Arch. Mech., 34, 4, pp. 445-464, Warszawa 1982

On the linear theory of thermo-viscoelastic materials with internal state variables

S. CHIRIȚĂ (IAȘI)

IN THIS PAPER we establish the basic equations for the linear theory of thermo-viscoelastic materials with internal state variables. We further prove a uniqueness theorem for the solution of initial-boundary value problems formulated in the context of this theory.

W pracy przedstawiono podstawowe równania liniowej teorii materiałów termo-lepkosprężystych z wewnętrznymi zmiennymi stanu. Dowiedziono następnie twierdzenia o jednoznaczności dla rozwiązań zagadnień początkowo-brzegowych sformułowanych w ramach tej teorii.

В работе представлены основные уравнения линейной теории термо-вязкоупругих материалов с внутренними переменными состояния. Затем доказана теорема единственности для решений начально-краевых задач, сформулированных в рамках этой теории.

1. Introduction

A GENERAL theory of thermo-viscoelastic material bodies with internal state variables has been formulated by COLEMAN and GURTIN [1], BOWEN [2] and VALANIS [3]. Under some particular constitutive assumptions, an isotropic linear theory was considered by MIHĂILESCU and SULICIU [4, 5] concerning the propagation of acceleration waves in thermo-viscoelastic materials with internal state variables.

The present work considers materials with internal state variables, attention being focussed on the linear theory of anisotropic and inhomogeneous thermo-viscoelastic media. In Sect. 2 we summarize the basic structure for a thermoelastic body with internal state variables [1]. Further, we establish the basic equations for the case of small thermoelastic deformations.

For the case of linear theory, in Sect. 3, we prove the uniqueness of the solution to the initial-boundary value problems appropriate to the dynamics of the thermo-viscoelastic bodies with internal state variables. The method of proof is one based upon a Gronwall type inequality.

The uniqueness results for the internal state variable approach of finite deformations of materials without heat conduction was obtained by NACHLINGER and NUNZIATO [6], in the one-dimensional case, and by KOSIŃSKI [7, 8], for the three-dimensional case.

2. Basic equations

In what follows we consider the linear theory of mechanics of continuous media with internal state variables.

We consider a body which, at time $t = 0$, occupies the properly regular region V of Euclidean three-dimensional space R^3 and is bounded by the piecewise smooth surface ∂V [9]. The configuration of the body at time $t = 0$ is taken as the reference configuration. The motion of the body is referred to a fixed system of rectangular Cartesian axes.

The integral forms of the law of linear momentum and the law of balance of energy are equivalent to the following differential equations [1]:

$$t_{ji,j}+\varrho F_i = \varrho \ddot{u}_i, \tag{2.1}$$

$$\varrho \dot{U} = t_{ij}\dot{\varepsilon}_{ij}+\varrho r+q_{i,i}, \tag{2.2}$$

where

$$2\varepsilon_{ij} = u_{i,j}+u_{j,i}, \tag{2.3}$$

and, within the linear approximation,

$$2\dot{\varepsilon}_{ij} = \dot{u}_{i,j}+\dot{u}_{j,i}. \tag{2.4}$$

In the above relations we have used the following notations: ϱ is the density mass, u_i are the components of the displacement vector, U is the internal energy per unit mass, F_i are the components of the body force vector per unit mass, r is the heat supply function per unit mass and unit time, t_{ij} are the components of the stress tensor and q_i are the components of the heat flux vector. Throughout this paper we shall use the following conventions: a superposed dot denotes the material time derivative; Latin indices have the range 1, 2, 3, while the Greek subscripts have the range $1, 2, \ldots, n$; summation over repeated subscripts is implied; subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate x_i.

