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Ray methods in multidimensional gasdynamics

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THE CHARACTERISTIC determinant leads to a hodograph transformation which interchanges the dependent and independent variables. The characteristic coordinates $\mathbf{N} = (n_t, n_x, n_y, n_z)$ are conjugate to (t, x, y, z) and satisfy the Hamiltonian equations. This establishes the ray form of gasdynamics analogous to geometrical mechanics. A one-to-one correspondence between differential operators and multiplication by the components of $\mathbf{N}$ is observed. Gasdynamics is interpreted as wave mechanics in the limit of zero wavelength, thus suggesting an extension of the ray formulation to statistical fluid mechanics.

Wyznacznik charakterystyczny prowadzi do transformacji hodografu zamieniającej zmienne zależne i niezależne. Współrzędne charakterystyczne $\mathbf{N} = (n_t, n_x, n_y, n_z)$ są sprzężone z (t, x, y, z) i spełniają równania Hamiltona. Nadaje to gazodynamice postać promieniową analogiczną do mechaniki geometrycznej. Stwierdza się jedno-jednoznaczną odpowiedniość operatorów różniczkowych oraz mnożenie przez składowe $\mathbf{N}$. Gazodynamikę interpretuje się jako mechanikę falową w granicznym przypadku zerowych długości fal co sugeruje rozszerzenie sformułowania promieniowego na statystyczną mechanikę cieczy.

Характеристический определитель приводит к преобразованию годографа, заменяющему зависимые и независимые переменные. Характеристические координаты $\mathbf{N} = (n_t, n_x, n_y, n_z)$ сопряжены с (t, x, y, z) и удовлетворяют уравнениям Гамильтона. Это придает газодинамике лучевой вид аналогичный геометрической механике. Констатируется одно-однозначное соответствие дифференциальных операторов и умножение через составляющие $\mathbf{N}$. Газодинамика интерпретируется как волновая механика в предельном случае нулевых длни волн, что предсказывает расширение лучевой формулировки на статистическую механику жидкости.

1. Introduction

THE THEORY of characteristics for multidimensional inviscid gasdynamics provides an exact connection between gasdynamical fields on the one hand and classical particle mechanics on the other. Further, the theory establishes a rigorous operator formalism, a one-to-one correspondence between differential and algebraic multiplication operators. These latter aspects of gasdynamics have not yet received enough attention, especially in view of the fact that the ray formulation of classical gasdynamics appears amenable to an extension to a stochastic wave theory.

The aim of this paper is to display the special role of the components of the characteristic normal both as the natural coordinates and as multiplication operators. The primitive variables will be expressed by a hodograph-like transformation in terms of the characteristic coordinates (the components of the characteristic normal), the latter, and the position coordinates, satisfying a set of Hamiltonian equations. Thus a formal connection between gasdynamical fields, wave motion and Hamiltonian particle mechanics is established. This connection, it is believed, will provide a starting point for an extension of the classical mechanics of gases to a new form of statistical fluid mechanics.

2. Theory of characteristics

In regions of space-time where the solution is continuous, the equations of conservation of mass, momentum and energy may be put in the form

$$
\text{(2.1)} \qquad \begin{bmatrix} \dfrac{D}{Dt}, & \varrho\nabla\cdot, & 0 \\ 0\ , & \varrho\dfrac{D}{Dt}, & \nabla \\ 0\ , & \gamma p\nabla\cdot, & \dfrac{D}{Dt} \end{bmatrix} \begin{Bmatrix} \varrho \\ \mathbf{u} \\ p \end{Bmatrix} = 0,
$$

where ϱ — mass density, $\mathbf{u}$ — fluid velocity vector, p — pressure, $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ — substantial derivative.

