Gamma = G/G_1 \gg 1$. Taking into account this type of anisotropy, one may conclude:

1. The penetration depth $\delta(\lambda, H, \Gamma)$ of the annular punch into transversely isotropic half-space is larger than the penetration depth into isotropic medium. For example, the ratio between the depth in the composite and in the isotropic semi-space for $\nu = 0.20$ and $\nu_1 = 0.10$ is $\delta(\lambda, 2, 2) = 2.04\delta_1$; $\delta(\lambda, 2, 20) = 5.12\delta_1$; $\delta(\lambda, 5, 10) = 6.00\delta_1$; $\delta(\lambda, 10, 5) = 6.82\delta_1$; $\delta(\lambda, 20, 20) = 14.25\delta_1$. The values of the depth increase slightly as λ increases.

2. The contact stresses, produced by flat-ended punch on a surface of the half-space are independent of the anisotropy of the material, and their values increase as the punch becomes thin. When the punch face is not flat, the contact stresses depend on the anisotropy of the material; as the transverse anisotropy increases, the contact stresses decrease.

3. When $H > H'$ and $\Gamma > \Gamma'$, then the penetration depth of the punch into a layer (H, Γ) is smaller and the contact stresses become larger than for the case of the half-space with the parameters H and Γ. In the case of not flat-ended annular punch, the contact region decreases. When $H < H'$ and $\Gamma < \Gamma'$, the contact region and penetration depth into the layer (H, Γ) are larger and the contact stresses are smaller in comparison with the same values in the half-space with the parameters H and Γ.

4. As the transverse material anisotropy of the layer increases for the same foundation or as the transverse anisotropy of the foundation increases for the same layer, the contact region and the penetration depth increase and the contact stresses decrease. As the transverse anisotropy of the layer or the foundation decrease, the contact region and the penetration depth decreases and the contact stresses increase.

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