

## Two-fluid theory of the condensation and evaporation effects in fluid flow

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A TWO-FLUID theory for the effects of condensation and evaporation in the fluid flow has been formulated. We consider a mixture of two fluids of which fluid 1 is a perfect gas while fluid 2 may exist in its liquid, vapor or a mixture of a liquid and its own vapor state. A dryness fraction  $\lambda$  is introduced. When  $\lambda = 1$ , fluid 2 is in gaseous state; when  $\lambda = 0$ , fluid 2 is in liquid state and when  $0 < \lambda < 1$ , fluid 2 is a mixture of a liquid and its own vapor. A relaxation time  $t_\lambda$  characterizes the relaxation process of evaporation or condensation. With this formulation we may investigate the relaxation zone in the condensation shock in a supersonic wind tunnel as well as the unsteady flow region due to condensation and evaporation when the mass concentration of water in moist air is very high. The general fundamental equations of this two-fluid theory are given. A special case of one-dimensional unsteady nozzle flow with condensation and evaporation is discussed in details. Finally, the sound speed of fluid 2 is briefly analyzed.

Sformułowano teorię dwóch płynów dla opisu zjawisk kondensacji i parowania w procesie przepływu. Rozpatrujemy mieszaninę dwóch płynów, z których jeden jest gazem doskonałym, a drugi może być w stanie ciekłym lub występować jako para albo mieszanina cieczy z jej parą. Wprowadzono współczynnik suchości  $\lambda$ ; gdy  $\lambda = 1$ , płyn 2 jest w stanie gazowym, a gdy  $0 < \lambda < 1$ , płyn 2 jest mieszaniną cieczy i jej pary. Czas relaksacji  $t_\lambda$  charakteryzuje relaksacyjny proces parowania lub kondensacji. Przy takim sformułowaniu możemy rozpatrywać obszar relaksacji w kondensacyjnej fali uderzeniowej w naddźwiękowym tunelu aerodynamicznym, jak również obszar przepływu niestacznego pojawiający się wskutek kondensacji i parowania przy wysokiej koncentracji wody w wilgotnym powietrzu. Podano ogólne równania podstawowe dla tej teorii. Szczegółowo przeanalizowano szczególnie przypadek jednowymiarowego niestacznego przepływu przez dyszę z uwzględnieniem kondensacji i parowania. Przeanalizowano również prędkość dźwięku w płynie 2.

Сформулирована теория двух жидкостей для описания явлений конденсации и испарения в процессе течения. Рассматривается смесь двух жидкостей, из которых одна является идеальным газом, а вторая может быть в жидком состоянии или выступать как пар или смесь жидкости с ее паром. Введен коэффициент сухости  $\lambda$ ; когда  $\lambda = 0$ , жидкость 2 находится в газовом состоянии, а когда  $0 < \lambda < 1$ , жидкость 2 является смесью жидкости и ее пара. Время релаксации  $t_\lambda$  характеризует релаксационный процесс испарения или конденсации. При такой формулировке можем рассматривать область релаксации в конденсационной ударной волне в сверхзвуковом аэродинамическом туннеле, как тоже область неустойчивого течения, появляющуюся вследствие конденсации и испарения при высокой концентрации воды во влажном воздухе. Приведены общие основные уравнения для этой теории. Подробно проанализирован частный случай одномерного неустойчивого течения через сопло с учетом конденсации и испарения. Проанализирована тоже скорость звука в жидкости 2.

### 1. Introduction

IN MANY practical fluid flow problems, the effects of condensation and/or evaporation are important. One of such problems is that associated with meteorology and another is the well-known condensation shock in a supersonic wind tunnel. If one considers

the flow of moist air in a de Laval nozzle with an arbitrary degree of humidity, three different types of flow are observed [1, 2]:

- a) the flow exhibits a steady and continuous behaviour;
- b) a steady shock occurs at the start of the condensation region, and
- c) the flow becomes unsteady even though the boundary conditions are steady.

There is no satisfactory theory to explain the possibility of these three different flow patterns, particularly the prediction of the starting point of condensation and the conditions when the unsteady periodic processes set in. Most of the theoretical analysis, starting with the classical work of OSWATITSCH [3] and those of WEGENER and his associates [4, 5], are concerned with both microscopic (formation of droplets) and macroscopic (continuous flow) points of view. It seems to the author that such analysis is rather inconsistent and difficult to deal with. In the present paper this problem will be investigated using the continuum theory only. By means of a relaxation time we formulate the problem using the two-fluid theory because the problem is quite similar to the well-known analysis of flow with chemical reaction in which a relaxation time is used. In our approach the effects of both condensation and evaporation will be considered depending on whether  $\lambda$  is increasing (evaporation) or decreasing (condensation). We are going to discuss first the essential features of our two-fluid theory in Sect. 2 and then the fundamental equations of this theory in Sect. 3, and finally the application of this theory to the steady and the unsteady flows in the one-dimensional nozzle flow of moist air in Sects. 4 to 6.

