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Noether's theorem for a nonholonomic system

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THE PURPOSE of this paper is to extend Noether's theorem to a conservative linear nonholonomic system whose position is specified by generalised coordinates. This extension is based upon the concept of admissible infinitesimal symmetry transformation under which the Lagrangian is divergence invariant. The existence of first integrals depends on the existence of solutions of a system of partial differential equations so-called Killing equations. All known results regarding the first integrals are deduced as special cases. Finally an example is given for the illustration of the theory.

Zadaniem pracy jest rozszerzenie twierdzenia Noethera na układ liniowy, zachowawczy i nieholonomiczny, którego położenie określone jest we współrzędnych uogólnionych. Rozszerzenie to oparte jest na koncepcji dopuszczalnej, infinitezymalnej transformacji symetrii, przy której lagrangian jest niezmienniczy ze względu na dywergencję. Istnienie pierwszych całek zależy od istnienia rozwiązań układu równań różniczkowych cząstkowych zwanych równaniami Killinga. Wszystkie znane wyniki dotyczące pierwszych całek wyprowadzono jako przypadki szczególne teorii. Podano na zakończenie przykład ilustrujący przedstawioną teorię.

Задачей работы является расширение теоремы Нётер на линейную, консервативную и неголономическую систему, положение которой определено в обобщенных координатах. Это расширение опирается на концепцию допустимогой, инфинитезимального преобразования симметрии, при котором лагранжиан инвариантный по отношению к дивергенции. Существование первых интегралов зависит от существования решений системы дифференциальных уравнений в частных производных, называемой уравнениями Киллинга. Все известные результаты, касающиеся первых интегралов, выведены как частные случаи теории. В заключение приведен пример иллюстрирующий представленную теорию.

1. Introduction

NOETHER's theorem [9] and its applications to finding first integrals play an important role in mathematical physics as it is clear from the recent publications [2, 3, 4, 5, 6, 7] in which Noether's theorem has been given for different types of dynamical systems. In all the said works no worthwhile attempt was made to extend Noether's theorem to a nonholonomic system. In [3] only, this theorem was given for a nonholonomic system whose position is specified by quasi-coordinates but in doing this, nothing was said about the symmetry transformation whether the changes in coordinates and time due to its application satisfy the equations of constraint or not. As a matter of principle, the infinitesimal changes of coordinates and time, i.e. δq and δt, due to the symmetry transformation must satisfy the constraint relations for a nonholonomic system. In view of the above, the assumption $\psi_j = 0$ made in [3, 8, 6] is unnecessary since it follows automatically from the above said property of the symmetry transformation. Hence Noether's theorem as given in [3] for a nonholonomic system is incorrect.

A transformation which is consistent with the constraints is called an admissible transformation. The object of this paper is to consider an admissible transformation for the correct formulation of Noether's theorem for a nonholonomic system.

In what follows we shall employ Einstein's summation convention. Time derivatives of a quantity are denoted by putting dots over it. Different indices and their ranges of values used in the sequel are as follows:

$$i, j, k = 1, 2, \dots, l, \quad \alpha, \beta = l+1, \dots, n, \quad p, r = 1, 2, \dots, n.$$

Let us consider a conservative nonholonomic system whose position is defined by n generalised coordinates $q_1, \dots, q_n$ and which is subject to $(n-l)$ linear constraints of the form

$$\dot{q}_\alpha = c_{\alpha i}\dot{q}_i, \tag{1.1}$$

where $c_{\alpha i}$ are functions of q only. If L is the Lagrangian of the system, the equations of motion obtained after embedding the constraints as in [10], are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + c_{\alpha i}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha}\right] = 0 \tag{1.2}$$

which, together with the relations (1.1), form a system of n equations for determining the quantities $q_1, q_2, \dots, q_n$ as functions of the time t.

2. Admissible symmetry transformation

Consider a continuous one-parameter transformation of generalised coordinates and time of the form

$$\bar{t} = t + \varepsilon\phi(t, q, \dot{q}), \tag{2.1}$$

$$\bar{q}_p = q_p + \varepsilon\psi_p(t, q, q), \tag{2.2}$$

where $q = (q_1, \dots, q_n)$, $\dot{q} = (\dot{q}_1, \dots, \dot{q}_n)$, ε is a small parameter whose squares and higher powers may be neglected and ϕ, ψ_p are functions of at least class C^2. A transformation is called a symmetry transformation if the transformed Lagrangian of the system $L(\bar{t}, \bar{q}, \dot{\bar{q}})$ yields equations of the same form as the original Lagrangian function $L(t, q, \dot{q})$. We assume that the transformation defined by Eqs. (2.1) and (2.2) is a symmetry transformation.