In the theory of characteristics we specify a parametric equation of a path, $t = t(s)$, $\mathbf{x} = \mathbf{x}(s)$, and substitute $p_t = p_s s_t$, $p_x = p_s s_x$, etc., thus changing the system (2.1) to

$$
\text{(2.2)} \qquad \begin{bmatrix} (s_t + \mathbf{n}\cdot\nabla s), & \varrho\nabla s\cdot & . & 0 \\ 0 & , & \varrho(s_t+\mathbf{u}\cdot\nabla s), & \nabla s \\ 0 & , & \gamma p\nabla s\cdot \quad , & (s_t+\mathbf{u}\cdot\nabla s) \end{bmatrix} \begin{Bmatrix} \varrho_s \\ \mathbf{u}_s \\ p_s \end{Bmatrix} = 0.
$$

Here we note, for further reference, the one-to-one correspondence between the differential operators $\partial/\partial t$, ∇, operating on ϱ, $\mathbf{u}$, p, and the multiplication operators s_t, ∇s, operating on the total derivatives ϱ_s, $\mathbf{u}_s$, p_s. This correspondence applies only if the system (2.2) yields a nontrivial solution, that is, if and only if the determinant of the coefficient matrix in the system (2.2) vanishes,

$$
\text{(2.3)} \qquad \varrho^3(s_t+\mathbf{u}\cdot\nabla s)^3[(s_t+\mathbf{u}\cdot\nabla s)^2 - a^2\nabla s\cdot\nabla s] = 0,
$$

where $a^2 = \gamma\, p/\varrho$ — square of the speed of sound, γ — ratio of specific heats.

All the factors of the above characteristic determinant are homogeneous in derivatives of the parameter s. We may, therefore, change the scale and units of s arbitrarily. For instance, one may normalize s so that $\nabla s = \mathbf{n}$ — space component of the characteristic normal, $|\mathbf{n}| = 1$, and then $s_t = n_t$ — time component of the characteristic normal, the latter obtained as a function of $\mathbf{u}$, $\mathbf{n}$, a and by equating each factor of the determinant to zero separately,

$$
n_t = -\mathbf{u}\cdot\mathbf{n}, \quad \text{and} \quad n_t = -(\mathbf{u}\cdot\mathbf{n} + a|\mathbf{n}|).
$$

The homogeneity of the determinant (2.3) implies that, by a change of scale and units, we may associate the frequency ω and wavenumber $\mathbf{k}$, or the energy H and momentum $\mathbf{p}$, with $-n_t$ and $\mathbf{n}$, respectively. Consequently, we have

$$
\begin{aligned} \omega &= \mathbf{u}\cdot\mathbf{k}, \quad & \omega &= (\mathbf{u}\cdot\mathbf{k} + a|\mathbf{k}|), \\ H &= \mathbf{u}\cdot\mathbf{p}, \quad & H &= (\mathbf{u}\cdot\mathbf{p} + a|\mathbf{p}|), \end{aligned}
$$

and the theory of characteristics for the case of the equations of gasdynamics may be paraphrased in the language of wave theory or the language of particle mechanics. Likewise, methods and results of the wave theory or of particle mechanics may be utilized directly. In particular, we would like to restate here the result reported by LIGHTHILL [1], namely, that if a ray is defined by the equation

$$\frac{d\mathbf{x}}{dt} = \frac{\partial\omega}{\partial\mathbf{k}}$$

with time t as a parameter, then along such a ray

$$\frac{d\mathbf{k}}{dt} = \frac{-\partial\omega}{\partial\mathbf{x}}, \quad \frac{d\omega}{dt} = \frac{\partial\omega}{\partial t},$$

where, in general, $\omega = \omega(\mathbf{k}, \mathbf{x}, t)$. Thus the disturbances (discontinuities in the normal derivatives ϱ_s, $\mathbf{u}_s$, p_s) propagate according to the Hamiltonian equations of motion. The Hamiltonian equations enable one to construct the rays if the flow field is known, that is, if $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $a = a(\mathbf{x}, t)$ are known simultaneously with $\mathbf{k}(\mathbf{x}, t)$ and $\omega(\mathbf{k}, \mathbf{x}, t)$.