## 2. Essential features of the two-fluid theory of condensation and evaporation effects in a fluid flow

We consider the flow of a mixture of two fluids, i.e. fluid 1 and fluid 2. Fluid 1 is always in the gaseous state in the whole flow field and subscript 1 or  $a$  is used for the value of any flow variables of fluid 1. Fluid 2 may be in vapor, liquid or coexistence of liquid and vapor state and subscript 2 or  $w$  is used for the value of any flow variables of fluid 2. For each of these two fluids we have, in general, six flow variables: 3 velocity components and three state variables, i.e.,

For fluid 1 we have  $\mathbf{q}_1(u_1, v_1, w_1)$ ,  $p_1$ ,  $\rho_1$ ,  $T_1$ ; and

for fluid 2 we have  $\mathbf{q}_2(u_2, v_2, w_2)$ ,  $p_2$ ,  $\rho_2$ ,  $T_2$  (or  $\lambda$ ),

where  $\mathbf{q}$  is the velocity vector with  $u$ ,  $v$ ,  $w$  as, respectively, the  $x$ -,  $y$ - and  $z$ -component of the velocity vector;  $p$  is the pressure,  $\rho$  is the density and  $T$  is the temperature. For fluid 2, because of the phase change in the flow field, it is convenient to introduce a dryness factor  $\lambda$  such that

$$(2.1) \quad \frac{1}{\rho_2} = V_2 = V_L + \lambda(V_v - V_L) = V_2(p, T, \lambda),$$

where  $V = 1/\rho$  is the specific volume of the fluid and the value of  $\lambda$  lies between 0 and 1 in the following manner:

when

$$(2.2) \quad \begin{aligned} T &> T_s(p), & \lambda &= 1, \\ T &< T_s(p), & \lambda &= 0, \\ T &= T_s(p), & 0 &\leq \lambda \leq 1. \end{aligned}$$

The saturated temperature of fluid 2,  $T_s$ , is a given function of the pressure  $p$ . Even though we use seven variables for fluid 2, at any given point in the flow field, only six variables of fluid 2 are independent variables.

For the two-fluid theory we have to distinguish the species density from the partial density [6]. Let us consider an element of the mixture of fluid 1 and fluid 2 with the total mass  $\bar{M} = \bar{M}_1 + \bar{M}_2$  and total volume  $\bar{V} = \bar{V}_1 + \bar{V}_2$ . The value for the mixture is that without subscript.

The species density of fluid 1 is

$$(2.3) \quad \varrho_1 = \frac{\bar{M}_1}{\bar{V}_1} = \varrho_a$$

and the species density of fluid 2 is

$$(2.4) \quad \varrho_2 = \frac{\bar{M}_2}{\bar{V}_2} = \varrho_w.$$

The partial density of fluid 1 is

$$(2.5) \quad \bar{\varrho}_1 = \frac{\bar{M}_1}{\bar{V}} = (1-Z)\varrho_a$$

and the partial density of fluid 2 is

$$(2.6) \quad \bar{\varrho}_2 = \frac{\bar{M}_2}{\bar{V}} = Z\varrho_w,$$

where

$$(2.7) \quad Z = \frac{\bar{V}_2}{\bar{V}} = \text{volume fraction of fluid 2 in the mixture.}$$

In the fundamental equations of the two-fluid theory, the partial density is always used. The total density of the mixture is

$$(2.8) \quad \varrho = \frac{\bar{M}}{\bar{V}} = \bar{\varrho}_1 + \bar{\varrho}_2 = (1-Z)\varrho_a + Z\varrho_w.$$

The total pressure of the mixture is

$$(2.9) \quad p = p_1 + p_2.$$

Since fluid 1 is always in the gaseous state, we assume that the perfect gas law holds for fluid 1, i.e.

$$(2.10) \quad p_1 = \bar{\varrho}_1 R_1 T_1 = (1-Z)\varrho_1 R_1 T_1 = (1-Z)p,$$

where  $R_1$  is the gas constant of fluid 1.

From Eqs. (2.9) and (2.10) we have

$$(2.11) \quad p_2 = Zp.$$

In our two-fluid theory it is convenient to use the following thirteen variables:

$$(2.12) \quad \mathbf{q}_1(u_1, v, w_1), T_1, \varrho_1, p, Z; \quad \mathbf{q}_2(u_2, v_2, w_2), T_2, \varrho_2, \lambda.$$

### 3. Fundamental equations for the two-fluid theory with condensation and evaporation effects

We have, in general, thirteen fundamental equations for the thirteen variables of Eq. (2.12). These fundamental equations are as follows:

1. Equation of state of fluid 1 is the perfect gas law of Eq. (2.10). i.e.

$$p = \rho_1 R_1 T_1.$$

2. Equation of state of fluid 2 is given by Eq. (2.1) in terms of the dryness fraction  $\lambda$ , i.e.

$$\frac{1}{\rho_2} = \frac{1}{\rho_L} + \lambda \left( \frac{1}{\rho_v} - \frac{1}{\rho_L} \right) = V_2(T_2, p, \lambda).$$

If the temperature  $T_2$  of fluid 2 is larger than  $T_s(p)$ , we have  $\lambda = 1$  and Eq. (2.1) becomes

$$(3.1) \quad \rho_2 = \rho_v = \frac{p}{R_v T_2},$$

where  $R_v$  is the gas constant of the vapor of fluid 2. We assume that the vapor obeys the perfect gas law as a first approximation. If the temperature  $T_2$  is less than  $T_s(p)$ , we have  $\lambda = 0$  and

$$(3.2) \quad \rho_2 = \rho_L = \text{constant}.$$

Here we assume that the density of fluid 2 in liquid state is a constant.