Now, corresponding to Eqs. (2.1) and (2.2), we have the infinitesimal transformation of the form

$$\delta t = \varepsilon\phi, \quad \delta q_p = \varepsilon\psi_p. \tag{2.3}$$

The infinitesimal transformation defined by Eq. (2.3) is called admissible if it satisfies the equations of constraint

$$\delta q_\alpha = c_{\alpha i}\delta q_i,$$

where δq are arbitrary variations of the generalised coordinates. Hence the necessary and sufficient conditions for the admissibility are

$$\psi_\alpha = c_{\alpha i}\psi_i. \tag{2.4}$$

Since Eq. (2.3) is a symmetry transformation by our assumption therefore, proceeding as in [3], the equations of constraint in the new coordinates and velocities assume the form

$$\dot{\bar{q}}_\alpha = c_{\alpha i}(\bar{q})\dot{\bar{q}}_i \tag{2.5}$$

where

$$\dot{\bar{q}}_\alpha = \frac{d\bar{q}_\alpha}{d\bar{t}}, \quad \dot{\bar{q}}_i = \frac{d\bar{q}_i}{d\bar{t}}.$$

We now express Eqs. (2.5) in terms of q, ϕ, ψ_p and their time derivatives. For doing this, we neglect squares and higher powers of ε. Using Eqs. (2.1) and (2.2) we get

$$\dot{\bar{q}}_p = \frac{d\bar{q}_p}{d\bar{t}} = \frac{d(q_p+\varepsilon\psi_p)}{d(t+\varepsilon\phi)} = \frac{\dot{q}_p + \varepsilon\dfrac{d\psi_p}{dt}}{1+\varepsilon\dfrac{d\phi}{dt}} = \dot{q}_p - \varepsilon\dot{q}_p\dot{\phi} + \varepsilon\frac{d\psi_p}{dt}$$

or

$$\dot{\bar{q}}_p = \dot{q}_p + \varepsilon\left(\frac{d\psi_p}{dt} - \dot{q}_p\dot{\phi}\right). \tag{2.6}$$

Expanding $c_{\alpha i}(\bar{q})$ in powers of ε, we obtain

$$c_{\alpha i}(\bar{q}) = c_{\alpha i}(q) + \varepsilon\frac{\partial c_{\alpha i}}{\partial q_p}\psi_p. \tag{2.7}$$

In view of the relations (2.6) and (2.7), the relation (2.5) yields

$$\dot{q}_\beta - \varepsilon\dot{q}_\beta\dot{\phi} + \varepsilon\frac{d\psi_\beta}{dt} = c_{\beta i}\dot{q}_i + \frac{\partial c_{\beta i}}{\partial q_p}\varepsilon\psi_p\dot{q}_i - \varepsilon c_{\beta i}\dot{q}_i\dot{\phi} + \varepsilon\frac{d\psi_1}{dt}c_{\beta i},$$

or using the constraint relations (1.1), we get

$$\frac{d\psi_\beta}{dt} = c_{\beta i}\frac{d\psi_i}{dt} + \frac{\partial c_{\beta i}}{\partial q_p}\psi_p\dot{q}_i. \tag{2.8}$$

Hence ψ_i should satisfy the relations (2.4) and (2.8). Now Eq. (2.8) in conjunction with Eqs. (1.1) and (2.4) yields the relations

$$\left(\frac{\partial c_{\beta i}}{\partial q_j} - \frac{\partial c_{\beta j}}{\partial q_i}\right)\dot{q}_i\psi_j = \left(c_{\alpha j}\frac{\partial c_{\beta i}}{\partial q_\alpha} - \frac{\partial c_{\beta j}}{\partial q_\alpha}c_{\alpha i}\right)\dot{q}_j\psi_i. \tag{2.9}$$

3. Fundamental invariance identity

Let us form the integral

$$J = \int_{t_1}^{t_2} L(t, q, \dot{q})dt \tag{3.1}$$

then, according to [7], J is said to be divergence invariant under the transformation (2.3) if there exists a function $F(t, q, \dot{q})$ such that

$$L(\bar{t}, \bar{q}, \dot{\bar{q}})\frac{d\bar{t}}{dt} - L(t, q, \dot{q}) = \varepsilon\frac{dF}{dt}. \tag{3.2}$$

