Arch. Mech., 34, 5-6, pp. 621-631, Warszawa 1982

Hamiltonian vortex models in the theory of turbulence

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THE WIDE-SPREAD approach in statistical theory of turbulence is the one using an application of statistical or field theoretical methods to dynamic systems generated by Fourier transform of Navier-Stokes equations. In this report the possibilities of another approach are studied. In our consideration the initial dynamic system describes the behaviour of small (point) vortices, approximating an instantaneous vorticity field.

Szeroko rozpowszechnione podejście w statystycznej teorii turbulencji polega na zastosowaniu mechaniki statystycznej lub metod teorii pola do układów dynamicznych generowanych przez transformacje fourierowskie z równań Naviera-Stokesa. W pracy omówiono możliwości zastosowania innego podejścia. W naszym ujęciu początkowy układ dynamiczny opisuje zachowanie się małych (punktowych) wirów przybliżających chwilowe pole wirowości.

Широко распространенный в статистической теории турбулентности подход заключается в применении методов статистической механики или теории поля к динамическим системам, полученным преобразованием Фурье из уравнений Навье-Стокса. В работе обсуждены возможности применения другого подхода. В нашем подходе исходная динамическая система описывает поведение малых (точечных) вихрей, приближающих мгновенное поле завихренности.

IN RECENT years in the works of a number of authors [2-7] a system of straightline vortex filaments approximating an instantaneous random field of vorticity has been considered as a statistical model of two-dimensional hydrodynamic turbulence. ONSAGER [1] was the first who pointed out the possibility of such an approach. One can assume that in the limit of large Reynolds numbers, when the domains of vorticity are relatively small and have the character of random impregnations in the potential flow, this approach in general outline reflects correctly the picture of turbulent fluid. Such a model is an alternative of the spectral models spread in the theory of turbulence.

The dynamics of vortex lines admits a description in the Hamiltonian formalism that allows one to apply the methods of the kinetic theory to the statistical ensemble of vortices. In this way a number of results has been obtained, which confirm the pithiness of the present model. Thus, in the works [3, 5] the possibility of realization of energy and entrophy cascades in the system of straightline vortex filaments was shown on the basis of numerical modelling. In a number of works by A. I. Chorin et al. a system of vortex filaments has been successfully used for modelling the plane flows of an incompressible fluid at large Reynolds numbers.

The obtained results stimulate further study of the model of straightline vortex filaments as well as the development of a rational model of the point vortices for the three-dimensional case.

In the present paper a closed evolutional equation for the vorticity distribution function is deduced and qualitative properties of its solution are studied proceeding from the Liouville equation for an ensemble of straightline vortex filaments with the use of the Prigogine–Balescu's diagram technique. Some results for the statistical model of the three-dimensional turbulence are also obtained on the basis of a dynamic system of small spatial vortices in an ideal fluid introduced in [9, 10].

1.

Equations of the motion of the system of the N straightline vortex filaments of equal circulation intensity $\varkappa$ have the form [11]

$$(1.1)\quad \frac{d\bar{r}_i}{dt} = -\varkappa\bar{e}\times\bar{\nabla}_i H_N, \quad H_N = -\sum_{i<j}^{N} V_{ij}, \quad V_{ij} = \ln|\bar{r}_i-\bar{r}_j|, \quad i,j = 1, \ldots, N.$$

$\bar{r}_i = (x_i, y_i)$ is the radius-vector of the i-th vortex, $\bar{\nabla}_i \equiv \partial/\partial\bar{r}_i$, $\bar{e}$ is the unit vector of the normal to the plane of the vortices motion.

An equivalent statistical description of the motion of the system (1.1) is given by the Liouville equation [12]

$$(1.2)\qquad \frac{\partial f_N}{\partial t} = -\varkappa\bar{e}\times\sum_{i<j}^{N}\bar{\nabla}_i V_{ij}(\bar{\nabla}_i-\bar{\nabla}_j)f_N = -\varkappa\sum_{i<j}^{N} L_{ij}f_N \equiv -\varkappa L f_N,$$

where $f_N(\bar{r}_1, \ldots, \bar{r}_N, t)$ is the probability density function of N vortices with the normalization

$$(1.3)\qquad \int d\bar{r}_1 \ldots d\bar{r}_N f_N = 1.$$

The Hamiltonian H_N in Eq. (1.1) contains only the terms of the form V_{ij} and the system of vortex filaments is strongly interacting. This fact excludes a direct application of ordinary methods of statistical mechanics [12, 13] to Eq. (1.2).