When  $T_2 = T_s(p)$ ,  $\lambda$  lies between 0 and 1, and there is a definite relation between  $T_2$  and  $p$ , i.e. the Clausius-Clapeyron relation [6]:

$$(3.3) \quad \left( \frac{dp}{dT} \right)_s = \frac{L}{\left( \frac{1}{\rho_v} - \frac{1}{\rho_L} \right) T_s} = \frac{L \rho_v}{\left( 1 - \frac{\rho_v}{\rho_L} \right) T_s},$$

where  $L$  is the latent heat of evaporation of fluid 2 which is a function of the saturated temperature  $T_s$ .

3. Equation for dryness fraction  $\lambda$

$$(3.4) \quad \frac{D\lambda}{Dt} = \frac{\partial \lambda}{\partial t} + (\mathbf{q}_2 \cdot \nabla) \lambda = \frac{1}{t_\lambda} (\lambda_e - \lambda),$$

where  $\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$  is the gradient operator and  $\lambda_e$  is the equilibrium value of  $\lambda$  which is a function of the species density  $\rho_2$  and the saturated temperature  $T_s(p)$ , i.e.

$$(3.5) \quad \lambda_e = \frac{\frac{1}{\rho_2} - \frac{1}{\rho_{LS}}}{\frac{1}{\rho_{vS}} - \frac{1}{\rho_{LS}}} = \lambda_e(\rho_2, T_s),$$

where the subscript  $s$  refers to the value at the saturated temperature  $T_s$  so that  $\rho_{LS}$  and  $\rho_{vS}$  are respectively the density of the saturated liquid and that of the saturated vapor of fluid 2. Both  $\rho_{LS}$  and  $\rho_{vS}$  are functions of the saturated temperature  $T_s$  only.

We introduce for the macroscopic theory a characteristic time or a relaxation time  $t_\lambda$  for evaporation (when  $\lambda$  increases) or condensation (when  $\lambda$  decreases). For a first approximation we may assume that  $t_\lambda$  is a constant for a given problem. The value of  $t_\lambda$  may be determined experimentally. For  $t_\lambda = 0$  we have  $\lambda = \lambda_e$ , the equilibrium flow of evaporation or condensation. For  $t_\lambda = \infty$ , we have  $\lambda = \text{constant}$ , the frozen flow of evaporation or condensation. The concept of the relaxation time  $t_\lambda$  is similar to that in the flow with chemical reaction.

4. *The equation of continuity of fluid 1 is*

$$(3.6) \quad \frac{\partial(1-Z)\varrho_a}{\partial t} + \nabla \cdot [(1-Z)\varrho_a \mathbf{q}_a] = \sigma_a = 0,$$

where  $\mathbf{q}_1 = \mathbf{q}_a$  and we assume that the source function of fluid 1 is zero, i.e.  $\sigma_a = 0$ .

5. *The equation of continuity of fluid 2 is*

$$(3.7) \quad \frac{\partial Z\varrho_2}{\partial t} + \nabla \cdot [Z\varrho_2 \mathbf{q}_2] = \sigma_2 = 0.$$

We also assume that the source function of fluid 2 is zero, i.e.  $\sigma_2 = 0$ . In general the density  $\varrho_2$  is given by Eq. (2.1).

If we add Eqs. (3.6) and (3.7), we have the *equation of continuity of the mixture of fluid 1 and fluid 2* as follows:

$$(3.8) \quad \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{q}) = 0,$$

where  $\varrho$  is the density of the mixture which is given by Eq. (2.8) and the velocity vector of the mixture as a whole,  $\mathbf{q}$  is defined as follows:

$$(3.9) \quad \varrho \mathbf{q} = (1-Z)\varrho_a \mathbf{q}_a + Z\varrho_2 \mathbf{q}_2.$$

Now we define a mass concentration of fluid 2 in the mixture as

$$(3.10) \quad k_2 = \frac{\bar{\varrho}_2}{\varrho} = \frac{Z\varrho_2}{\varrho}$$

and the mass concentration of fluid 1 is  $k_1 = 1 - k_2$ . We also define the diffusion velocity of the  $r$ -th species as  $\mathbf{w}_r$ , i.e.

$$(3.11) \quad \mathbf{w}_r = \mathbf{q}_r - \mathbf{q}.$$

Substituting Eqs. (3.10) and (3.11) into Eq. (3.7), we have the diffusion equation for fluid 2 as follows:

$$(3.12) \quad \frac{\partial \varrho k_2}{\partial t} + \nabla \cdot (k_2 \varrho \mathbf{q}) = -\nabla \cdot (k_2 \varrho \mathbf{w}_2).$$

From Eqs. (3.9) and (3.11) we find the following relation between  $\mathbf{w}_a$  and  $\mathbf{w}_2$ :

$$(3.13) \quad (1-Z)\varrho_a \mathbf{w}_a + Z\varrho_2 \mathbf{w}_2 = 0.$$

From Eqs. (3.8), (3.10), (3.12) and (3.13) we have

$$(3.14) \quad \varrho \frac{Dk_2}{Dt} = \nabla \cdot [(1-Z)\varrho_a \mathbf{w}_a].$$

The diffusion velocity may be expressed in terms of the diffusion coefficient  $D_{a_2}$  between fluid  $a$  and fluid 2 as follows:

$$(3.15) \quad (1-Z)\rho_a \mathbf{w}_a = \rho D_{a_2} \nabla C_a,$$

where  $C_a$  is the number concentration of the gaseous species  $a$  in the mixture, i.e.

$$(3.16) \quad C_a = \frac{n_a}{n},$$

where  $n_a$  is the number density of the gas  $a$  and  $n$  is the number density of the mixture.