Let us now assume that J is divergence invariant, then differentiating Eq. (3.2) with respect to ε and afterwards putting $\varepsilon = 0$, we get

$$L\left[\frac{\partial}{\partial\varepsilon}\left(\frac{d\bar{t}}{dt}\right)\right]_0 + \frac{\partial L}{\partial t}\left[\frac{\partial\bar{t}}{\partial\varepsilon}\right]_0 + \frac{\partial L}{\partial q_p}\left[\frac{\partial\bar{q}_p}{\partial\varepsilon}\right]_0 + \frac{\partial L}{\partial\dot{q}_p}\left[\frac{\partial\dot{\bar{q}}_p}{\partial\varepsilon}\right]_0 = \frac{dF}{dt}, \tag{3.3}$$

where $[\ \ ]_0$ denotes the operation of putting $\varepsilon = 0$ after the operation of differentiation is performed. Now the relations (2.1), (2.2) and (2.6) yield

$$\left[\frac{\partial}{\partial\varepsilon}\left(\frac{\partial\bar{t}}{\partial t}\right)\right]_0 = \frac{d\phi}{dt}, \quad \left[\frac{\partial\bar{t}}{\partial\varepsilon}\right]_0 = \phi, \quad \left[\frac{\partial\bar{q}_p}{\partial\varepsilon}\right]_0 = \psi_p,$$

$$\left[\frac{\partial\dot{\bar{q}}_p}{\partial\varepsilon}\right]_0 = \frac{d\psi_p}{dt} - \dot{q}_p\frac{d\phi}{dt},$$

and consequently the relation (3.3) becomes

$$L\frac{d\phi}{dt} + \frac{\partial L}{\partial t}\phi + \frac{\partial L}{\partial q_p}\psi_p + \frac{\partial L}{\partial\dot{q}_p}\left(\frac{d\psi_p}{dt} - \dot{q}_p\frac{d\phi}{dt}\right) = \frac{dF}{dt}, \tag{3.4}$$

which is the required fundamental invariance identity.

4. Noether's theorem and conservation laws

For the deduction of conservation laws we first prove the theorem

THEOREM 1. *If the fundamental integral J is divergence invariant under the transformation* (2.3), *then the following identity holds*:

$$E_p(L)(\psi_p - \phi\dot{q}_p) = \frac{d}{dt}\left[L\phi + \psi_p\frac{\partial L}{\partial\dot{q}_p} - \phi\dot{q}_p\frac{\partial L}{\partial\dot{q}_p} - F\right], \tag{4.1}$$

where

$$E_p(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial q_p}\right) - \frac{\partial L}{\partial q_p}.$$

Proof. To establish Eq. (4.1) we substitute from the following relations in Eq. (3.4)

$$\frac{\partial L}{\partial t}\phi = \frac{d}{dt}(L\phi) - \frac{\partial L}{\partial q_p}\dot{q}_p\phi - \frac{\partial L}{\partial\dot{q}_p}\ddot{q}_p\phi - L\frac{d\phi}{dt},$$

$$\frac{\partial L}{\partial\dot{q}_p}\frac{d\psi_p}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_p}\psi_p\right) - \psi_p\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_p}\right),$$

$$\frac{\partial L}{\partial\dot{q}_p}\dot{q}_p\dot{\phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_p}\dot{q}_p\phi\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_p}\right)\dot{q}_p\phi - \frac{\partial L}{\partial\dot{q}_p}\ddot{q}_p\phi$$

and consequently we obtain the identity (4.1).

COROLLARY 1. If we use the relations (1.1) and (2.4) in Eq. (4.1), we get

$$(4.2)\quad [\psi_i-\phi\dot{q}_i].[E_i(L)+c_{\alpha i}\cdot E_\alpha(L)] = \frac{d}{dt}\left[L\phi+(\psi_i-\phi\dot{q}_i)\left(\frac{\partial L}{\partial\dot{q}_i}+c_{\alpha i}\frac{\partial L}{\partial\dot{q}_\alpha}\right)-F\right].$$