Let us make use of the Prigogine-Balescu's method modification [14] in the version of the resolvent formalism [13]. A system of notations coincides with the one adopted in [13]. Fourier analysis in the class of the periodical in square of the area Ω functions is an original apparatus. For the distribution function of N vortices f_N as well as the potential of intervortex interaction let us represent the expansion in Fourier series in the form

$$(1.4)\qquad f_N(\bar{r}_1, \ldots, \bar{r}_N) = \Omega^{-N}\left[\varrho_0+\frac{4\pi^2}{\Omega}\sum_{j=1}^{N}\sum_{\bar{k}}{}' \varrho(\bar{k}, t)e^{i\bar{k}\bar{r}_j}+ \ldots\right],$$

$$(1.5)\qquad V(|\bar{r}_m-\bar{r}_j|) = 4\pi^2\Omega^{-1}\sum_{\bar{l}} V(|l|)e^{-i\bar{l}(\bar{r}_m-\bar{r}_j)},$$

$\bar{k}_j = 2\pi\Omega^{-1/2}\bar{n}_j$; $\bar{n}_j$ is the integer vector. The normalization condition (1.3) yields $\varrho_0 = 1$. Fourier-coefficients in the expansion (1.4) are related with the S-vortex distribution functions

$$f_S(\bar{r}_1, \ldots, \bar{r}_s, t) = \frac{N!}{(N-s)!}\int d\bar{r}_{s+1} \ldots d\bar{r}_N f_N.$$

In particular,

$$f(\bar{r}, t) = c\Big(1+\int d\bar{k}\, e^{i\bar{k}\bar{r}}\varrho(\bar{k}, t)\Big), \quad c = \lim_{N,\Omega\to\infty}\frac{N}{\Omega}.$$

By using Eqs. (1.4) and (1.5) one can obtain Fourier-representation of the formal solution of the Liouville equation (1.2) in the form of a series of the perturbation theory:

$$\text{(1.6)}\quad \varrho(\{\bar{k}\}, t) = (2\pi)^{-1}\oint dz\, e^{-izt}\sum_{n=0}^{\infty}(-\varkappa)^n\sum_{\{\bar{k}'\}}\left(\frac{4\pi^2}{\Omega}\right)^{\nu'-\nu} < \{\bar{k}\}\,|R_0(z)$$
$$\times [LR_0(z)]^n|\,\{\bar{k}'\} > \varrho(\{\bar{k}'\}, 0).$$

Matrix elements in Eq. (1.6) are determined on the basis of the plane waves. In particular,

$$\text{(1.7)}\quad \begin{aligned} &\langle\{\bar{k}\}\,|L|\,\{\bar{k}'\}\rangle = \sum_{n<j}\langle\{\bar{k}\}\,|L_{n,j}|\,\{\bar{k}'\}\rangle,\\ &\langle\{\bar{k}\}\,|L_{n,j}|\,\{\bar{k}'\}\rangle = 4\pi^2\Omega^{-1}\bar{e}\times i(\bar{k}_n'-\bar{k}_n)V(|k_n'-\bar{k}_n|)\,i(\bar{k}_n'-\bar{k}_n)\\ &\qquad\times\delta_{\bar{k}_n'+\bar{k}_j'-\bar{k}_n-\bar{k}_j}\cdot\prod_{p\neq n,j}\delta_{\bar{k}_p'-\bar{k}_p}.\end{aligned}$$

The matrix element of the undisturbed resolvent $R_0(z)$ in the given case is trivial:

$$\text{(1.8)}\quad \langle\{\bar{k}\}\,|R_0(z)|\,\{\bar{k}'\}\rangle = -\frac{1}{iz}\delta_{\{\bar{k}\}-\{\bar{k}'\}}.$$

The structure (1.6), (1.7), (1.8) enables one to use for graphic representations of these relations the same diagrams as in [13]. It follows from the expression (1.7) for the matrix element of the interaction operator that in the given case (Cf. [13]) there are three non-trivial one-vertex diagrams (Fig. 1).