Since the vanishing of the appropriate factor of the determinant (2.3) is necessary for the existence of the disturbance, and since such factors are linear in $\mathbf{u}$, a and in $\mathbf{k}$, ω, we proceed as follows. It was shown by RUSANOV [2] that an independent set of five characteristic relations for ϱ, $\mathbf{u}$, p may be obtained using the conservation of the entropy statement,

$$p/\varrho^{\gamma} = \text{const} \quad \text{on} \quad \frac{d\mathbf{x}}{dt} = \mathbf{u}, \tag{2.4}$$

and four additional relations corresponding to the quadratic factor in Eq. (2.3) written for four different waves. The quadratic factor, factored out into linear factors, written out for four wavenumber vectors $\mathbf{k}^{(r)}$, $r = 1, \ldots, 4$, $k^{(r)} = |\mathbf{k}|^{(r)}$, gives

$$\begin{bmatrix} k_1^{(1)}, & k_2^{(1)}, & k_3^{(1)}, & k^{(1)} \\ k_1^{(2)}, & k_2^{(2)}, & k_3^{(2)}, & k^{(2)} \\ k_1^{(3)}, & k_2^{(3)}, & k_3^{(3)}, & k^{(3)} \\ k_1^{(4)}, & k_2^{(4)}, & k_3^{(4)}, & k^{(4)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ a \end{Bmatrix} = \begin{Bmatrix} \omega^{(1)} \\ \omega^{(2)} \\ \omega^{(3)} \\ \omega^{(4)} \end{Bmatrix}. \tag{2.5}$$

Provided that the end points of the unit vectors $\mathbf{n}^{(r)} = \mathbf{k}^{(r)}/k^{(r)}$, $r = 1, \ldots, 4$, do not lie in a common plane, the determinant of the matrix in the relations (2.5) does not vanish and one may solve uniquely for the vector (u_1, u_2, u_3, a) at points of intersection of the four rays. Then, with $a^2 = \gamma p/\varrho$ and the entropy relation (2.4) written along a fifth ray, all of the field variables, viz. ϱ, $\mathbf{u}$, and p, may be expressed as functions of the characteristic coordinates $(n_t, \mathbf{n})$, or $(-\omega, \mathbf{k})$, or $(-H, \mathbf{p})$, enabling one to proceed with the simultaneous solution of the Hamiltonian equations.

We note that (u_1, u_2, u_3, a) may be obtained as a function of space time $(\mathbf{x}, t)$ only after the network of the characteristic rays is constructed by integration of the set of Hamiltonian equations. This is to be expected since in the nonlinear case the dependent variables and the characteristic coordinates are coupled and must be solved for simultaneously. The new result of the present paper is the hodograph transformation (2.5) which is a linear algebraic system in both the primitive and the characteristic variables and not a nonlinear partial differential equation.

The invertible relation (2.5) is in a nature of hodograph transformation exchanging the roles of the dependent and the independent variables. This transformation is unexpectedly simple as a consequence of the use of the characteristic coordinates rather than the conjugate coordinates $\mathbf{x}$ and t. A numerical method based on the ray formulation of gasdynamics was proposed by the author [3]. An example of a three-dimensional time-dependent solution using the proposed method was calculated by I. H. PARPIA [4]. The main advantage of the ray formulation is that it involves integration of a system of ordinary differential equations in the Hamiltonian form, and that it allows for the construction of the flow field and, simultaneously, of the wave fronts with due regard to the domain of dependence of the solution.

For completeness we give below an independent set of characteristic relations corresponding, according to RUSANOV [2], to a given choice of the orientation of the space component of the characteristic normal, the unit vector $\mathbf{n}$:

$$
\begin{aligned}
&\frac{D}{Dt}(p/\varrho^{\gamma}) = 0,\\
&\mathbf{q}\cdot\left[\varrho\frac{D\mathbf{u}}{Dt}+\nabla p\right] = 0,\\
&\mathbf{r}\cdot\left[\varrho\frac{D\mathbf{u}}{Dt}+\nabla p\right] = 0,\\
\left[\varrho a\mathbf{n}\cdot\frac{D\mathbf{u}}{Dt}\pm a\mathbf{n}\nabla\cdot\overset{s}{u}\right]&\pm\left[\frac{Dp}{Dt}\pm a\mathbf{n}\cdot\nabla p\right] = 0.
\end{aligned}
\tag{2.6}
$$