Let L be the Lagrangian which is obtained from L after eliminating the dependent velocities $\dot{q}_\alpha$ with the help of Eq. (1.1), then

$$\frac{\partial\bar{L}}{\partial\dot{q}_i} = \frac{\partial L}{\partial\dot{q}_i}+c_{\alpha i}\cdot\frac{\partial L}{\partial\dot{q}_\alpha}$$

and consequently the relation (4.2) becomes

$$(4.3)\quad [\psi_i-\phi\dot{q}_i]\,[E_i(L)+c_{\alpha i}E_\alpha(L)] = \frac{d}{dt}\left[\bar{L}\phi+(\psi_i-\phi\dot{q}_i)\frac{\partial\bar{L}}{\partial\dot{q}_i}-F\right].$$

Now the equations of motion (1.2) can be written in the form

$$E_i(L)+c_{\alpha i}E_\alpha(L) = 0,$$

and consequently Eq. (4.3) qives

$$\frac{d}{dt}\left[\bar{L}\phi+(\psi_i-\phi\dot{q}_i)\frac{\partial\bar{L}}{\partial\dot{q}_i}-F\right] = 0,$$

or

$$(4.4)\quad \bar{L}\phi+(\psi_i-\phi\dot{q}_i)\frac{\partial\bar{L}}{\partial\dot{q}_i}-F = \text{constant}.$$

The relation (4.4) represents a conservation law or first integral of the equations of motion (1.2). Now we can state the theorem:

THEOREM 2. (Noether's theorem). *If the fundamental integral J is divergence invariant under the one-parameter continuous transformation* (2.3) *where ψ satisfy the relations* (2.4) *and* (2.8), *then the relation* (4.4) *gives the first integral of the nonholonomic system.*

COROLLARY 2. Let the integral J be absolutely invariant under the transformation (2.3), then the relation (4.4) yields the first integral of the form

$$(4.5)\quad \bar{L}\phi+(\psi_i-\phi\dot{q}_i)\frac{\partial\bar{L}}{\partial\dot{q}_i} = \text{constant}.$$

5. Generalised Killing equations

If in the fundamental invariance identity (3.4) we put the values of $\frac{d\phi}{dt}$ and $\frac{dF}{dt}$, we get the identity

$$L\frac{\partial\phi}{\partial t}+L\frac{\partial\phi}{\partial q_p}\dot{q}_p+L\frac{\partial\phi}{\partial\dot{q}_p}\ddot{q}_p+\frac{\partial L}{\partial t}\phi+\frac{\partial L}{\partial q_p}\psi_p+\frac{\partial L}{\partial\dot{q}_p}\frac{\partial\psi_p}{\partial t}$$
$$+\frac{\partial L}{\partial\dot{q}_p}\frac{\partial\psi_p}{\partial q_r}\dot{q}_r+\frac{\partial L}{\partial\dot{q}_p}\frac{\partial\psi_p}{\partial\dot{q}_r}\ddot{q}_r-\dot{q}_p\frac{\partial L}{\partial\dot{q}_p}\frac{\partial\phi}{\partial t}-\dot{q}_p\frac{\partial L}{\partial\dot{q}_p}\frac{\partial\phi}{\partial q_r}\dot{q}_r$$
$$-\dot{q}_p\frac{\partial L}{\partial\dot{q}_p}\frac{\partial\phi}{\partial\dot{q}_r}\ddot{q}_r-\frac{\partial F}{\partial t}-\frac{\partial F}{\partial q_p}\dot{q}_p-\frac{\partial F}{\partial\dot{q}_p}\ddot{q}_p = 0.$$

Since this identity is true for arbitrary values of $\ddot{q}_r$, it follows that

$$L\frac{\partial\phi}{\partial t}+L\frac{\partial\phi}{\partial q_p}\dot{q}_p+\frac{\partial L}{\partial t}\phi+\frac{\partial L}{\partial q_p}\psi_p+\frac{\partial L}{\partial \dot{q}_p}\frac{\partial\psi_p}{\partial t}+\frac{\partial L}{\partial \dot{q}_p}\frac{\partial\psi_p}{\partial q_r}\dot{q}_r$$

$$(5.1)\qquad -\dot{q}_p\frac{\partial L}{\partial \dot{q}_p}\frac{\partial\phi}{\partial t}-\dot{q}_p\dot{q}_r\frac{\partial L}{\partial \dot{q}_p}\frac{\partial\phi}{\partial q_r}-\frac{\partial F}{\partial t}-\frac{\partial F}{\partial q_p}\dot{q}_p=0,$$

$$L\frac{\partial\phi}{\partial \dot{q}_p}+\frac{\partial L}{\partial \dot{q}_r}\frac{\partial\psi_r}{\partial \dot{q}_p}-\dot{q}_r\frac{\partial L}{\partial \dot{q}_r}\frac{\partial\phi}{\partial \dot{q}_p}-\frac{\partial F}{\partial \dot{q}_p}=0;$$