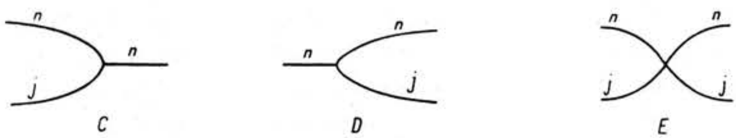

FIG. 1.

The corresponding matrix elements have the form

C. $\langle\bar{k}_n, \bar{k}_j|L_{nj}|\bar{k}_n'\rangle = 4\pi^2\Omega^{-1}\bar{e}\times i(\bar{k}_n'-\bar{k}_n)V(|\bar{k}_n'-\bar{k}_n|)\,i\bar{k}_n'\,\delta_{\bar{k}_n'-\bar{k}_n-\bar{k}_j}$,

D. $\langle\bar{k}_n|L_{nj}|\bar{k}_n', \bar{k}_j'\rangle = 4\pi^2\Omega^{-1}\bar{e}\times i(\bar{k}_n'-\bar{k}_n)V(|\bar{k}_n'-\bar{k}_n|)\,i(\bar{k}_n'-\bar{k}_{ij}')\,\delta_{\bar{k}_n'-\bar{k}_j-\bar{k}_n}$,

E. $\langle\bar{k}_n, \bar{k}_j|L_{nj}|\bar{k}_n', \bar{k}_j'\rangle = 4\pi^2\Omega^{-1}\bar{e}\times i(\bar{k}_n'-\bar{k}_n)V(|\bar{k}_n'-\bar{k}_n|)\,i(\bar{k}_n-\bar{k}_j')\,\delta_{\bar{k}_n'+\bar{k}_j'-\bar{k}_n-\bar{k}_j}$.

In virtue of these relations as well as the representation (1.6), the topological indices of the vertices C, D, E are equal to 0, 1, 0, respectively, [18].

Consider characteristic times for the given model. As the two-dimensional turbulence is simulated, one can take the rates ε and η in energy and entrophy cascades [15] as characteristics of the stochastization processes. The combination of parameters $\varkappa$, c, η and ε

yields a dimensionless complex $\Gamma = c^{1/2}\eta^{-1/2}\varepsilon^{1/2}$. The values having the time dimension are represented in the form

$$T_m = \varepsilon^{1/2}(\varkappa^2 c\eta)^{-1/2}\Gamma^m. \tag{1.9}$$

For $m = -1$ from Eq. (1.9) we have $\tau_i = (\varkappa c)^{-1}$ and for $m = 0$, $\tau_r = (\varkappa^2 c)^{-1/2}\varepsilon^{1/2}\eta^{-1/2}$. The time τ_i is the period of circulation of a pair of vortices at a mean distance $c^{-1/2}$ between them and naturally is identified with the characteristic time of interaction. The time τ_r containing rates of cascade processes can be considered as the characteristic time of relaxation. The character of dependence τ_i, τ_r on $\varkappa$ and c was used for the choice of interactions from Eq. (1.6) determining the convective transfer and relaxation.

Let us consider all possible contributions to $\varrho(\bar{k}, t)$ of the order $(\varkappa c)^n(\varkappa^2 c)^m$, $n, m = 0.1, \ldots$. The corresponding diagram representation has the form Fig. 2. Here the rectangles denote the infinite sums of all possible prolongations made up from the vertices of the type D and "pseudodiagonal" fragments [13] Fig. 3.

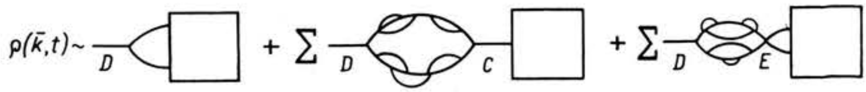

FIG. 2.

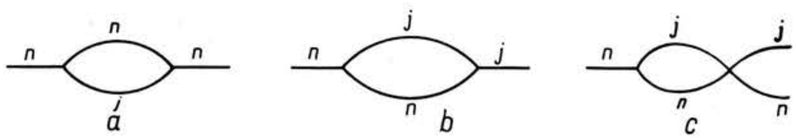

FIG. 3.