The entropy production inequality has the local form

$$-\varrho\dot{\psi}-\eta\dot{T}+t_{ij}\dot{\varepsilon}_{ij}+\frac{1}{T}q_i T_{,i} \geqslant 0, \tag{2.5}$$

where η is the entropy per unit volume, T is the absolute temperature which is assumed to be always positive, and ψ is the Helmholtz free energy function

$$\varrho\psi = \varrho U-T\eta. \tag{2.6}$$

According to the theory of [1], we define a linear thermo-viscoelastic material with internal state variables by the following constitutive equations:

$$\begin{aligned}
\psi &= \psi(\varepsilon_{mn};\, T;\, T_{,r};\, \xi_\beta;\, x_s),\\
t_{ij} &= t_{ij}(\varepsilon_{mn};\, T;\, T_{,r};\, \xi_\beta;\, x_s),\\
\eta &= \eta(\varepsilon_{mn};\, T;\, T_{,r};\, \xi_\beta;\, x_s),\\
q_i &= q_i(\varepsilon_{mn};\, T;\, T_{,r};\, \xi_\beta;\, x_s),\\
\dot{\xi}_\alpha &= f_\alpha(\varepsilon_{mn};\, T;\, T_{,r};\, \xi_\beta;\, x_s),
\end{aligned} \tag{2.7}$$

the functions from the set (2.7) being consistent with the assumptions of the linear theory. In the above equations the scalars ξ_α ($\alpha = 1, 2, \ldots, n$) represent the internal state variables [1, 2].

From the relations (2.5) and (2.7) it follows that

$$t_{ij} = \varrho \frac{\partial \psi}{\partial \varepsilon_{ij}}, \quad \eta = -\varrho \frac{\partial \psi}{\partial T}, \quad \frac{\partial \psi}{\partial T_{,r}} = 0, \tag{2.8}$$

$$-\sigma_\alpha f_\alpha + \frac{1}{T} q_i T_{,i} \geqslant 0, \tag{2.9}$$

where

$$\sigma_\alpha = \varrho \frac{\partial \psi}{\partial \xi_\alpha} \tag{2.10}$$

is called the chemical affinity of x_i [2].

Taking into account Eqs. (2.8)–(2.10), the relation (2.2) becomes

$$\sigma_\alpha f_\alpha + T\dot{\eta} = \varrho r + q_{i,i}. \tag{2.11}$$

In the linear theory we consider the temperature θ measured from the absolute temperature T_0 of the initial state and the internal state variables ω_α measured from the internal state variables ξ^0_α of the initial state. Thus we have

$$T = T_0 + \theta, \quad \xi_\alpha = \xi^0_\alpha + \omega_\alpha. \tag{2.12}$$

Therefore we suppose that the initial state of the body is characterized by the following:

$$\varepsilon_{ij} = 0, \quad T = T_0, \quad T_{,i} = 0, \quad \xi_\alpha = \xi^0_\alpha. \tag{2.13}$$

The initial state of the body is said to be an equlibrium state for the material if

$$f_\alpha(0, T_0, 0, \xi^0_\beta, x_s) = 0. \tag{2.14}$$

The initial state is a strong equilibrium state if it is an equlibrium state for which we have

$$\sigma_\alpha(0, T_0, 0, \xi^0_\beta, x_s) = 0. \tag{2.15}$$

In our subsequent development we will suppose that the initial state is a strong equilibrium state. In this case, from the inequality (2.9) we get [2, 10]

$$q_i(0, T_0, 0, \xi^0_\beta, x_s) = 0. \tag{2.16}$$

In the linear theory of an anisotropic thermo-viscoelastic material with internal state variables, we assume

$$\varrho\psi = \frac{1}{2} C_{ijrs}\varepsilon_{ij}\varepsilon_{rs} + \frac{1}{2} D_{\alpha\beta}\omega_\alpha\omega_\beta - \frac{1}{2} a\theta^2 - E_{ij}\varepsilon_{ij}\theta + F_{ij\alpha}\varepsilon_{ij}\omega_\alpha + G_\alpha\theta\omega_\alpha, \tag{2.17}$$

$$f_\alpha = m_{ij\alpha}\varepsilon_{ij} + n_\alpha\theta + p_{\alpha\beta}\omega_\beta + r_{i\alpha}\theta_{,i}, \tag{2.18}$$

$$q_i = f_{ijk}\varepsilon_{jk} + g_i\theta + h_{i\alpha}\omega_\alpha + k_{ij}\theta_{,j}. \tag{2.19}$$