With the help of Eq. (3.15), the diffusion equation (3.14) becomes

$$(3.17) \quad \rho \frac{Dk_2}{Dt} = \nabla \cdot [\rho D_{a_2} \nabla C_a].$$

If the diffusion velocity is negligible, Eq. (3.17) becomes

$$(3.18)_1 \quad \frac{Dk_2}{Dt} = 0$$

or

$$(3.18)_2 \quad k_2 = \text{constant}.$$

When the diffusion phenomenon is negligible, the mass concentration  $k_2$  is a constant in the whole flow field even if we have phase change. Since  $k_a + k_2 = 1$ , the mass concentration of the gaseous species is also a constant when the diffusion phenomenon is negligible.

6. *The equation of motion of fluid  $a$  is*

$$(3.19) \quad (1-Z)\rho_a \frac{D\mathbf{q}_a}{Dt} = -(1-Z)\nabla p + \nabla \cdot \boldsymbol{\tau}_a + \mathbf{F}_{b_a} + \mathbf{F}_a,$$

where  $\boldsymbol{\tau}_a$  is the viscous stress tensor of the fluid  $a$ ;  $\mathbf{F}_{b_a}$  is the body force of fluid  $a$ , such as the gravitational force, and  $\mathbf{F}_a$  is the interaction force between fluids  $a$  and 2. We may write

$$(3.20) \quad \mathbf{F}_a = K_F(\mathbf{q}_2 - \mathbf{q}_a).$$

The interaction factor  $K_F$  depends on the flow conditions between the two fluids  $a$  and 2 (see Ref. [6]).

7. *The equation of motion of fluid 2 is*

$$(3.21) \quad Z\rho_2 \frac{D\mathbf{q}_2}{Dt} = -Z\nabla p + \nabla \cdot \boldsymbol{\tau}_2 + \mathbf{F}_{b_2} - \mathbf{F}_a,$$

where  $\boldsymbol{\tau}_2$  is the viscous stress tensor of fluid 2 and  $\mathbf{F}_{b_2}$  is the body force on fluid 2.

We may obtain the equation of motion of the mixture by adding Eqs. (3.19) and (3.21). The final equation of motion of the mixture would be identical in form as that of the single fluid theory if we define a proper viscous stress tensor of the mixture including the effects of diffusion velocity (see reference [6]).

8. Equation of energy of fluid  $a$  is

$$(3.22) \quad \frac{\partial}{\partial t} \left[ (1-Z)\varrho_a \left( \bar{U}_{ma} + \frac{1}{2} q_a^2 + \phi_a \right) \right] + \frac{\partial}{\partial x^j} \left[ (1-Z)\varrho_a u_a^j \left( \bar{U}_{ma} + \frac{1}{2} q_a^2 + \phi_a \right) - u_a^i \tau_a^{ij} + \delta^{ij} u_a^i p_a - \dot{Q}_{ca}^j \right] = K_T(T_2 - T_a) + \varepsilon_a,$$

where  $U_{ma}$  is the internal energy per unit mass of fluid  $a$ . For simplicity, we may assume  $U_{ma} = C_{va} T_a$  where the specific heat at a constant volume for fluid  $a$  may be assumed as a constant;  $q_a$  is the magnitude of the velocity vector of fluid  $a$  and  $\phi_a$  is the potential energy per unit mass of fluid  $a$ .  $Q_{ca}^j$  is the  $j$ -th component of the heat conductive flux of fluid  $a$  such that  $Q_{ca}^j = -K_a \frac{\partial T_a}{\partial x^j}$  and  $K_a$  is the coefficient of heat conduction of fluid  $a$ ;  $K_T$  is the thermal friction coefficient between fluids  $a$  and 2;  $\varepsilon_a$  is the energy source of fluid  $a$  and  $\delta^{ij} = 0$  if  $i \neq j$  and  $\delta^{ij} = 1$  if  $i = j$ .

9. Equation of energy of fluid 2 is

$$(3.23) \quad \frac{\partial}{\partial t} \left[ Z\varrho_2 \left( \bar{U}_{m_2} + \frac{1}{2} q_2^2 + \phi_2 \right) \right] + \frac{\partial}{\partial x^j} \left[ Z\varrho_2 u_2^j \left( \bar{U}_{m_2} + \frac{1}{2} q_2^2 + \phi_2 \right) - u_2^i \tau_2^{ij} + \delta^{ij} u_2^i p_2 - Q_{c_2}^j \right] = K_T(T_a - T_2) + \varepsilon_2.$$

The symbols in Eq. (3.23) have the same meaning as those in Eq. (3.22) except that subscript 2 refers to the corresponding values for fluid 2.

We may obtain the energy equation for the mixture by adding Eqs. (3.22) and (3.23). The final form of the energy equation of the mixture will be identical in form as that of the single fluid theory if we define the proper internal energy and the heat conductive flux of the mixture by including the effects of diffusion velocities.