$$(5.2)\qquad \psi_\alpha=c_{\alpha l}\psi_l;$$

$$(5.3)\qquad \left(\frac{\partial c_{\beta l}}{\partial q_j}-\frac{\partial c_{\beta j}}{\partial q_l}\right)\dot{q}_l\psi_j=\left(c_{\alpha j}\frac{\partial c_{\beta l}}{\partial q_\alpha}-\frac{\partial c_{\beta j}}{\partial q_\alpha}c_{\alpha l}\right)\dot{q}_j\psi_l.$$

The $(n+1)$ equations (5.1) together with $(2n-2l)$ equations (5.2) and (5.3) form the system of generalised Killing equations for the nonholonomic system. These equations can be used for finding ϕ and $\psi_p$and then using these values to obtain the first integrals by means of Noether's theorem. In this connection we can formulate the theorem:

THEOREM 3. *If the generalised Killing equations* (5.1), (5.2) *and* (5.3) *admit a solution in ϕ and ψ_p, then the equations of motion of the system* (1.2) *have the first integral* (4.5).

6. Applications of Killing equations and Noether's theorem

6.1. The energy integral

Let us assume that the Lagrangian L does not depend explicitly on time, i.e. $\partial L/\partial t = 0$. In this case one of the possible solutions of the Killing equations (5.1) is either $\phi=\phi_0=$ constant, $\psi_p=0$, for $F=0$ or $\phi=0$, $\psi_p=\dot{q}_p$ for $F=L$. The first solution obviously satisfies Eqs. (5.2) and (5.3); the second solution satisfies Eq. (5.3) automatically and it also satisfies Eq. (5.2) in view of the constraints (1.1). Corresponding to these values of ϕ and ψ_p we have the infinitesimal transformations

$$\bar{t}=t+\varepsilon\phi_0,\quad \bar{q}_p=q_p\quad \text{for}\quad F=0,$$
$$\bar{t}=t,\quad \bar{q}_p=q_p+\varepsilon\dot{q}_p\quad \text{for}\quad F=L.$$

Hence the integral J remains invariant under the above two transformations. Consequently Noether's theorem gives the first integral

$$\dot{q}_l\frac{\partial\bar{L}}{\partial q_l}-\bar{L}=\text{const},$$

which is in fact the energy integral of the nonholonomic system.

6.2. The momentum integral

Let us assume that L is independent of a particular coordinate say q_1, so that $\partial L/\partial q_1=0$, then Eqs. (5.1) are satisfied by the values $\phi=0$, $\psi_1=$ constant, $\psi_2=\psi_3=\dots\psi_n=0$, $F=0$. Putting these values in Eq. (5.2), we get $c_{\beta 1}=0$ and similarly Eqs. (5.3) yield

$$\frac{\partial c_{\beta l}}{\partial q_1}\dot{q}_l=0.$$

Taking into account the independence of $\dot{q}_1, \ldots, \dot{q}_l$, the last relations give

$$\frac{\partial c_{\beta l}}{\partial q_l} = 0,$$

i.e. the constraint coefficients $c_{\beta l}$ are independent of the coordinate q_1. Corresponding to the above values of ϕ and ψ, we have the infinitesimal transformation.

$$\bar{t} = t, \quad \bar{q}_1 = q_1 + \varepsilon, \quad \bar{q}_2 = q_2, \ldots, \bar{q}_n = q_n.$$

Hence using Noether's theorem, we get the first integral

$$\frac{\partial \bar{L}}{\partial \dot{q}_1} = \text{constant},$$

which is the usual momentum integral corresponding to the coordinate q_1. These considerations lead to the theorem:

THEOREM 4. *Sufficient conditions for the existence of momentum integral corresponding to a coordinate* q_1 *are* (i) L *is independent of* q_1, (ii) *the constraint equations* (1.1) *do not contain the velocity components* $\dot{q}_1$ *and* (iii) *the constraint coefficients* $c_{\beta i}$ *are independent of* q_1.

This theorem was first proved by C. AGOSTINELLI [4].

6.3. An example of a nonholonomic system

Let us consider the free motion of a rigid body in a horizontal plane in the case when the projection of the centre of mass coincides with the point of contact of a sharp-edged wheel and the plane (a special case of Chaplygin's sleigh).