In the relations (2.17)–(2.19) the coefficients $C_{ijrs}, D_{\alpha\beta}, E_{ij}, a, F_{ij\alpha}, G_\alpha, m_{ij\alpha}, n_\alpha, p_{\alpha\beta}, r_{i\alpha}, f_{ijk}, g_i, h_{i\alpha}$ and k_{ij} are functions of x_s, which characterize the thermo-viscoelastic properties of the material with internal state variables. For a homogeneous material these quantities are constants. They satisfy the symmetry relations

$$\begin{gathered} C_{ijrs} = C_{rsij} = C_{jirs}, \quad D_{\alpha\beta} = D_{\beta\alpha}, \quad E_{ij} = E_{ji}, \\ F_{ij\alpha} = F_{ji\alpha}, \quad m_{ij\alpha} = m_{ji\alpha}, \quad f_{ijk} = f_{ikj}. \end{gathered} \tag{2.20}$$

In view of the relation (2.17), from Eqs. (2.8) and (2.10) we deduce

$$t_{ij} = C_{ijrs}\varepsilon_{rs} - E_{ij}\theta + F_{ij\alpha}\omega_\alpha, \quad \eta = E_{ij}\varepsilon_{ij} + a\theta - G_\alpha\omega_\alpha, \tag{2.21}$$

and

$$\sigma_\alpha = F_{ij\alpha}\varepsilon_{ij} + G_\alpha\theta + D_{\alpha\beta}\omega_\beta. \tag{2.22}$$

According to the linear approximation, Eq. (2.11) becomes

$$T_0\dot{\eta} = \varrho r + q_{i,i}. \tag{2.23}$$

If we substitute the relations (2.3) and (2.21) into the relations (2.1), $(2.7)_5$ and (2.23), we get

$$(C_{ijrs}u_{r,s})_{,j} - (E_{ji}\theta)_{,j} + (F_{ji\alpha}\omega_\alpha)_{,j} + \varrho F_i = \varrho\ddot{u}_i, \tag{2.24}$$

$$T_0(E_{ij}\dot{u}_{i,j} + a\dot{\theta} - G_\alpha\dot{\omega}_\alpha) = \varrho r + (f_{ijk}u_{j,k})_{,i} + (g_i\theta)_{,i} + (h_{i\alpha}\omega_\alpha)_{,i} + (k_{ij}\theta_{,j})_{,i}, \tag{2.25}$$

$$\dot{\omega}_\alpha = m_{ij\alpha}u_{i,j} + n_\alpha\theta + p_{\alpha\beta}\omega_\beta + r_{i\alpha}\theta_{,i}. \tag{2.26}$$

To these equations we adjoin the initial conditions and the boundary conditions. In our hypotheses we assume the following initial conditions:

$$u_i(x_s, 0) = 0, \quad \dot{u}_i(x_s, 0) = 0, \quad \theta(x_s, 0) = 0, \quad \omega_\alpha(x_s, 0) = 0, \quad \text{on } \bar{V}. \tag{2.27}$$

We supplement the above equations with the prescribed boundary conditions

$$u_i = \bar{u}_i \quad \text{on} \quad \partial V_1 \times [0, t_0], \quad t_i = t_{ji}\nu_j = \bar{t}_i \quad \text{on} \quad \partial V_2 \times [0, t_0],$$
$$\theta = \bar{\theta} \quad \text{on} \quad \partial V_3 \times [0, t_0], \quad q_i\nu_i = \bar{q} \quad \text{on} \quad \partial V_4 \times [0, t_0], \tag{2.28}$$

where $\bar{u}_i$, $\bar{t}_i$, $\bar{\theta}$ and $\bar{q}$ are prescribed functions of x_s and t, and ∂V_1, ∂V_2 and ∂V_3, ∂V_4 denote subsets of ∂V such that $\partial V_1 \cup \partial V_2 = \partial V_3 \cup \partial V_4 = \partial V$ and $\partial V_1 \cap \partial V_2 = \partial V_3 \cap \partial V_4 = \phi$; and ν_i are the components of the unit outward normal to ∂V.

By a solution of the considered initial-boundary value problems, we mean the state of deformation $(u_i, \theta, \omega_\alpha)(x_s, t)$ satisfying Eqs. (2.24)–(2.26), the designated initial conditions (2.27) and the boundary conditions (2.28).

3. A uniqueness theorem

In this section we establish the uniqueness of solution to the initial-boundary value problems defined by Eqs. (2.24)–(2.26), the initial conditions (2.27) and the boundary conditions (2.28).