We are going to solve Eqs. (2.1), (2.10), (2.16), (3.6), (3.7), (3.19), (3.21), (3.22) and (3.23) for the variables  $\varrho_a$ ,  $\varrho_2$ ,  $Z$ ,  $p$ ,  $q_a$ ,  $q_2$ ,  $T_a$ ,  $T_2$  and  $\lambda$  with proper initial and boundary conditions. Equations (3.6) and (3.7) may be replaced by Eqs. (3.8) and (3.17).

#### 4. One-dimensional unsteady inviscid flow through a nozzle with condensation and evaporation effects

We are going to apply our fundamental equations of Sect. 3 to a special case of one-dimensional flow of moist air through a de Laval nozzle with a given cross-sectional area  $A(x)$  from a reservoir of constant pressure  $p$ , at constant temperature  $T$ , with a specific humidity:

$$k_{20} = \frac{\bar{\varrho}_{20}}{\varrho_{a0} + \bar{\varrho}_{20}} = \frac{\varrho_{20}}{\varrho_0} = \frac{z\varrho_{20}}{\varrho_0}.$$

The  $x$  is the distance along the axis of the nozzle. We shall consider only the main effect of heat release or absorption due to the condensation or the evaporation of the water

in the mixture in the flow field. Hence we may neglect the effects of viscosity, heat conduction and diffusion phenomena.

Our variables are:

- 1)  $p$  — pressure of the mixture,
- 2)  $T = T_a = T_w$  — temperature of the mixture,
- 3)  $\rho$  — density of the mixture,
- 4)  $U = U_a = U_w$  — velocity of the mixture in the direction along the axis of the nozzle,
- 5)  $z$  — volume fraction of the water in the mixture,
- 6)  $\lambda$  — dryness fraction of the water.

Here we assume that the flow is in equilibrium as far as the fluid dynamic variables are concerned, i.e.  $U = U_a = U_w$  and  $T = T_a = T_w$ , where subscript  $a$  refers to the value of the air and subscript  $w$  refers to the value of the water which may be in vapor, liquid or coexistence of vapor and liquid. Without the diffusion effect,  $k_2 = k_{20}$  is a constant in the whole flow field. Hence the volume fraction  $z$  is related to the other variables by Eq. (3.10).

For one-dimensional unsteady flow of moist air in a de Laval nozzle, the fundamental equations for the six variables,  $p$ ,  $\rho$ ,  $T$ ,  $U$ ,  $Z$  and  $\lambda$  are as follows:

$$\begin{aligned}
 p &= \frac{\rho R_M T}{1 - Z}, \\
 \frac{\partial \rho}{\partial t} + \frac{1}{A} \frac{\partial \rho u A}{\partial x} &= 0, \\
 \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} &= - \frac{\partial p}{\partial x}, \\
 \rho \frac{\partial U_m}{\partial t} + \rho u \frac{\partial \bar{U}_m}{\partial x} &= \frac{p}{\rho} \left( \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} \right), \\
 \frac{\partial \lambda}{\partial t} + u \frac{\partial \lambda}{\partial x} &= \frac{1}{t_\lambda} (\lambda_e - \lambda), \\
 Z &= \frac{k_{20} \rho}{\rho_w(p, T, \lambda)},
 \end{aligned}
 \tag{4.1}$$

where

$$R_M = (1 - k_{20}) R_a \tag{4.2}$$

and  $R_a$  is the gas constant of the air.

The internal energy of the mixture per unit mass is

$$U_m = k_{20} U_{mw} + (1 - k_{20}) C_{va} T, \tag{4.3}$$

where the internal energy of the water per unit mass is

$$U_{mw} = U_{mwl} + \lambda (U_{mwv} - U_{mwl}) \tag{4.4}$$

and  $U_{mwl} = C_{vwl} T$  is the internal energy per unit mass of the water in the liquid state, and  $U_{mwv} = C_{vzv} T$  is the internal energy per unit mass of the water in vapor state:  $C_{vwl}$  and  $C_{vzv}$  are respectively the specific heat at constant volume of the water in liquid and



in vapor state. For simplicity, we may assume  $C_{vwl}$  and  $C_{vwp}$  as constants for a first approximation.

Equation (4.1) should be solved for given initial and boundary conditions. In order to demonstrate the special feature of our two-fluid theory, we consider the following case:

We have a de Laval nozzle of a given cross section.

At the entrance section,  $x = 0$ , we have a uniform flow of moist air with the following values. We denote the value of the flow variables at  $x = 0$  by a subscript 0.

I.  $\lambda_0 = 1$ . The water is in the vapor state.

II. The mass concentration  $k_{20}$  is given and it will remain at this value in the whole flow field.

III. The pressure is kept at the value  $p_0$  for all times.

IV. The temperature is kept at the value  $T_0$  and  $T_0 > T_s(p_0)$  where  $T_s(p_0)$  is the saturated temperature of the water at  $p_0$ .