In the second and third groups of contributions the external sums are taken over all possible "head" diagrams. A direct summation of contributions (Fig. 2) by the methods [13, 16] leads to an equation reversible in the time which does not describe the relaxation process to the stationary state and contains diverging terms of the order $0(t)$ at $t \to \infty$. The appearance of divergence is associated with an attempt to describe a strongly interacting system by a succession of binary processes divided in time. The divergence can be suppressed if the collective character of the interaction of vortices is taken into account. This can be done by means of the renormalization of the Green's function (propagator) [14] for which a closed equation arises thereat.

Let us consider the following diagram representation of the renormalized propagator (Fig. 4). Here are considered the contributions of all possible fragments connected in an arbitrary number, which are formed by introducing the elementary diagrams (Fig. 3a, b) between the vertices of the type D and C.

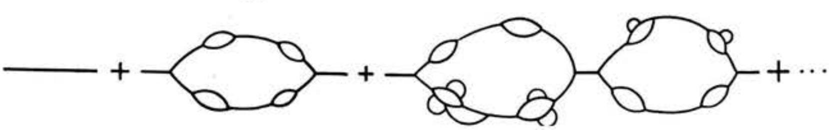

FIG. 4.

The corresponding analytical expression of the operator series has the form

$$(1.10)\quad G(\bar{k}, t) = (2\pi)^{-1} \oint dz\, e^{-izt} \frac{1}{-iz} \sum_{n=0}^{\infty} \left[\Phi(\bar{k}, z) \frac{1}{-iz} \right]^n G(\bar{k}, 0), \qquad G(k, 0) = I,$$

where I is the unity operator, and

$$\Phi(\bar{k}, z) = \sum^{\infty} (-\varkappa)^{2(m+1)} \langle k | L_D [R_0(z) L]^{2m} R_0(z) L_c | \bar{k} \rangle .$$

After differentiating Eq. (1.10) with respect to time with the use of the propagator representation (1.10) as well as the convolution theorem for the Laplace transformation, we have

$$(1.11)\quad \frac{\partial G(\bar{k}, t)}{\partial t} = (2\pi)^{-1} \oint dz\, e^{-izt} \Phi(\bar{k}, z) \frac{1}{-iz} \sum_{n=0}^{\infty} \left[\Phi(\bar{k}, z) \frac{1}{-iz} \right]^n \cdot G(\bar{k}, 0)$$

$$= \int_0^t d\theta Z(\bar{k}, t-\theta) G(\bar{k}, \theta),$$

$$(1.12)\quad Z(\bar{k}, t) = (2\pi)^{-1} \oint dz\, e^{-izt} \Phi(\bar{k}, z).$$

With the help of the Resibois' factorization theorem [17] the operator series (1.12) can be represented in the form

$$Z(\bar{k}_i, t) = \sum_{j=1}^{N} \sum_{\bar{l}} (-\varkappa)^2 \langle \bar{k}_i | L_{ij} | \bar{k}_i - \bar{l}_i, \bar{l}_j \rangle G(\bar{k}_i - \bar{l}_i, t) G(\bar{l}_j, t) \langle \bar{k}_i - \bar{l}_i, \bar{l}_j | L_{ij} | \bar{k}_i \rangle .$$

As a result, when passing to the "thermodynamic" limit $N \to \infty$, $\Omega \to \infty$, $N/\Omega \to c$ we derive the equation for the renormalized propagator

$$(1.13)\quad \frac{\partial G(\bar{k}, t)}{\partial t} = -4\pi^2 \varkappa^2 c \int_0^t d\theta P(\bar{k}, t-\theta) G(\bar{k}, \theta),$$

$$P(\bar{k}, \theta) = \int d\bar{l} [\bar{e}\, \bar{l} (\bar{k} - 2\bar{l}) V^2(|\bar{l}|) \bar{e}\, \bar{l}\, \bar{k}] G(\bar{k} - \bar{l}, \theta) G(\bar{l}, \theta).$$

