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Stationary singular surfaces in materials with scalar internal variables

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IT IS SHOWN that there can exist stationary discontinuity surfaces of the second deformation gradient in a three-dimensional material with scalar internal variables. Explicit expressions for the growth and decay of the discontinutites are obtained. For a material with an unknown number of scalar internal variables, a method is proposed to find a lower bound for the number of the scalar internal variables, by observing the variation of such discontinuities.

Wykazano, że w trójwymiarowym materiale ze skalarnymi zmiennymi wewnętrznymi mogą istnieć stacjonarne powierzchnie nieciągłości drugiego gradientu deformacji. Otrzymano jawne wyrażenia na wzrost i zanikanie nieciągłości. Dla materiału o nieznanej liczbie skalarnych zmiennych wewnętrznych zaproponowano metodę znajdywania kresu dolnego dla liczby skalarnych zmiennych wewnętrznych drogą obserwacji zmienności takich nieciągłości.

Доказано, что в трехмерном материале со скалярными внутренними переменными могут существовать стационарные поверхности разрыва второго градиента деформации. Получены явные выражения для роста и исчезания разрыва. Для материала с неизвестным количеством скалярных внутренних переменных предложен метод нахождения нижней грани для количества скалярных внутренних переменных путем наблюдения переменности таких разрывов.

1. Introduction

INTERNAL variables are adopted in the constitutive equations of several classes of materials, e.g. (elastic-) plastic materials, chemically reacting materials, mixtures, materials with phase transitions etc. In general, internal variables can not be observed directly, so that it is important to analyse various behaviours of the materials in order to verify concerned assumptions on the constitutive equations. Acceleration waves in materials with internal variables have been studied in [1–6], while shock waves have been investigated in [3] and [7]. Since the histories of external variables should be known to solve the differential equations for the internal variables, the materials with internal variables may be regarded as special types of materials with memory, cf. [8] and [9]. Acceleration and shock waves in materials with memory have been studied in [10–14].

All the waves analysed above were singular surfaces or singular points which have nonvanishing velocities. On the other hand, recently the author [15] has shown that there may exist stationary singular points in a one-dimensional material with internal variables.

This paper considers a three-dimensional material with scalar internal variables. In the next section the constitutive equations are given, and proved is the existence of tationary singular surfaces across which the spatial derivatives of the deformation gradient and the internal variables are discontinuous. Section 3 derives and solves the amplitude equations for the discontinuities across the stationary singular surfaces. As shown in [5], the amplitude of an acceleration wave in a three-dimensional material with internal variables is governed by a Bernoulli's differential equation. Hence the variation of the amplitude is characterized by only two parameters independently of the number of the internal variables. In the meantime, the amplitude of a stationary singular surface is governed by a system of linear first-order differential equations, where the number of parameters equals the square of the number of the internal variables. Thus the variation of the amplitude of the stationary singular surface is much more intricate than that of the acceleration wave. In Section 4, for a material with an unknown number of scalar internal variables, a method is proposed to find a lower bound for the number of the internal variables, by observing the amplitude of a stationary singular surface, i.e. the jump of the second deformation gradient across the surface.

2. Constitutive equations and existence of stationary singular surfaces

A material with scalar internal variables is described by the constitutive equations

$$\mathbf{T}_{\varkappa} = \boldsymbol{\phi}(\mathbf{F}, \mathbf{a}), \tag{2.1}$$

$$\dot{\mathbf{a}} = \boldsymbol{\psi}(\mathbf{F}, \mathbf{a}), \tag{2.2}$$

where $\mathbf{T}_{\varkappa}$ denotes the Piola–Kirchhoff stress, $\mathbf{F}$ the deformation gradient, and $\mathbf{a}$ is the column vector of N scalar internal variables. A superposed dot means the partial differentiation with respect to time. We assume that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are continuously differentiable with respect to their arguments. When the external forces do not exist, the balance law of linear momentum is given by

$$T_{\varkappa}{}^{iJ}{}_{,J} = \varrho_{\varkappa}\dot{v}^{i}, \tag{2.3}$$

where $\mathbf{v}$ denotes the velocity and $\varrho_{\varkappa}$ the material density in a reference configuration.

We consider a plane S which may be propagating in the material and which satisfies the following conditions:

(i) The displacement $\mathbf{u}$, $\mathbf{F}$, $\mathbf{v}$ and $\mathbf{a}$ are continuous with respect to the time and the coordinates.

(ii) The first derivatives of $\mathbf{v}, \mathbf{F}$ and $\mathbf{a}$ suffer finite jump discontinuities across S in a time interval.

(iii) Each of $\mathbf{v}, \mathbf{F}, \mathbf{a}$ and the jumps of their first derivatives are uniformly constant over S at any instant.