Let (x, y) be the coordinates of the centre of mass referred to fixed rectangular axes Oxy in the plane, and θ the angle between the plane of the wheel and the axis Ox. Let K be the radius of gyration of the body, supposed to be of unit mass, about an axis passing through the point of contact of the wheel and the plane and perpendicular to the plane, then

$$L = T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + K^2\dot{\theta}^2). \tag{6.1}$$

Since the wheel has a sharp edge, the velocity of the centre of mass is always directed along the plane of the wheel. This qives rise to the nonholonomic constraint

$$\dot{y} = \dot{x}\tan\theta.$$

Taking $q_1 = \theta$, $q_2 = x$, $q_3 = y$, the relations (2.4) and (2.8) for the problem under consideration become respectively

$$\psi_3 = \psi_2 \tan\theta, \tag{6.3}$$

$$\frac{d\psi_3}{dt} = \tan\theta \frac{d\psi_2}{dt} + \psi_1 \dot{x} \sec^2\theta, \tag{6.4}$$

or using Eq. (6.3) in Eq. (6.4), we get

$$\psi_1 \dot{x} = \psi_2 \dot{\theta}. \tag{6.5}$$

Assuming that ϕ and ψ are functions of q only and $F = 0$, then the relations (5.1) yield

(6.6)
$$\frac{\partial L}{\partial \dot{q}_p}\frac{\partial \psi_p}{\partial q_r}\dot{q}_r - L\dot{q}_p\frac{\partial \phi}{\partial q_p} = 0.$$

Using the constraint relations (6.2) after substituting for L from Eq (6.1) in Eq. (6.6) and then equating the coefficients of different powers of $\dot{q}$ equal to zero, we obtain the system of partial differential equations

$$\frac{\partial \phi}{\partial \theta} = 0, \quad \frac{\partial \phi}{\partial x} + \tan\theta\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \psi_1}{\partial \theta} = 0,$$

$$K^2\frac{\partial \psi_1}{\partial x} + K^2\tan\theta\frac{\partial \psi_1}{\partial y} + \frac{\partial \psi}{\partial \theta} + \tan\theta\frac{\partial \psi_3}{\partial y} = 0,$$

$$\frac{\partial \psi_2}{\partial x} + \tan\theta\frac{\partial \psi_2}{\partial y} + \tan\theta\frac{\partial \psi_3}{\partial x} + \tan\theta\frac{\partial \psi_3}{\partial y} = 0.$$

Possible solutions of these equations are: (i) $\phi = \text{constant}$, $\psi_1 = \psi_2 = \psi_3 = 0$, (ii) $\phi = 0$, $\psi_1 = \omega$ (constant), $\psi_2 = A\cos\theta$, $\psi_3 = A\sin\theta$, where A is a constant. Corresponding to the first solution the relation (4.5) gives the energy integral

$$\bar{L} = \text{constant}$$

or

(6.7)
$$K^2\dot{\theta}^2 + \dot{x}^2\sec^2\theta = C_1$$

and the second solution similarly gives the integral

(6.8)
$$\omega K^2\dot{\theta} + A\dot{x}\sec\theta = C_2.$$

Putting the values of ψ_1, ψ_2, ψ_3 from the solution (ii) in Eq. (6.5), we get

(6.9)
$$\dot{x} = A/\omega\cos\theta\dot{\theta},$$

and consequently the relation (6.2) yields

(6.10)
$$\dot{y} = A/\omega\sin\theta\dot{\theta}.$$

Putting the value of $\dot{x}$ from Eq. (6.9) into Eq. (6.8), we get

$$\dot{\theta}(\omega K^2 + A^2/\omega) = C_2.$$

Choosing $C_2 = \omega^2K^2 + A^2$, the last relation gives

$$\dot{\theta} = \omega,$$

or

(6.11)
$$\theta = \omega t,$$

assuming that the constant of integration is zero. Substituting from Eq. (6.11) into Eqs. (6.9) and (6.10), we obtain

$$\dot{x} = A\cos\omega t, \quad \dot{y} = A\sin\omega t,$$

or by integration

$$x = A/\omega\sin\omega t + b_2, \quad y = -A/\omega\sin\omega t + b_3,$$

where b_2 and b_3 are constants. Thus the centre of mass moves with a constant speed A in a direction that rotates with a constant angular velocity ω, i.e. the centre of mass describes a circle of radius A/ω. This agrees with the solution of the problem given in [8].

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