In order to prove this we shall need the following assumptions:

(a) the mass density $\varrho(x_s)$ is strictly positive, i.e.

$$\varrho(x_s) \geqslant \varrho_0 > 0, \quad \text{on } \bar{V}; \tag{3.1}$$

(b) the specific heat $a(x_s)$ is strictly positive, i.e.

$$a(x_s) \geqslant a_0 > 0, \quad \text{on } \bar{V}; \tag{3.2}$$

(c) $C_{ijkl}(x_s)$ is positive definite in the sense that there exists a positive constant λ such that

$$\int_V C_{ijkl}\xi_{ij}\xi_{kl}dV \geqslant \lambda \int_V \xi_{ij}\xi_{ij}dV, \tag{3.3}$$

for all second-order symmetric tensors ζ_{ij};

(d) the symmetric part $\tilde{k}_{ij}$ of the thermal conductivity tensor k_{ij}, is positive definite in the sense that there exists a positive constant μ such that

$$\int_V \frac{1}{T_0}\tilde{k}_{ij}\zeta_i\zeta_j dV \geqslant \mu \int_V \zeta_i\zeta_i dV, \tag{3.4}$$

for all vectors ζ_i.

The above restrictions are currently used in the classical theory of thermoelasticity in order to establish the uniqueness and thermoelastic stability (see e.g. [11], [12]).

Because of the linearity of the problems, it suffices to prove that the considered initial-boundary value problems in which $F_i = r = 0$ and $\bar{u}_i = \bar{t}_i = \bar{\theta} = \bar{q} = 0$ imply that $u_i = \theta = \omega_\alpha = 0$ in $\bar{V}\times[0, t_0]$, provided that the hypotheses (3.1)–(3.4) hold. Therefore we consider the problem P_0 defined by the following equations:

$$t_{ji,j} = \varrho\ddot{u}_i, \tag{3.5}$$

$$T_0\dot{\eta} = q_{i,i}, \tag{3.6}$$

$$\dot{\omega}_\alpha = f_\alpha, \tag{3.7}$$

$$\begin{aligned} t_{ij} &= C_{ijrs}\varepsilon_{rs} - E_{ij}\theta + F_{ij\alpha}\omega_\alpha, \\ \eta &= E_{ij}\varepsilon_{ij} + a\theta - G_\alpha\omega_\alpha, \\ q_i &= f_{ijk}\varepsilon_{jk} + g_i\theta + h_{i\alpha}\omega_\alpha + k_{ij}\theta_{,j}, \\ f_\alpha &= m_{ij\alpha}\varepsilon_{ij} + n_\alpha\theta + p_{\alpha\beta}\omega_\beta + r_{i\alpha}\theta_{,i}, \end{aligned} \tag{3.8}$$

with the initial conditions

$$u_i(x_s, 0) = 0, \quad \dot{u}_i(x_s, 0) = 0, \quad \theta(x_s, 0) = 0, \quad \omega_\alpha(x_s, 0) = 0, \quad \text{on} \quad \bar{V}, \tag{3.9}$$

and the boundary conditions

$$\begin{aligned} u_i &= 0 \quad \text{on} \quad \partial V_1\times[0, t_0], \quad t_i = t_{ji}\nu_j = 0 \quad \text{on} \quad \partial V_2\times[0, t_0], \\ \theta &= 0 \quad \text{on} \quad \partial V_3\times[0, t_0], \quad q_i\nu_i = 0 \quad \text{on} \quad \partial V_4\times[0, t_0]. \end{aligned} \tag{3.10}$$

In order to prove the uniqueness of solution of the problem P_0, it suffices to show that the function $y(t)$ defined by

$$y(t) = \int_V (\dot{u}_i\dot{u}_i + \varepsilon_{ij}\varepsilon_{ij} + \theta^2 + \omega_\alpha\omega_\alpha)dV \tag{3.11}$$

vanishes on $[0, t_0]$. Assume to the contrary that $y(t) \not\equiv 0$ on $[0, t_0]$. Then we have the following:

LEMMA 1. If $(u_i, \theta, \omega_\alpha)(x_s, t)$ is a solution of the problem P_0 then

$$(3.12)\quad \int_V \left(\frac{1}{2}\varrho \dot{u}_i \dot{u}_i + \frac{1}{2} C_{ijrs}\varepsilon_{ij}\varepsilon_{rs} + \frac{1}{2} a\theta^2 + F_{ij\alpha}\varepsilon_{ij}\omega_\alpha\right) dV = \int_0^t \int_V \left[(G_\alpha\theta + F_{ij\alpha}\varepsilon_{ij})\dot{\omega}_\alpha - \frac{1}{T_0} q_i\theta_{,i}\right] dV d\tau, \qquad t \in [0, t_0].$$

Proof. By using Eqs. (3.5)–(3.8), the boundary conditions (3.10), the geometric relations (2.3) and the symmetry relations (2.20), we get

$$(3.13)\quad \frac{d}{dt}\int_V \left(\frac{1}{2}\varrho \dot{u}_i \dot{u}_i + \frac{1}{2} C_{ijrs}\varepsilon_{ij}\varepsilon_{rs} + \frac{1}{2} a\theta^2 + F_{ij\alpha}\varepsilon_{ij}\omega_\alpha\right) dV$$
$$= \int_V (\varrho\dot{u}_i\ddot{u}_i + C_{ijrs}\varepsilon_{rs}\dot{\varepsilon}_{ij} + a\theta\dot{\theta} + F_{ij\alpha}\dot{\varepsilon}_{ij}\omega_\alpha + F_{ij\alpha}\varepsilon_{ij}\dot{\omega}_\alpha) dV$$
$$= \int_V [\dot{u}_i t_{ji,j} + (C_{ijrs}\varepsilon_{rs} + F_{ij\alpha}\omega_\alpha)\dot{u}_{i,j} + \theta(\dot{\eta} - E_{ij}\dot{\varepsilon}_{ij} + G_\alpha\dot{\omega}_\alpha)$$
$$+ F_{ij\alpha}\varepsilon_{ij}\dot{\omega}_\alpha] dV = \int_V \left[(G_\alpha\theta + F_{ij\alpha}\varepsilon_{ij})\dot{\omega}_\alpha - \frac{1}{T_0} q_i\theta_{,i}\right] dV.$$

We now integrate on $[0, t]$, $t \in [0, t_0]$ and we use the initial conditions (3.9) so that from Eq. (3.13) the identity (3.12) follows. This completes the proof.

LEMMA 2. Let $(u_i, \theta, \omega_\alpha)(x_s, t)$ be a solution of the problem P_0. We assume the hypothesis (d) to be satisfied. Then there exist positive constants m_1 and m_2 so that

$$(3.14)\quad \int_V \left[(G_\alpha\theta + F_{ij\alpha}\varepsilon_{ij})\dot{\omega}_\alpha - \frac{1}{T_0} q_i\theta_{,i}\right] dV \leqslant -m_1 \int_V \theta_{,i}\theta_{,i} dV + m_2 \int_V (\varepsilon_{ij}\varepsilon_{ij} + \theta^2 + \omega_\alpha\omega_\alpha) dV, \qquad t \in [0, t_0].$$

Proof. By using the relations (3.7) and $(3.8)_{3,4}$, we can write

$$(3.15)\quad \int_V \left[(G_\alpha\theta + F_{ij\alpha}\varepsilon_{ij})\dot{\omega}_\alpha - \frac{1}{T_0} q_i\theta_{,i}\right] dV = -\int_V \frac{1}{T_0}\tilde{k}_{ij}\theta_{,i}\theta_{,j} dV$$
$$+ \int_V (H_{ijrs}\varepsilon_{ij}\varepsilon_{rs} + I\theta^2 + J_{ij}\varepsilon_{ij}\theta + K_{ij\alpha}\varepsilon_{ij}\omega_\alpha + L_\alpha\omega_\alpha\theta$$
$$+ M_{ijk}\varepsilon_{ij}\theta_{,k} + N_i\theta_{,i}\theta + P_{i\alpha}\theta_{,i}\omega_\alpha) dV,$$

where we have used the notations

$$H_{ijrs} = \frac{1}{2}(F_{rs\alpha}m_{ij\alpha} + F_{ij\alpha}m_{rs\alpha}), \qquad I = G_\alpha n_\alpha,$$
$$(3.16)\quad J_{ij} = m_{ij\alpha}G_\alpha + F_{ij\alpha}n_\alpha, \qquad K_{ij\alpha} = F_{ij\beta}p_{\beta\alpha}, \qquad L_\alpha = G_\beta p_{\beta\alpha},$$
$$M_{ijk} = F_{ij\alpha}r_{k\alpha} - \frac{1}{T_0} f_{kij}, \qquad N_i = G_\alpha r_{i\alpha} - \frac{1}{T_0} g_i, \qquad P_{i\alpha} = -\frac{1}{T_0} h_{i\alpha}.$$