The vectors $\mathbf{q}$ and $\mathbf{r}$ are arbitrary nonparallel vectors orthogonal to $\mathbf{n}$, e.g. $\mathbf{q} = \mathbf{c}\times\mathbf{n}$, $\mathbf{r} = \mathbf{n}\times(\mathbf{c}\times\mathbf{n})$ with $\mathbf{c}$ arbitrary.

3. Operator formalism in gasdynamics

In Sect. 2 we have drawn attention to the apparent correspondence between the differential and multiplication operators,

$$\frac{\partial}{\partial t}\leftrightarrow n_t,\qquad \nabla\leftrightarrow\mathbf{n}. \tag{3.1}$$

We observe the same correspondence in Gauss' Divergence Theorem,

$$\iiint\limits_V \nabla^* f\,dV = \iint\limits_A \mathbf{n}^* f\,dA,$$

where $*$ denotes any allowable multiplication, f is a scalar, vector, or a tensor, $\mathbf{n}$ is a unit normal to the surface area A bounding a volume V.

In regions of space-time where the solution itself is discontinuous, we shall demonstrate the same correspondence which holds on surfaces of discontinuity Σ. The surfaces Σ are the boundaries between regions G^+ and G^- in which the solution is continuous, differentiable, and possesses appropriate limits as Σ is approached from G^+ and G^- sides.

TRUESDELL [5] quotes Hadamard's result that, to the general conservation law in the divergence form,

$$\frac{\partial\psi}{\partial t}+\nabla\cdot(\psi\mathbf{u}) = 0,$$

for a property of a fluid per unit volume, ψ, valid in G^+ and G^-, and written as

$$\left(\frac{\partial}{\partial t},\nabla\right)\cdot(\psi,\psi\mathbf{u}) = 0,$$

there corresponds

$$n_t[\psi]+\mathbf{n}\cdot[\psi\mathbf{u}] = 0$$

which may be written as

$$(n_t,\mathbf{n})\cdot[\psi,\psi\mathbf{u}] = 0 \tag{3.2}$$

and which holds only on Σ, the boundary between G^+ and G^-, and where $[\psi]\equiv\psi^+-\psi^-$ = jump in ψ across Σ. If the fourvector $\mathbf{N} = (n_t,\mathbf{n})$, which is normal to the hypersurface $\Sigma(\mathbf{x}, t)$, is normalized so that its space component $\mathbf{n}$ becomes a unit spatial vector normal to the intersection of the hypersurfaces Σ with space, then n_t = time component of $\mathbf{N} = -U$, where U = the speed of propagation of Σ through space. Thus, if a hypersurface of discontinuity exists, then there is a one-to-one correspondence between the differential operator $(\partial/\partial t,\nabla)$, operating on the fourvector of densities and their fluxes, and the multiplication operator $(-U,\mathbf{n})$ operating on the fourvector of differences across Σ (jumps) of densities and their fluxes.

A typical surface of discontinuity in gasdynamics is a shock wave across which the Rankine–Hugoniot jump conditions hold. In a general case of moving shock, these conditions take the form of (3.2).

We have demonstrated here how one may, through the operator formalism and correspondence rules (3.1), pass directly from the differential form of conservation laws to the algebraic shock conditions. The converse, namely changing of the shock conditions into a differential form of the conservation law by using the correspondence rules (3.1), is self-evident. However, the application of the correspondence rules (3.1) to the characteristic surfaces is not as simple. One of the reasons is the fact that the characteristic relations (2.6) contain $\mathbf{n}$ explicitly as a parameter, while $\mathbf{n}$ will also serve as an operator corresponding to the vector differential operator. We shall adopt the convention that $\mathbf{n}$ will denote a parameter while $\mathbf{k}/|\mathbf{k}|$ will be used to denote operators.