After similar transformations the corresponding equation of Fig. 2 for $\varrho(\bar{k}, t)$ takes the form

$$\frac{\partial \varrho(\bar{k}, t)}{\partial t} = 4\pi^2 \varkappa^2 c \bar{e} \times \int d\bar{k}_1 i(\bar{k}_1 - \bar{k}) V(|\bar{k} - \bar{k}_1|) i(2\bar{k}_1 - \bar{k}) \varrho(\bar{k}_1, t) \varrho(\bar{k} - \bar{k}_1, t)$$

$$-4\pi^2 \varkappa^2 c \int_0^t d\theta P(\bar{k}, \theta) \varrho(\bar{k}, t-\theta) - 4\pi^2 \varkappa^2 c \int_0^t d\theta \int d\bar{k}_1 P_1(\bar{k}, \bar{k}_1, \theta)$$

$$(1.14)\quad \times \varrho(\bar{k}_1, t-\theta) \varrho(\bar{k} - \bar{k}_1, t-\theta),$$

$$P_1(\bar{k}, \bar{k}_1, \theta) = \int d\bar{l}\, \bar{e} \times \bar{l} V(|\bar{l}|) (\bar{k} - 2\bar{l}) G(\bar{k} - \bar{l}, \theta) G(\bar{l}, \theta) \bar{e} \times (\bar{k} - \bar{l}) V(|\bar{k}_1 - \bar{l}|) (2\bar{k}_1 - \bar{k}).$$

Equations (1.13) and (1.14) are of non-Markovian character and yield a closed description of the vortex evolution at times $t \sim \tau_r$. We shall assume $|\bar{k}| \sim |\bar{l}|$, i.e. the characteristic dimension of the nonuniformities and the effective length of their interaction are close what corresponds to the ideas of the phenomenological theory of turbulence. Assuming that $G(\bar{k}, t) = G(\alpha|\bar{k}|t)$ substitute into Eq. (1.13) $G(\bar{l}, \theta) \simeq G(\bar{k}, \theta)$, $G(\bar{k}-\bar{l}, \theta) \simeq I$.

Thereat Eq. (1.13) is transformed into the equation with the integral of the convolution type solving which with the help of Laplace transformation one can obtain

$$(1.15)\qquad \begin{aligned} &G(\bar{k}, t) = (\alpha|\bar{k}|t)^{-1}J_1(2\alpha|\bar{k}|t), \quad \alpha^2 = 4\pi^2\varkappa^2 ca^2,\\ &a^2 = \int d\bar{l}V^2(|\bar{l}|)[1-\cos^2(\bar{k}, \bar{l})]|\bar{l}|^2, \quad a^2 > 0. \end{aligned}$$

$J_1(x)$ is a Bessel function of the first kind of the first order. Now one can pass on in a usual manner [16] to the Markovian form (1.4). Then the contribution of the product of propagators with regard to Eq. (1.15) is

$$(1.16)\qquad \int_0^\infty d\theta G(\bar{k}-\bar{l}, \theta)G(\bar{l}, \theta) \simeq (\alpha|\bar{l}|)^{-1}.$$

Returning to the initial variables with the help of the inverse Fourier transformation

$$(1.17)\qquad \int d\bar{k}e^{i\bar{k}\bar{r}}\varrho(\bar{k}, t) = c^{-1}[f(\bar{r}, t)-c] = n(\bar{r}, t).$$

For the one-point distribution function of the vorticity in the plane case, we derive the equation

$$\frac{\partial f(\bar{r}, t)}{\partial t} + \bar{U}(\bar{r}, t)\bar{\nabla}f(\bar{r}, t) = (\varkappa^2 c)^{1/2}a_1 a^{-1}\Delta f(\bar{r}, t) + (\varkappa^2 c)^{1/2}a^{-1}\int d\bar{r}_1 n(\bar{r}_1, t)$$
$$\times \bar{\bar{B}}(\bar{r}-\bar{r}_1): \bar{\nabla}\bar{\nabla}f(\bar{r}, t) - (\varkappa^2 c)^{1/2}a^{-1}n(\bar{r}, t)\int d\bar{r}_1 \bar{\bar{B}}(\bar{r}-\bar{r}_1): \bar{\nabla}_1\bar{\nabla}_1 f(\bar{r}_1, t),$$

where $\bar{\bar{B}}(\bar{r}-\bar{r}_1)$ is the tensor of the second rank.