We now make use of the hypothesis (d). An application of the Schwarz inequality and the arithmetic-geometric mean inequality

$$ab \leqslant \frac{1}{2}\left(\frac{a^2}{\pi^2}+b^2\pi^2\right), \tag{3.17}$$

to the last terms in Eq. (3.15) gives, for arbitrary positive constants π_1, π_2 and π_3,

$$2\int_V \left[(G_\alpha\theta+F_{ij\alpha}\varepsilon_{ij})\dot{\omega}_\alpha-\frac{1}{T_0}q_i\theta_{,i}\right]dV \leqslant (-2\mu+\pi_1^2+\pi_2^2+\pi_3^2) \times \int_V \theta_{,i}\theta_{,i}dV+\left(\frac{M_1^2}{\pi_1^2}+M_4^2+M_5^2+M_6^2\right)\int_V \varepsilon_{ij}\varepsilon_{ij}dV + \left(\frac{M_2^2}{\pi_2^2}+M_7^2+2\right)\int_V \theta^2 dV+\left(\frac{M_3^2}{\pi_3^2}+M_8^2+1\right)\int_V \omega_\alpha\omega_\alpha dV. \tag{3.18}$$

In the above inequality we have used the notations

$$\begin{aligned} &M_1^2 = \max(M_{ijk}M_{ijk})(x_s), && M_2^2 = \max(N_iN_i)(x_s),\\ &M_3^2 = \max(P_{i\alpha}P_{i\alpha})(x_s), && M_4^2 = 2\max[(H_{ijmn}H_{ijmn})(x_s)],^{1/2}\\ &M_5^2 = \max(J_{ij}J_{ij})(x_s), && M_6^2 = \max(K_{ij\alpha}K_{ij\alpha})(x_s),\\ &M_7^2 = 2\max|I(x_s)|, && M_8^2 = \max(L_\alpha L_\alpha)(x_s), \quad \text{on } \bar{V}. \end{aligned} \tag{3.19}$$

We choose the arbitrary constants π_1, π_2 and π_3 so that the quantity m_1 defined by

$$m_1 = \mu-\frac{1}{2}(\pi_1^2+\pi_2^2+\pi_3^2) \tag{3.20}$$

is strictly positive. Thus, from Eq. (3.18) we deduce the inequality (3.14), provided we choose

$$m_2 = \frac{1}{2}\max\left(\frac{M_1^2}{\pi_1^2}+M_4^2+M_5^2+M_6^2, \quad \frac{M_2^2}{\pi_2^2}+M_7^2+2, \quad \frac{M_3^2}{\pi_3^2}+M_8^2+1\right). \tag{3.21}$$

The proof of the lemma is complete.

LEMMA 3. Let $(u_i, \theta, \omega_\alpha)(x_s, t)$ be a solution of the problem P_0. We assume the hypotheses (a)–(d) to be satisfied. Then there is a positive constant m_3 so that

$$\int_V (\dot{u}_i\dot{u}_i+\varepsilon_{ij}\varepsilon_{ij}+\theta^2+\omega_\alpha\omega_\alpha)dV \leqslant m_3\int_0^t\int_V (\dot{u}_i\dot{u}_i+\varepsilon_{ij}\varepsilon_{ij} +\theta^2+\omega_\alpha\omega_\alpha)dVd\tau, \quad t\in[0, t_0]. \tag{3.22}$$