Substituting $-\omega$ and $\mathbf{k}$ for $\partial/\partial t$ and ∇ in the characteristic relations (2.6), we obtain their algebraic equivalents:

$$\begin{gathered}(-\omega+\mathbf{u}\cdot\mathbf{k})(p/\varrho^{\gamma}) = 0,\\ \mathbf{q}\cdot[\varrho(-\omega+\mathbf{u}\cdot\mathbf{k})\mathbf{u}+\mathbf{k}p] = 0,\\ \mathbf{r}\cdot[\varrho(-\omega+\mathbf{u}\cdot\mathbf{k})\mathbf{u}+\mathbf{k}p] = 0,\\ \varrho a\mathbf{n}\cdot[(-\omega+\mathbf{u}\cdot\mathbf{k})\mathbf{u}\pm a\mathbf{n}(\mathbf{k}\cdot\mathbf{u})]\pm[(-\omega+\mathbf{u}\cdot\mathbf{k})p\pm a\mathbf{n}\cdot\mathbf{k}p] = 0.\end{gathered} \tag{3.3}$$

The first of the above equations is obviously satisfied when the linear factor of the determinant (2.3) vanishes, i.e. when $(-\omega+\mathbf{u}\cdot\mathbf{k}) = 0$. Since $\mathbf{q}$ and $\mathbf{r}$ are orthogonal to

$\mathbf{n}$ and, therefore, also to $\mathbf{k}$, the terms $\mathbf{q}\cdot\mathbf{k}p$ and $\mathbf{r}\cdot\mathbf{k}p$ drop out and Eqs. $(3.3)_2$ and $(3.3)_3$ are also satisfied when $(-\omega+\mathbf{u}\cdot\mathbf{k}) = 0$. Rearranging Eq. $(3.3)_4$ and using $\mathbf{k}\cdot\mathbf{u} = k(\mathbf{n}\cdot\mathbf{u})$, $\mathbf{n}\cdot\mathbf{k}p = (\mathbf{k}\cdot\mathbf{k})p/k = kp$, we have

$$[-\omega+\mathbf{u}\cdot\mathbf{k}\pm ak](\varrho a\mathbf{n}\cdot\mathbf{u}\pm p) = 0. \tag{3.4}$$

On the characteristic surfaces corresponding to the quadratic factor of the determinant (2.3), the expression in square brackets vanishes and Eq. $(3.3)_4$ is satisfied. We observe that the particular form of Eq. (3.4) arrived at was obtained by moving $\varrho a\mathbf{n}$ to the right of the operator $[-\omega+\mathbf{u}\cdot\mathbf{k}\pm ak]$ and by interpreting $\mathbf{n}\cdot\nabla$ as corresponding to the improper operator $k = (\mathbf{k}\cdot\mathbf{k})^{1/2}$.

With the extension of the classical gasdynamics to statistical fluid mechanics in mind, we observe that averaging of the characteristic relations (2.6) corresponds to averaging of the various factors of the determinant (2.3). This could be accomplished by an introduction of distributions and subsequent integration in wavenumber space. In order to derive the averaged differential operators formally, one has to determine the rules of the ordering of the multiplication operators because the corresponding differential operators do not commute. The particular operand which will give a correct characteristic relation is also of interest. We turn now to the problem of deriving the characteristic relations (2.6) from the factors of the determinant (2.3).

The triple linear factor of Eq. (2.3) gives three independent relations corresponding to a single choice of the parameter $\mathbf{n}$, viz. Eqs. $(2.6)_1$–$(2.6)_3$. The first of the relations is obtained simply by operating with $(-\omega+\mathbf{u}\cdot\mathbf{k})$ on p/ϱ^γ and substituting in accord with rule (3.1),

$$(-\omega+\mathbf{u}\cdot k)p/\varrho^\gamma \leftrightarrow \left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)(p/\varrho^\gamma).$$