In this paper we call S simply a singular surface. The first-order compatibility conditions for $\mathbf{v}$, $\mathbf{F}$ and $\mathbf{a}$ across S become then

$$[\dot{v}^{i}] = U^{2}e^{i}, \quad [\dot{F}^{i}{}_{J}] = -Ue^{i}n_{J}, \quad [F^{i}{}_{J,K}] = e^{i}n_{J}n_{K}, \tag{2.4}$$

$$[\dot{a}^{\alpha}] = -Ub^{\alpha}, \quad [a^{\alpha}{}_{,J}] = b^{\alpha}n_{J}, \tag{2.5}$$

where

$$e^{i} \equiv [F^{i}{}_{J,K}]n^{J}n^{K}, \quad b^{\alpha} \equiv [a^{\alpha}{}_{,J}]n^{J}, \tag{2.6}$$

and where $[\cdot]$ denotes the jump of a quantity, $\mathbf{n}$ is a unit normal on S, U the speed of S in the n direction. Here and henceforth Latin indices run from 1 to 3, and Greek ones from 1 to N. If $U = 0$, S is called a stationary singular surface. If $U \neq 0$, S is an acceleration wave.

Substituting Eq. (2.1) into Eq. (2.3), we get

$$\frac{\partial \phi^{iJ}}{\partial F^p{}_Q} F^p{}_{Q,J} + \frac{\partial \phi^{iJ}}{\partial a^\alpha} a^\alpha{}_{,J} = \varrho_\varkappa \dot{v}^i, \tag{2.7}$$

where we have omitted all the arguments of functions for simplicity. Taking the jump of Eq. (2.7) across S and using Eqs. $(2.4)_{1,3}$ and $(2.5)_2$ yield

$$(Q^i{}_p - \varrho_\varkappa U^2 \delta^i{}_p) e^p + R^i{}_\alpha b^\alpha = 0, \tag{2.8}$$

where

$$Q^i{}_p \equiv \frac{\partial \phi^{iJ}}{\partial F^p{}_Q} n_J n_Q, \qquad R^i{}_\alpha \equiv \frac{\partial \phi^{iJ}}{\partial a^\alpha} n_J. \tag{2.9}$$

The $\mathbf{Q}$ is called the instantaneous acoustic tensor corresponding to $\mathbf{F}$, $\mathbf{a}$ and $\mathbf{n}$. The jump relation of Eq. (2.2) across S becomes by use of Eq. $(2.5)_1$

$$U b^\alpha = 0. \tag{2.10}$$

Thus there are two possibilities: In the case when $U = 0$, S is a stationary singular surface, where $\mathbf{b}$ need not vanish. Then Eq. (2.8) reduces to

$$Q^i{}_p e^p + R^i{}_\alpha b^\alpha = 0. \tag{2.11}$$

In the other case when $\mathbf{b} = \mathbf{0}$, Eq. (2.8) takes the form

$$(Q^i{}_p - \varrho_\varkappa U^2 \delta^i{}_p) e^p = 0. \tag{2.12}$$

Thus in order for S to be a singular surface, $\varrho_\varkappa U^2$ must be a real and nonnegative eigenvalue of $\mathbf{Q}$ and $\mathbf{e}$ must be a real eigen vector of $\mathbf{Q}$ belonging to the eigenvalue. A positive eigenvalue corresponds to an acceleration wave, and a null eigenvalue to a stationary singular surface. Hence we see that there may exist two types of stationary singular surfaces in the material .The first type can always exist whatever value $\mathbf{Q}$ takes, but the second type can exist only when $\mathbf{Q}$ has a null eigen value.

According to the discussions [16, 17] about acoustic tensors for elastic materials, in general the eigen values of the acoustic tensors need not be real, nonnegative or positive. If the acoustic tensors have null eigenvalues, stationary singular surfaces may also exist in the elastic materials. Henceforth we assume that the acoustic tensor $\mathbf{Q}$ does not have any null eigenvalue, and hence we consider the first type of stationary singular surfaces only.

3. Growth and decay of amplitudes of stationary singular surfaces

The second-order compatibility conditions for $\mathbf{v}$, $\mathbf{F}$ and $\mathbf{a}$ across a stationary singular surface S are given by

$$[\ddot{v}^i] = 0, \quad [\dot{v}^i{}_{,J}] \equiv [\ddot{F}^i{}_J] = 0, \quad [\ddot{a}^\alpha] = 0, \tag{3.1}$$

$$(3.2) \qquad [\dot{F}^i{}_{J,K}] = \left(\frac{\delta}{\delta t}\, e^i\right) n_J n_K, \quad [\dot{a}^\alpha{}_{,J}] = \left(\frac{\delta}{\delta t}\, b^\alpha\right) n_J,$$

where $\delta/\delta t$ means the time differentiation of a jump quantity. Differentiating Eq. (2.2) with respect to X^J and taking the jump of the result across S, by use of Eqs. $(2.4)_3$ and $(2.5)_2$ we get

$$(3.3) \qquad [\dot{a}^\alpha{}_{,J}] = A^\alpha{}_p n_J e^p + B^\alpha{}_\beta n_J b^\beta,$$

where

$$(3.4) \qquad A^\alpha{}_p \equiv \frac{\partial \psi^\alpha}{\partial F^p{}_Q}\, n_Q, \quad B^\alpha{}_\beta \equiv \frac{\partial \psi^\alpha}{\partial a^\beta}.$$