$$\bar{\bar{B}} = \bar{e}\times\bar{\nabla}|\bar{r}-\bar{r}_1|\bar{e}\times\bar{\nabla}V(|\bar{r}-\bar{r}_1|),$$
$$\bar{U}(\bar{r}, t) = \varkappa c\bar{e}\times\bar{\nabla}\int dr_1\, V(|\bar{r}-\bar{r}_1|)n(\bar{r}_1, t)$$

is the average hydrodynamic velocity induced by the locally noncompensated vorticity, the colon is the symbol of the scalar product of tensors, a_1 is determined analogously to a in Eq. (1.15) differing from it only by an additional multiplier $|\bar{l}|^{-1}$ in the integrand.

A substitution in a, a_1 of the generalized Fourier-image of the potential $V(|\bar{l}|) = (2\pi|l|^2)^{-1}$ leads to divergent integrals. To eliminate these divergences one can make use of techniques of the theory of Coulomb plasma [13]. As the explicit expressions a, a_1 are not needed here, we shall assume them to be simply limited.

Let us consider some qualitative properties of the equation derived. Obviously the l.h.s of Eq. (1.17) is the Helmholz operator for the vorticity field in the two-dimensional case. Since $a, a_1 > 0$, then the first term in the r.h.s. of Eq. (1.17) containing the Laplace operator determines the positive diffusion of the vorticity distribution. The quadratic form being formed by the components of the tensor $\bar{\bar{B}}(\bar{r}-\bar{r}_1)$ is

$$(1.18)\qquad (\bar{\bar{B}}\bar{\xi}, \bar{\xi}) = e_{ik}\frac{\partial|\bar{r}-\bar{r}_1|}{\partial x_k}e_{jn}\frac{\partial V(|\bar{r}-\bar{r}_1|)}{\partial x_n}\xi_i\xi_j,$$

where $\bar{r} = (x_1, x_2)$, e_{ik} is the antisymmetric tensor ($e_{12} = -e_{21} = 1; e_{11} = e_{22} = 0$) and the summation from 1 to 2 with respect to the repeated indices is implied. It is directly checked that the form (1.18) is nonnegative. While making use of the structure (1.18) and its non-negativity for continuous $n(\bar{r}, t)$, one can show that the first and second terms in the r.h.s of Eq. (1.17) determine the elliptic operator of the second order whose sign depends on the distribution $f(\bar{r}, t)$ and can be different at various points of the flow region. The sign of the latter term is determined by the local value $n(\bar{r}, t)$. This means that Eq. (1.12) is capable of describing the local processes of destruction (positive diffusion) and appearance (negative diffusion) of large vortex structures. As known from [18], the latter effect is identified with the phenomenon of negative viscosity.

Let us consider the production of information entropy

$$S(t) = -\int d\bar{r} f(\bar{r}, t) \ln f(\bar{r}, t) \tag{1.19}$$

in the process of evolution. We shall assume that $f(\bar{r}, t)$ at $|\bar{r}| \to \infty$ decreases sufficiently fast together with the derivatives. Multiply Eq. (1.17) by $-(1+\ln f(\bar{r}, t))$ and integrate over the whole space. After a series of transformations the equation of entropy balance is reduced to the form

$$\begin{aligned}\frac{dS}{dt} = &\int d\bar{r}\bar{\nabla}\bar{U}(\bar{r}, t) f(\bar{r}, t) + (\varkappa^2 c)^{1/2} a_1 a^{-1} \int d\bar{r} [f(\bar{r}, t)]^{-1} [\bar{\nabla} f(\bar{r}, t)]^2 \\ &+ \tfrac{1}{2}(\varkappa^2 c)^{1/2} a^{-1} \iint d\bar{r} d\bar{r}_1 \bar{\nabla}\bar{\nabla} : \bar{\bar{B}}(\bar{r}-\bar{r}_1) [\ln f(\bar{r}, t) - \ln f(\bar{r}_1, t)] [f(r, t) - f(\bar{r}_1, t)] \\ &+ (\varkappa^2 c)^{1/2} a^{-1} \iint d\bar{r} d\bar{r}_1 \bar{\bar{B}}(\bar{r}-\bar{r}_1) : \bar{\nabla}_1 f(\bar{r}_1, t) \bar{\nabla}_1 f(\bar{r}_1, t) [f(\bar{r}, t)]^{-1} n(\bar{r}, t) \\ &+ (\varkappa^2 c)^{1/2} a^{-1} \iint d\bar{r} d\bar{r}_1 \bar{\nabla}\bar{\nabla} : \bar{\bar{B}}(\bar{r}-\bar{r}_1) f(\bar{r}_1, t) n(\bar{r}, t).\end{aligned} \tag{1.20}$$