Proof. In view of the hypotheses (a)–(c), we note that

$$m_0\int_V (\dot{u}_i\dot{u}_i+\varepsilon_{ij}\varepsilon_{ij}+\theta^2)dV \leqslant \int_V (\varrho\dot{u}_i\dot{u}_i+C_{ijrs}\varepsilon_{ij}\varepsilon_{rs}+a\theta^2)dV, \tag{3.23}$$

where

$$m_0 = \min(\varrho_0, \lambda, a_0). \tag{3.24}$$

Further, we use the Schwarz inequality and the arithmetic-geometric mean inequality (3.17) so that

$$(3.25)\qquad 2\left|\int_V F_{ij\alpha}\varepsilon_{ij}\omega_\alpha dV\right| \leqslant \pi_4^2\int_V \varepsilon_{ij}\varepsilon_{ij}dV+\frac{M_9^2}{\pi_4^2}\int_V \omega_\alpha\omega_\alpha dV,$$

for an arbitraty constant π_4, where

$$(3.26)\qquad M_9^2 = \max(F_{ij\alpha}F_{ij\alpha})(x_s) \quad \text{on } \overline{V}.$$

If we now take into account the relations (3.14), (3.23) and (3.25), from the identity (3.12) we get

$$(3.27)\qquad m_0\int_V (\dot{u}_i\dot{u}_i+\varepsilon_{ij}\varepsilon_{ij}+\theta^2)dV \leqslant \pi_4^2\int_V \varepsilon_{ij}\varepsilon_{ij}dV+\frac{M_9^2}{\pi_4^2}\int_V \omega_\alpha\omega_\alpha dV$$
$$-m_1\int_0^t\int_V \theta_{,i}\theta_{,i}dVd\tau+m_2\int_0^t\int_V(\varepsilon_{ij}\varepsilon_{ij}+\theta^2+\omega_\alpha\omega_\alpha)dVd\tau, \qquad t\in[0,t_0].$$

On the other hand, by using the initial conditions (3.9) and the relations (3.7) and $(3.8)_4$, we obtain

$$(3.28)\qquad \int_V \omega_\alpha\omega_\alpha dV = \int_0^t \frac{d}{d\tau}\left(\int_V \omega_\alpha\omega_\alpha dV\right)d\tau = 2\int_0^t\int_V \omega_\alpha\dot{\omega}_\alpha dVd\tau$$
$$= 2\int_0^t\int_V (m_{ij\alpha}\varepsilon_{ij}\omega_\alpha+n_\alpha\omega_\alpha\theta+p_{\alpha\beta}\omega_\alpha\omega_\beta+r_{i\alpha}\omega_\alpha\theta_{,i})dVd\tau.$$

An application of the Schwarz inequality and the arithmetic-geometric mean inequality to the left side of the relation (3.28) gives

$$(3.29)\qquad 2\int_V (m_{ij\alpha}\varepsilon_{ij}\omega_\alpha+n_\alpha\omega_\alpha\theta+p_{\alpha\beta}\omega_\alpha\omega_\beta+r_{i\alpha}\omega_\alpha\theta_{,i})dV$$
$$\leqslant m_4\int_V(\varepsilon_{ij}\varepsilon_{ij}+\theta^2+\omega_\alpha\omega_\alpha)dV+\pi_5^2\int_V \theta_{,i}\theta_{,i}dV,$$

for an arbitrary constant π_5 and

$$m_4 = \max\left(M_{13}^2,\ 1,\ \frac{M_{10}^2}{\pi_5^2}+M_{11}^2+M_{12}^2+1\right),$$
$$(3.30)\qquad M_{10}^2 = \max(r_{i\alpha}r_{i\alpha})(x_s), \qquad M_{11}^2 = 2\max[(p_{\alpha\beta}p_{\alpha\beta})(x_s)]^{1/2},$$
$$M_{12}^2 = \max(n_\alpha n_\alpha)(x_s), \qquad M_{13}^2 = \max(m_{ij\alpha}m_{ij\alpha})(x_s) \quad \text{on } \overline{V}.$$

From the relations (3.28) and (3.29) we obtain, for an arbitrary constant π_6,

$$(3.31)\qquad \pi_6^2\int_V \omega_\alpha\omega_\alpha dV \leqslant \pi_6^2 m_4\int_0^t\int_V(\varepsilon_{ij}\varepsilon_{ij}+\theta^2+\omega_\alpha\omega_\alpha)dVd\tau+\pi_6^2\pi_5^2\int_0^t\int_V \theta_{,i}\theta_{,i}dVd\tau.$$