Multiplying Eq. (3.3) by n^J and using Eq. $(3.2)_2$ yields

$$(3.5) \qquad \frac{\delta}{\delta t}\, b^\alpha = A^\alpha{}_p e^p + B^\alpha{}_\beta b^\beta.$$

Since we have assumed that $\mathbf{Q}$ does not have null eigenvalues, the characteristic equation for $\mathbf{Q}$ has no null roots:

$$(3.6) \qquad \det Q \neq 0,$$

which implies that $\mathbf{Q}$ is invertible. Multiplying Eq. (2.11) by $Q^{-1q}{}_i$ and then substituting the result into Eq. (3.5) to eliminate $\mathbf{e}$, we obtain a system of linear differential equations for $\mathbf{b}$:

$$(3.7) \qquad \frac{\delta}{\delta t}\, b^\alpha = C^\alpha{}_\beta b^\beta,$$

where

$$(3.8) \qquad C^\alpha{}_\beta = -A^\alpha{}_p Q^{-1p}{}_i R^i{}_\beta + B^\alpha{}_\beta.$$

When $\mathbf{F}$ and $\mathbf{a}$ are known in time at S, by the definition $\mathbf{C}$ is also known in time, where recall that $\mathbf{F}$ and $\mathbf{a}$ are assumed to be uniformly constant over S at each instant. Then under the initial condition

$$(3.9) \qquad b^\beta(t_0) = b^\beta{}_0.$$

Eq. (3.7) can be solved as

$$(3.10) \qquad b^\beta(t) = P^\alpha{}_\beta(t) b^\beta{}_0,$$

where

$$(3.11) \qquad P^\alpha{}_\beta(t) = \delta^\alpha{}_\beta + \int_{t_0}^{t} C^\alpha{}_\beta(\tau)\,d\tau + \int_{t_0}^{t} C^\alpha{}_\gamma(\tau_1) \int_{t_0}^{\tau_1} C^\gamma{}_\beta(\tau_2)\,d\tau_2\,d\tau_1 + \ldots .$$

Let λ denote the least upper bound of $||\mathbf{C}(\cdot)||$ in $[t_0, t]$. Then Eq. (3.11) implies that for each t ($\geqslant t_0$)

$$(3.12) \qquad ||\mathbf{P}(t)|| \leqslant e^{\lambda(t-t_0)}.$$

Substituting Eq. (3.10) into Eq. (2.11) and then multiplying the result by $Q^{-1q}{}_i$, we get

$$(3.13) \qquad e^q = -Q^{-1q}{}_i R^i{}_\alpha P^\alpha{}_\beta b^\beta{}_0,$$

where note that $\mathbf{Q}$ and $\mathbf{R}$ are known matrices in time. From Eqs. (3.10), (3.12) and (3.13) we have the following theorem:

THEOREM 1. *If* $\mathbf{Q}$ *does not have any null eigenvalue for every* $t \geqslant t_0$, $\mathbf{e}$ *and* $\mathbf{b}$ *are bounded for each* $t \geqslant t_0$.

In a special case where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are linear in $\mathbf{F}$ and $\mathbf{a}$, or where $\mathbf{F}$ and $\mathbf{a}$ are constant in time at S, from Eqs. (2.9), (3.4) and (3.8) $\mathbf{C}$ reduces to a constant matrix $\mathbf{C}_0$. Then $\mathbf{P}$ given by Eq. (3.11) takes the form:

$$\mathbf{P}(t) = e^{(t-t_0)\mathbf{C}_0}. \tag{3.14}$$

Let μ be the minimum of the real parts of all eigenvalues of $\mathbf{C}_0$. Then it is well known that there is a positive number M such that

$$||e^{(t-t_0)\mathbf{C}_0}|| \leqslant Me^{\mu(t-t_0)}. \tag{3.15}$$

Notice that $\mathbf{Q}$ and $\mathbf{R}$ are constant matrices in this case. Thus Eqs. (3.10), (3.13), (3.14) and (3.15) are combined to imply the next theorem.