In virtue of the fluid incompressibility $\bar{\nabla} \cdot \bar{U}(\bar{r}, t) = 0$, the convective term makes a zero contribution to the entropy production. The positivity of the second term contribution in the r.h.s. (1.20) is obvious. The third term is positively determined, too, as

$$\bar{\nabla}\bar{\nabla} : \bar{\bar{B}}(\bar{r}-\bar{r}_1) = |\bar{r}-\bar{r}_1|^{-3} \geqslant 0 \tag{1.21}$$

and the following inequality holds:

$$[\ln f(\bar{r}, t) - \ln f(\bar{r}_1, t)] [f(\bar{r}, t) - f(\bar{r}_1, t)] \geqslant 0.$$

In virtue of the positive determinancy (1.18) and (1.21), the sign of the latter two integrals in Eq. (1.20) depends only on the behaviour of the function $n(\bar{r}, t)$. Under the condition that a total moment is equal to zero

$$\int d\bar{r} n(\bar{r}, r) = 0,$$

the estimates show that a negative contribution to these integrals will be collected mainly from the flow region periphery where the remaining parts of the integrands tend to zero sufficiently fast. Hence a complete contribution of these integrals to the entropy production will be either nonnegative or, at least, close to zero with respect to the model.

Thus, in the process of evolution $f(\bar{r}, t)$ satisfying Eq. (1.17) the entropy (1.19) increases monotonically, in the broad sense of the word. On the other hand, from convergence of the integral

$$\int d\bar{r} f(\bar{r}, t) < \infty$$

one can easily derive that the functional (1.13) is limited. This fact enables one to draw the conclusion that in the process of evolution the solution (1.17) tends to some stationary distribution.

Analysis of the results of numerical experiments [4, 6] whose conditions are close to assumptions employed here shows that Eq. (1.17) reflects correctly the character of evolution of large systems of point vortices on the plane. In particular, it can be employed for estimating the dissipative properties of the algorithms of the type [8] as it is made when analyzing the properties of the difference schemes with the help of differential approximations [19].

2.

The construction of a dynamic model of point vortices in the three-dimensional case entails great difficulties because of a more complex behaviour of the vortex fields in the spatial flows [11]. In particular it is displayed in the absence of the universal point carrier of vorticity in the three-dimensional case which is a straightline vortex filament for the plane case.

In [9] and independently in [10] it has been proposed to describe the dynamics of a system of small vortices in the three-dimensional case in terms of the canonical variables conjugate with respect to the Hamiltonian

$$\begin{gathered} H = \sum_{i=1}^{N} T(\bar{p}_i) + \sum_{i=1}^{N} \bar{V}(\bar{r}_i)\cdot\bar{p}_i + \varepsilon \sum_{i<j}^{N} \Phi_{ij}, \\ \Phi_{ij} = -\frac{1}{4\pi}\left[\frac{\bar{p}_i\bar{p}_j}{|\bar{r}_{ij}|^3} - \frac{(\bar{p}_i\cdot\bar{r}_{ij})(\bar{p}_j\cdot\bar{r}_{ij})}{|\bar{r}_{ij}|^5}\right]. \end{gathered} \tag{2.1}$$

Here $\bar{r}_i$ are physical coordinates of vortices, $\bar{p}_i$ — their LAMB's [11] momenta, $T(\bar{p}_i)$ is the energy of solitary vortices, $\bar{V}(\bar{r})$ is the velocity field of external potential flow, Φ_{ij} is the energy of interaction of separate vortices interacting as dipoles with variable dipole momenta.