The projections of the momentum equation onto the two vectors $\mathbf{q}$ and $\mathbf{r}$ lying in the plane normal to $\mathbf{n}$, viz. Eqs. $(2.6)_2$ and $(2.6)_3$ may be obtained by a multiplication of $(-\omega+\mathbf{u}\cdot\mathbf{k})$ by $\mathbf{q}\varrho$ or $\mathbf{r}\varrho$ on the left and by $\mathbf{u}$ on the right. Since $\mathbf{q}$ and $\mathbf{r}$ are orthogonal to $\mathbf{k}$, we may add an arbitrary term proportional to $\mathbf{q}\cdot\mathbf{k}$ or $\mathbf{r}\cdot\mathbf{k}$. The arbitrariness of such a term may be removed if one requires that, as shown by RUSANOV [2], Eqs. $(2.6)_2$ and $(2.6)_3$ may be written as linear combinations of Eq. $(2.6)_4$ each evaluated for a different independent choice of the normal $\mathbf{n}$, e.g. such that $\mathbf{q} = \mathbf{n}_1-\mathbf{n}_2$, $\mathbf{r} = \mathbf{n}_3-\mathbf{n}_4$. Another approach is possible. One may treat $\mathbf{n}$ in $\mathbf{q} = \mathbf{c}\times\mathbf{n} = \mathbf{n}\times\mathbf{c}$ as an operator and write

$$0 = \mathbf{n}\times\mathbf{c}\cdot\varrho(-\omega+\mathbf{u}\cdot\mathbf{k})\mathbf{u} \leftrightarrow \mathbf{c}\cdot\nabla\times\left[\varrho\frac{D\mathbf{u}}{Dt}\right] = 0,$$

which represents the projection of the curl of the momentum equation on the constant vector $\mathbf{c}$.

To the quadratic factor of the determinant (2.3), viz. Eq. (3.5),

$$(-\omega+\mathbf{u}\cdot\mathbf{k})^2-a^2\mathbf{k}\cdot\mathbf{k} = (-\omega+\mathbf{u}\cdot\mathbf{k}+ak)(-\omega+\mathbf{u}\cdot\mathbf{k}-ak) = 0, \tag{3.5}$$

where $k = |\mathbf{k}|$, there corresponds the second-order differential equation, the potential equation of gasdynamics, viz.

$$\left[\left(\frac{D}{Dt}\right)^{(2)} - a^2\nabla^2\right]\phi = 0 \tag{3.6}$$

for the velocity potential ϕ, $\nabla\phi = \mathbf{u}$, where the notation $(D/Dt)^{(2)}$ implies that the convective velocity $\mathbf{u}$ in D/Dt is kept constant when operated upon by D/Dt, that is

$$\left(\frac{D}{Dt}\right)^{(2)} = \frac{\partial^2}{\partial t^2} + 2\mathbf{u}\cdot\nabla\frac{\partial}{\partial t} + \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla.$$

The correspondence rules (3.1) applied to the quadratic factor (3.5) lead directly to the second-order differential operator in Eq. (3.6). The choice of the operand follows from the recognition of Eq. (3.6) as a familiar equation. However, since Eq. (3.6) is a single equation of second order, one has the choice of factoring the operator in Eq. (3.6), or of considering Eq. (3.6) as a matrix equation for a two-component column vector formally equivalent to two first-order equations, or to obtain two first-order equations corresponding to each of the linear factors in Eq. (3.5). This task is simplified because we have shown earlier that the characteristic relations $(2.6)_4$ are satisfied if Eq. (3.5) holds. It suffices to multiply the linear factor $(-\omega+\mathbf{u}\cdot\mathbf{k}\pm ak)$ on the right by $(\varrho a\mathbf{n}\cdot\mathbf{u}\pm p)$ in order to obtain Eq. (3.4) which is equivalent to Eq. $(3.3)_4$. This determines the operand which has $\mathbf{n}$ as parameter. In order to arrive at Eq. $(2.6)_4$, we need only to reverse the steps in the derivation of Eq. $(3.3)_4$. We keep the acoustic impedance ϱa constant, use the correspondence rules (3.1) for the proper operators and treat the improper operator $k = (\mathbf{k}\cdot\mathbf{k})^{1/2}$ as follows:

$$\begin{gathered} k(\mathbf{n}\cdot\mathbf{u}) = k\left(\frac{\mathbf{k}}{k}\cdot\mathbf{u}\right) = \mathbf{k}\cdot\mathbf{u} \leftrightarrow \nabla\cdot\mathbf{u}, \\ kp = \frac{\mathbf{k}\cdot\mathbf{k}}{k}p = \mathbf{n}\cdot\mathbf{k}p \leftrightarrow \mathbf{n}\cdot\nabla p. \end{gathered} \tag{3.7}$$

With these steps we recover the characteristic relation $(2.6)_4$. As a consequence we have obtained two relations $(2.6)_4$ instead of a single potential equation (3.6), or, what is equivalent, we have split the second-order potential equation into two characteristic relations corresponding to the same but arbitrary choice of the normal $\mathbf{n}$.

To simplify the differential equation corresponding to Eq. (3.4), we may write

$$\left(\frac{D}{Dt} \pm a\mathbf{n}\cdot\nabla\right)(\varrho_0 a_0\mathbf{n}\cdot\mathbf{u}\pm p) = 0,$$

which is equivalent to the characteristic relations $(2.6)_4$ if we adopt the convention that $\varrho_0 a_0$ is kept constant when operated upon, and if we change $(\mathbf{n}\cdot\nabla)(\mathbf{n}\cdot\mathbf{u})$ to $(\mathbf{n}\cdot\mathbf{n})(\nabla\cdot\mathbf{u}) = \nabla\cdot\mathbf{u}$. Unfortunately, the operand used above, having an appearance of a Riemann invariant, does not remain constant on the rays $d\mathbf{x}/dt = \mathbf{u}\pm a\mathbf{n}$.

The rules (3.7) may be restated in a simpler form:

The improper multiplication operator $k = (\mathbf{k}\cdot\mathbf{k})^{1/2}$ *corresponds to the differential operator* $(\mathbf{n}\cdot\nabla)$ *with the proviso that when* $(\mathbf{n}\cdot\nabla)$ *operates on* $\mathbf{n}$, *then* $\mathbf{n}$ *and* ∇ *commute, i.e.*

$$(\mathbf{n}\cdot\nabla)\mathbf{n} = (\mathbf{n}\cdot\mathbf{n})\nabla = \nabla.$$

4. Summary

The present paper formulates the method of characteristics, Section 2, in the form of ray mechanics for the dual purpose of, 1) facilitating the computations in multidimensional cases, 2) displaying the analogy (isomorphism) between the Euler's field equations of inviscid gasdynamics and Hamiltonian equations of particle mechanics or of the geometrical (ray) wave theory.

The kinematics of wave motion in gasdynamics determines, it was shown, not only the ray network but also the primitive variables as solutions of the Euler's system. The characteristic coordinates (components of the characteristic normal) and the primitive variables are related linearly, a fact which facilitates computations. The Hamiltonian form of the ray formulation implies that gasdynamics, which was shown to be analogous to particle mechanics, corresponds to the zero wavelength limit of wave mechanics. Naturally, one is tempted to investigate the possibility of an extention of the deterministic gasdynamics to the nonclassical realm of finite wavelengths and statistical theories in a manner classical mechanics was extended to wave mechanics. The present formulation provides a "classical limit" to which gasdynamic wave mechanics must reduce in the zero wavelength limit.

Further, they ray formulation displays a familiar operator formalism, somewhat a surprise in a highly nonlinear theory. It is shown in Section 3 that such a formalism is intimately related to and determined by the gasdynamic wave surfaces or by the internal fluid boundaries. A set of rules, allowing for a formal passage from differential to algebraic relations, and vice versa, was developed rigorously as a consequence of ray formulation based on the theory of characteristics. The results will be useful as guides in the development of a statistical gasdynamics based on probability distributions of gasdynamic waves.

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