THEOREM 2. *Suppose that* $\boldsymbol{\phi}$ *and* $\boldsymbol{\psi}$ *are linear in* $\mathbf{F}$ *and* $\mathbf{a}$, *or that* $\mathbf{F}$ *and* $\mathbf{a}$ *are constant in time at S. If* $\mathbf{Q}$ *does not have any null eigenvalue, and if the real parts of all eigenvalues of* $\mathbf{C}_0$ *are negative, then* $\mathbf{e}$ *and* $\mathbf{b}$ *tend to* $\mathbf{0}$ *as* $t \to \infty$.

4. Determination of a lower bound for the number of scalar internal variables

Consider a material with an unknown number of scalar internal variables, which has the constitutive equations (2.1) and (2.2). We assume that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are linear in $\mathbf{F}$ and $\mathbf{a}$, or that $\mathbf{F}$ and $\mathbf{a}$ are constant in time at a stationary singular surface S. Let e denote a component of $\mathbf{e}$, then from Eqs. (3.10), (3.13) and (3.14) it can be represented in the form

$$e(t) = \mathbf{c} \cdot (e^{(t-t_0)\mathbf{C}_0}\mathbf{b}_0), \tag{4.1}$$

where $\mathbf{b}_0$ and $\mathbf{c}$ are constant vectors in R^N and $\mathbf{C}_0$ is a constant $N \times N$ matrix. The expression (4.1) is of the same form as that for the jump of the second deformation gradient across a stationary singular point in a one-dimensional material with internal variables [15]. Thus we can apply the method proposed there to find a lower bound for the number of internal variables. In what follows we shall review the procedure briefly. Differentiating Eq. (4.1) at $t = t_0$, we get

$$e_i = \mathbf{c} \cdot (\mathbf{C}_0{}^i\mathbf{b}_0) \quad (i = 0, 1, \ldots), \tag{4.2}$$

where

$$e_i \equiv \frac{d^i}{dt^i} e(t)|_{t=t_0}. \tag{4.3}$$

Let $g(\cdot)$ denote the minimal polynomial with degree $K(\leqslant N)$ of $\mathbf{C}_0$, then it follows from the definition that

$$g(\mathbf{C}_0) \equiv \mathbf{C}_0{}^K + d_{K-1}\mathbf{C}_0{}^{K-1} + \ldots + d_0\mathbf{I} = \mathbf{0}, \tag{4.4}$$

where $d_0, \ldots, d_{K-1}$ are constant scalars. Multiplying Eq. (4.4) by $\mathbf{C}_0{}^i\mathbf{b}_0$ from the right and by $\mathbf{c}$ from the left, and then using Eq. (4.2), we get

$$e_{K+i} + d_{K-1}e_{K+i-1} + \ldots + d_0 e_i = 0 \quad (i = 0, 1, \ldots). \tag{4.5}$$

Next suppose that we can determine the time derivatives e_i up to and including the L-th order by observing the stationary singular surface. Consider a system of linear equations for $(y_0, \ldots, y_{n-1})$ for each $n = 1, 2, \ldots, L$:

$$e_i y_0 + e_{i+1} y_1 + \ldots + e_{i+n-1} y_{n-1} = e_{i+n} \quad (i = 0, 1, \ldots, L-n). \tag{4.6}$$

If $L \geqslant K$, from Eq. (4.5) the equations (4.6) for $n = K$ have the solution

$$(y_0, \ldots, y_{K-1}) = (-d_0, \ldots, -d_{K-1}). \tag{4.7}$$

Conversely, if Eq. (4.6) has no solutions for every n up to and including a positive integer N', N' should be smaller than K and hence the number of internal variables, N, is larger than N'. For each $n = 1, 2, \ldots, L$, define

$$\mathbf{G}_n \equiv \begin{bmatrix} e_0 & e_1 & \ldots & e_{n-1} \\ e_1 & e_2 & \ldots & e_n \\ \cdot & \cdot & \cdot & \cdot \\ e_{L-n} & e_{L-n+1} & \ldots & e_{L-1} \end{bmatrix}, \tag{4.8}$$

$$\mathbf{H}_n \equiv \begin{bmatrix} e_0 & e_1 & \ldots & e_n \\ e_1 & e_2 & \ldots & e_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ e_{L-n} & e_{L-n+1} & \ldots & e_L \end{bmatrix}. \tag{4.9}$$

Then the above assertion can be expressed as the next theorem.

THEOREM 3. *If N' is the maximum integer such that rank $\mathbf{G}_n \neq$ rank $\mathbf{H}_n$ for every $n \leqslant N'$, then the number of the internal variables is larger than N'.*

So far we have paid our attention to a component of the jump quantities $e^i \equiv [F^i{}_{J,K}] n^J n^K$. If we observe other components of e^i or other stationary singular surfaces in the same material, Theorem 3 may give several lower bounds for the number of the internal variables. In this case we may employ the maximum of the lower bounds as the most accurate lower bound for the number of the scalar internal variables of the material.

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