V. The density of the mixture is  $\varrho_0$ , i.e.

$$(4.5)_1 \quad \varrho_0 = (1 - Z_0)\varrho_{a0} + Z_0\varrho_{w0}.$$

The species density of the air  $\varrho_{a0}$  is given by Eq. (2.10), i.e.

$$(4.5)_2 \quad p_0 = \varrho_{a0} R_a T_0.$$

We assume that the species density of the water vapor also satisfy the perfect gas law (2.10), i.e.

$$(4.5)_3 \quad \varrho_{w0} = \varrho_{wv0} \approx \frac{p_0}{R_v T_0}.$$

The volume fraction of the water vapor at the entrance section is then

$$(4.5)_4 \quad Z_0 = \frac{k_{20}\varrho_0}{\varrho_{w0}} = \frac{k_{20} \frac{R_v}{R_a}}{(1 - k_{20}) + k_{20} \frac{R_v}{R_a}}.$$

At the exit section,  $x = b$ , there might be a steady uniform state or not. We will check whether such a steady uniform state exists or not in Section 5: The values of the flow variables at the exit section are denoted by a subscript  $\infty$ . These values may be a function of time if a steady uniform state does not exist.

I.  $\lambda_\infty$  may be a given constant or a function of  $t$  but its value lies between 0 and 1.

II. The pressure at the exit is  $p_\infty$  while the temperature is  $T_\infty$ . We have the following relations between  $\lambda_\infty$ ,  $p_\infty$  and  $T_\infty$ :

$$(4.6)_1 \quad \begin{aligned} \lambda_\infty &= 1, \\ T_\infty &\geq T_s(p_\infty), \quad 0 \leq \lambda_\infty \leq 1, \\ \lambda_\infty &= 0. \end{aligned}$$

III. The density of the mixture at the exit is

$$(4.6)_2 \quad \varrho_\infty = (1 - Z_\infty)\varrho_{a\infty} + Z_\infty\varrho_{w\infty},$$

where

$$(4.6)_3 \quad \varrho_{a\infty} = \frac{p_\infty}{R_a T_\infty},$$

and

$$(4.6)_4 \quad \frac{1}{\varrho_{w\infty}} = \frac{1}{\varrho_{wl\infty}} + \lambda_\infty \left( \frac{1}{\varrho_{wv\infty}} - \frac{1}{\varrho_{wl\infty}} \right).$$

When  $0 \leq \lambda_\infty \leq 1$ ,  $\varrho_{wL} = \varrho_{wL}(T_s)$  — the density of saturated liquid of water and  $\varrho_{wV} = \varrho_{wV}(T_s)$  — the density of the saturated vapor of water. When  $\lambda_\infty = 0$ ,  $\varrho_{w\infty} = \varrho_{wL\infty}$  and when  $\lambda_\infty = 1$ ,  $\varrho_{w\infty} = \varrho_{wV\infty}$ .

IV. The volume fraction of the water in the mixture is

$$(4.6)_5 \quad Z_\infty = \frac{k_{20} \varrho_\infty}{\varrho_{w\infty}}.$$

V. The internal energy of the mixture  $U_{m\infty}$  is

$$(4.6) \quad U_{m\infty} = k_{20} U_{mw\infty} + (1 - k_{20}) C_{va} T_\infty,$$

where

$$(4.6)_7 \quad U_{mw\infty} = U_{mwl\infty} + \lambda_\infty (U_{mwv\infty} - U_{mwl\infty}),$$

$$(4.6)_8 \quad U_{mwl\infty} = C_{vwl\infty} T_\infty, \quad U_{mwv\infty} = C_{vuv\infty} T_\infty.$$

The relation  $T_s(p_\infty)$  is given by Eq. (3.3).

It is convenient to reduce Eqs. (4.1) and (3.3) in terms of the nondimensional forms by introducing the following nondimensional variables:

$$(4.7) \quad \bar{x} = \frac{x}{b}, \quad \bar{t} = \frac{t}{b/u_0}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_0}, \quad \bar{p} = \frac{p}{p_0}, \quad \bar{T} = \frac{T}{T_0},$$

$$\bar{T}_s = \frac{T_s}{T_0}, \quad \bar{U}_m = \frac{U_m}{C_{va} T_0}, \quad \bar{\varrho}_w = \frac{\varrho_w}{\varrho_0}, \quad \bar{L} = \frac{L}{R_w T_0},$$

where the bar refers to the nondimensional quantities.

Substituting Eqs. (4.7) into Eqs. (3.3) and (4.1), we have

$$\bar{p} = \bar{\varrho} \bar{T} \frac{(1 - Z_0)}{(1 - Z)},$$

$$\frac{\partial \bar{\varrho}}{\partial \bar{t}} = -\bar{u} \frac{\partial \bar{\varrho}}{\partial \bar{x}} - \bar{\varrho} \frac{\partial \bar{u}}{\partial \bar{x}} - \frac{\bar{\varrho} \bar{u}}{A} \frac{dA}{d\bar{x}},$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} = -\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} - \frac{1}{\gamma_a M_0^2} \frac{1}{\bar{\varrho}} \frac{\partial \bar{p}}{\partial \bar{x}},$$

$$(4.8) \quad \frac{\partial \bar{U}_m}{\partial \bar{t}} = -\bar{u} \frac{\partial \bar{U}_m}{\partial \bar{x}} - \frac{(1 - k_{20})(\gamma_a - 1)}{(1 - Z_0)} \frac{\bar{p}}{\bar{\varrho}} \left( \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{1}{A} \frac{dA}{d\bar{x}} \right),$$

$$\frac{\partial \bar{\lambda}}{\partial \bar{t}} = -\bar{u} \frac{\partial \bar{\lambda}}{\partial \bar{x}} + \frac{1}{R_{\lambda a}} (\lambda_e - \bar{\lambda}),$$

$$(4.8) \quad Z = \frac{k_{20} \bar{\rho}}{\bar{\rho} w},$$

[cont.]