Now, from the inequalities (3.27) and (3.31) we deduce

$$(3.32)\qquad m_0\int\limits_V(\dot{u}_i\dot{u}_i+\theta^2)dV+(m_0-\pi_4^2)\int\limits_V\varepsilon_{ij}\varepsilon_{ij}dV+\left(\pi_6^2-\frac{M_9^2}{\pi_4^2}\right)\int\limits_V\omega_\alpha\omega_\alpha dV$$
$$\leqslant-(m_1-\pi_6^2\pi_5^2)\int\limits_0^t\int\limits_V\theta_{,i}\theta_{,i}dVd\tau+(m_2+\pi_6^2m_4)\int\limits_0^t\int\limits_V(\varepsilon_{ij}\varepsilon_{ij}+\theta^2+\omega_\alpha\omega_\alpha)dVd\tau,\qquad t\in[0,t_0].$$

Then, we choose the arbitrary constants π_4, π_5 and π_6 so that

$$(3.33)\qquad m_5\equiv m_0-\pi_4^2>0,\quad m_6\equiv\pi_6^2-\frac{M_9^2}{\pi_4^2}>0,\quad m_7\equiv m_1-\pi_6^2\pi_5^2>0.$$

With these in mind, the inequality (3.22) follows from the inequality (3.32), provided

$$(3.34)\qquad m_3\equiv(m_2+\pi_6^2m_4)/\min(m_0,m_5,m_6).$$

The proof of the lemma is now complete.

Obviously the inequality (3.22) and Gronwall's lemma [13] imply

$$y(t)=\int\limits_V(\dot{u}_i\dot{u}_i+\varepsilon_{ij}\varepsilon_{ij}+\theta^2+\omega_\alpha\omega_\alpha)dV=0\quad\text{on }[0,t_0],$$

which contradicts our initial assumption concerning the uniqueness.

Thus we have

THEOREM 1. *Under the hypotheses* (a)–(d) *there is at the most one solution of the initial--boundary value problems defined by Eqs.* (2.24)–(2.26), *with the initial conditions* (2.27) *and the boundary conditions* (2.28).

References

1. B. D. COLEMAN and M. E. GURTIN, *Thermodynamics with internal state variables*, J. Chem. Phys., **47**, 597–613, 1967.
2. R. M. BOWEN, *On the thermochemistry of reacting materials*, J. Chem. Phys., **49**, 1625–1637, 1968.
3. K. C. VALANIS, *Unified theory of thermomechanical behaviour of viscoelastic materials*, Symp. Behav. Mater. Dyn. Loads, 1967, Springer-Verlag 1968, 343–364.
4. M. MIHĂILESCU and I. SULICIU, *Finite and symmetric thermomechanical waves in materials with internal state variables*, Int. J. Solids Structures, **12**, 559–575, 1976.
5. I. SULICIU and M. MIHĂILESCU–SULICIU, *On the semilinear theory for materials with internal state variables*, An. St. Univ. Iași, Matematică, **24**, 403–416, 1978.
6. R. R. NACHLINGER and J. W. NUNZIATO, *Wave propagation and uniqueness theorems for elastic materials with internal state variables*, Int. J. Engng Sci., **14**, 31–38, 1976.
7. W. KOSIŃSKI, *A uniqueness theorem for the dynamic initial-displacement boundary-value problem in the theory with internal state variables*, Quart. Appl. Math., **38**, 129–134, 1980.
8. W. KOSIŃSKI, *On weak solutions, stability and uniqueness in dynamics of dissipative bodies*, Arch. Mech., **33**, 2, 317–321, 1981.
9. G. FICHERA, *Existence theorems in elasticity*, in: Handbuch der Physik VIa/2, Springer, Berlin, Heidelberg, New York 1972.
10. N. CRISTESCU and I. SULICIU, *Viscoplasticity* [in Romanian], Ed. tehnică, Bucharest 1976.

11. R. J. KNOPS and E. W. WILKES, *Theory of elastic stability*, in: Handbuch der Physik VIa/3, Springer, Berlin, Heidelberg, New York 1973.
12. C. M. DAFERMOS, *On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity*, Arch. Rat. Mech. Anal., **29**, 241–271, 1968.
13. J. DIEUDONNÉ, *Foundations of modern analysis*, Academic Press, New York 1969.

MATHEMATICAL SEMINARIUM
UNIVERSITY OF IASI, ROMANIA.

Received September 15, 1981.