$$\frac{d\bar{p}_2}{d\bar{T}_s} = \frac{\bar{L}(1-Z_0) \frac{R_v}{R_a}}{\left( \frac{1}{\bar{\rho}_{wvs}} - \frac{1}{\bar{\rho}_{wls}} \right) \bar{T}_s (1-k_{20})},$$

where

$$(4.9) \quad M_0 = \frac{u_0}{\sqrt{\gamma_0 P_0 / \rho_0}}$$

and  $a_0 = \sqrt{\gamma_0 P_0 / \rho_0}$  is the sound speed of the mixture when both  $z$  and  $k_{20}$  are negligibly small. In general, the sound speed of the mixture is different from  $a_0$  (see Ref. [6]):

$$(4.10) \quad R_{t\lambda} = \frac{t_\lambda u_0}{b},$$

where  $R_{t\lambda}$  is the nondimensional relaxation time for condensation and evaporation.

From Eqs. (4.8) we see that the important nondimensional parameters in our problem are:

1. The initial Mach number  $M_0$ .
2. The nondimensional relaxation time for condensation and evaporation  $R_{t\lambda}$ .
3. The mass concentration of water in the mixture  $k_{20}$ .
4. The nondimensional latent heat of water  $\bar{L}$ .

We may calculate the one-dimensional unsteady or steady flow under various values of  $M_0$ ,  $R_{t\lambda}$ ,  $k_{20}$  and  $\bar{L}$  for given initial and boundary conditions. Before one makes any numerical calculations, one should investigate the problem whether there is another steady uniform state at a section far downstream for a given uniform state at the entrance section. To investigate the possibility of a steady uniform state far downstream, it is convenient to solve this problem by comparing the analysis of our two-fluid theory with the conventional single fluid theory.

## 5. Comparison of the two-fluid theory with the conventional single fluid theory

In the conventional single fluid theory of our problem [1], only the heat addition effect due to condensation is studied. To compare our theory with the corresponding single-fluid theory, we first reduce Eqs. (4.8) into forms similar to the corresponding single fluid theory with heat addition. For simplicity, in this section, we consider the case of a nozzle with a constant cross-sectional area, i.e.  $A(x) = \text{constant}$ .

In the ordinary single theory, the following approximations are usually made:

1. The volume fraction of the water is assumed to be negligibly small, i.e.  $z \cong 0$ ,  $z_0 \cong 0$ .
2. The density of the mixture  $\rho$  is approximately equal to that of the air, i.e.  $\rho \cong \rho_a$ ,  $R_M \cong R_a$ .

3. We do not calculate how the dryness fraction  $\lambda$  varies in the flow field, that is, we do not calculate the relaxation zone of condensation and evaporation but consider the limiting equilibrium cases before and after the relaxation zone. Hence we take  $\lambda = \lambda_0 = 1$  before the relaxation zone and  $\lambda = \lambda_\infty = 0$  after the relaxation zone.

With the above assumptions, for the case of steady flow, Eqs. (4.8) are reduced to the following forms:

$$\begin{aligned} p &= \rho R_a T, \\ \rho_a u &= \rho_{a0} u_0 = \text{constant}, \\ (5.1) \quad \rho_a u^2 + p &= \rho_{a0} u_0^2 + p_0 = \text{constant}, \\ H_a + \frac{1}{2} u^2 &= H_{a0} + \frac{1}{2} u_0^2 + k_{20} L, \end{aligned}$$

where

$$(5.2) \quad L = H_{wv0} - H_{wl}$$

is the latent heat of condensation of water in the mixture and  $H$  is the enthalpy. Equations (5.1) are the equations of the single fluid theory [1] with heat addition due to the condensation of water vapor,  $Q_w = k_{20} L$ . It is well known that if the initial Mach number  $M_0$  is supersonic, for a given  $Q_w$  below a critical value  $Q_{wc}$ , the final Mach number  $M_\infty$  decreases with the increase of  $Q_w$  and when  $Q_w = Q_{wc}$ ,  $M_\infty = 1$ . When  $Q_w > Q_{wc}$ , there will be no steady state solution for the flow in this nozzle (see Ref. [1]).

The corresponding steady state equations of our two-fluid theory are as follows:

$$\begin{aligned} p &= \frac{\rho R_M T}{1 - Z}, \\ \rho u &= \rho_0 u_0 = \text{constant}, \\ \rho u^2 + p &= \rho_0 u_0^2 + p_0 = \text{constant}, \\ (5.3) \quad k_{20} U_{mv} + (1 - k_{20}) C_{va} T + \frac{p}{\rho} + \frac{1}{2} u^2 &= \text{constant} \\ &= k_{20} U_{mrv} + (1 - k_{20}) C_{va} T_0 + \frac{p_0}{\rho_0} + \frac{1}{2} u_0^2, \\ u \frac{\partial \lambda}{\partial x} &= \frac{1}{t_\lambda} (\lambda_e - \lambda), \\ Z &= \frac{k_{20} \rho}{\rho_w(p, T, \lambda)}. \end{aligned}$$

The main difference between Eqs. (5.2) and (5.1) is that by the two-fluid theory, Eqs. (5.2), we may calculate the flow field in the relaxation zone. Furthermore, the assumptions (1) and (2) for the single fluid theory, Eqs. (5.1), will introduce some errors in the analysis if  $k_{20}$ ,  $z$  or  $z_0$  is not negligibly small. Since the heat addition  $Q_w$  is proportional to  $k_{20}$ , it is interesting to note that the flow will be unsteady for the steady uniform entrance condition when  $k_{20}$  and then  $z_0$  is sufficiently large. For a large value of  $k_{20}$ , Eqs. (5.2) would be more accurate than those of Eqs. (5.1). At the final equilibrium con-

dition far downstream, the dryness fraction  $\lambda_\infty$  may be a constant but different from zero. From Eqs. (5.3)<sub>1</sub> to (5.3)<sub>4</sub> we may calculate a critical  $\lambda_{\infty c}$  such that  $M_\infty = 1$  for any given large value of  $k_{20}$ . Hence, for  $\lambda_\infty > \lambda_{\infty c}$ , the steady state uniform solution will always exist far downstream. We will not be able to determine this  $\lambda_{\infty c}$  in the conventional single fluid theory of Eqs. (5.1). When  $Q_w > Q_{wc}$  or  $\lambda_\infty < \lambda_{\infty c}$ , we have to use Eqs. (4.8) to study the unsteady flow field. The results of the calculation of the unsteady flow field will be reported in another paper.

## 6. Sound speed of a fluid in the coexistence state of its liquid and vapor

It should be noticed that the sound speed of the second fluid which may be a mixture of the liquid and its own vapor behaves quite differently from ordinary gas in a certain domain. Let us discuss briefly this point as follows:

The sound speed  $a$  of a medium is defined as

$$(6.1) \quad a = \sqrt{\left(\frac{dp}{d\rho}\right)_s} = V \sqrt{-\left(\frac{dp}{dV}\right)_s},$$

where  $V = \frac{1}{\rho}$  is the specific volume of the medium and subscript  $S$  refers to an isentropic process.  $S$  is entropy.

In general, we may consider that the pressure is a function of the specific volume  $V$  and the temperature  $T$ , i.e.  $p = p(V, T)$ . Then

$$(6.2) \quad \left(\frac{dp}{dV}\right)_s = \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial T}\right)_v \left(\frac{dT}{dV}\right)_s.$$

The first law of thermodynamics gives for isentropic process

$$(6.3) \quad TdS = \left(\frac{\partial U_m}{\partial T}\right)_v dT + \left[\left(\frac{\partial U_m}{\partial V}\right)_T + p\right]dV = C_v dT + \left[\left(\frac{dU_m}{dV}\right)_T + p\right]dV = 0.$$

From Eq. (6.3) we have

$$(6.4) \quad \left(\frac{dT}{dV}\right)_s = -\frac{1}{C_v} \left[\left(\frac{dU_m}{dV}\right)_T + p\right].$$

Since  $\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}$ , we have from Eq. (6.3)

$$(6.5) \quad \left(\frac{\partial U_m}{\partial V}\right)_T + p = T \left(\frac{dp}{dT}\right)_v.$$

From Eqs. (6.4) and (6.5) we have

$$(6.6) \quad \left(\frac{dT}{dV}\right)_s = -\left(\frac{\partial p}{\partial T}\right)_v \frac{T}{C_v}.$$

From Eqs. (6.1), (6.2) and (6.6) we have

$$(6.7) \quad a^2 = -V^2 \left[ \left(\frac{dp}{dV}\right)_T - \left(\frac{dp}{dT}\right)^2 \frac{T}{C_v} \right] = \left(\frac{\partial p}{\partial \rho}\right)_T + \frac{1}{\rho^2} \left(\frac{\partial p}{\partial T}\right)_\rho^2 \frac{T}{C_v}.$$

For a perfect gas with the equation of state

$$(6.8) \quad p = \rho RT,$$

Eq. (6.7) becomes

$$(6.9) \quad a^2 = RT \left( 1 + \frac{R}{C_v} \right) = \gamma RT.$$

Equation (6.9) is the well-known result of the sound speed of a perfect gas.

In the coexistence region of the liquid and its own vapor, the pressure  $p$  is a function of temperature only, and Eq. (6.7) becomes

$$(6.10) \quad a = \frac{1}{\rho} \left( \frac{dp}{dT} \right)_e \left( \frac{T}{C_v} \right)^{1/2}.$$

In this coexistence region,  $p = p(T)$  is given by the Clausius–Clapeyron equation (3.3). Hence, from Eqs. (3.3) and (6.10), we have

$$(6.11) \quad a = \left( \frac{\rho_v}{\rho} \right) \frac{L}{\left( 1 - \frac{\rho_v}{\rho_L} \right)} \left( \frac{1}{C_v T} \right)^{1/2}.$$

If we consider the case near the saturated vapor line,  $\rho_v \cong \rho$  and  $\rho_v \ll \rho_L$  and  $L \cong \text{constant}$ , Eq. (6.11) becomes

$$(6.12) \quad a = L \left( \frac{1}{C_v T} \right)^{1/2}.$$

Now the sound speed of the medium is inversely proportional to  $\sqrt{T}$  rather than proportional to  $\sqrt{T}$  in the case of a perfect gas.

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