As follows from [10], $\sum_{i=1}^{N} p_i^2$ is not an integral of the system (2.1) and the model suggested contains an essential (in the three-dimensional case) effect of stretching the vortex filaments. It is known [11] that the energy of a solitary vortex and its velocity depend in a complex way on the topology of vortex filaments inside it. For simplicity we shall make use of the model representation of the form([1])

$$T(\bar{p}) = A|\bar{p}|^{1/k}, \tag{2.2}$$

The dependence (2.2) is obtained by approximating the numerical results [20] for the one-parametric family of vortices. The index $k \in [5/7, 2]$ depends monotonically on the

([1]) The results of this section were obtained by the authors jointly with V. B. Levinsky.

parameter. Thereat $k = 5/7$ corresponds to a spherical Hill's vortex and $k = 2$ to the limit of thin rings [11].

If we take the ratio of the characteristic dimension of vortices to the mean distance between them proportional to the ratio of the microscale of turbulence to its integral scale, then in Eq. (2.1) $\varepsilon \ll 1$ and the methods of statistical systems with a weak interaction are applicable to the model. Here the one-point distribution function of vortices with respect to momenta and coordinates $f(\bar{p}, \bar{r}, t)$ obeys the kinetic equation

$$\frac{\partial f}{\partial t} + (\bar{u}+\bar{w}+\bar{V})\frac{\partial f}{\partial \bar{r}} - \bar{p}\left(\frac{\partial f}{\partial \bar{p}} \cdot \frac{\partial}{\partial \bar{r}}\right)(\bar{w}+\bar{u}) = J(f,f). \tag{2.3}$$

Here $\bar{u} = \partial T/\partial \bar{p}$, $\bar{V}(\bar{r})$ is the velocity of external potential flow,

$$\bar{w}(\bar{r}_1, t) = \int d\bar{r}_2\, d\bar{p}_2 \frac{\partial \Phi_{12}}{\partial \bar{p}_1} f(\bar{p}_2, \bar{r}_2, t)$$

is the velocity induced at the point by distribution of vortices.

$$J(f,f) = \frac{\partial}{\partial p_r}\int d\bar{p}_1 \frac{[\bar{p}\times\bar{u}_{12}][\bar{p}_1\times\bar{u}_{12}]}{8\pi b_0^4|\bar{u}/2|^7}[\bar{u}_{12}^2\delta_{rs} - u_{12,r}u_{12,s}]\left(\frac{\partial}{\partial p_s} - \frac{\partial}{\partial p_{1,r}}\right) f(\bar{p}) f(\bar{p}_1) \tag{2.4}$$

is the collision integral in the Landau's form [21], summation is implied with respect to repeated indices.

Let us outline the relationship between the moment equations which can be derived from Eq. (2.3) and the Reynolds equations for the averaged values of the phenomenological theory of turbulence. Within the framework of 10-moment approximation let us approximate the distribution function as

$$f(\bar{p}, \bar{r}, t) = \exp\left(-\frac{T(\bar{p})}{\langle T\rangle}\right)[a_0(\bar{r}, t) + \bar{a}_1\cdot\bar{u} + \bar{\bar{a}}_2\cdot\bar{u}\bar{u}], \tag{2.5}$$

where the exponential factor reduces to zero the integral of collisions (2.4), angular brackets mean averaging with respect to $f(\bar{p}, \bar{r}, t)$.

Coefficients in Eq. (2.5) can be expressed by the following averaged values:

$$n(\bar{r}, t) = \int d\bar{p} f(\bar{p}, \bar{r}, t), \qquad \langle\bar{p}\rangle = n^{-1}\int d\bar{p}\bar{p} f(\bar{p}, \bar{r}, t),$$

$$\langle\bar{u}\rangle = n^{-1}\int d\bar{p}\bar{u} f(\bar{p}, \bar{r}, t), \qquad \langle\bar{\bar{P}}\rangle = n^{-1}\int d\bar{p}\bar{u}\bar{p} f(\bar{p}, \bar{r}, t).$$

For the components $\langle\bar{p}\rangle$ and $\langle\bar{\bar{P}}\rangle$ the following system of equations is obtained:

$$\frac{\partial n\langle p_i\rangle}{\partial t} + \frac{\partial}{\partial x_j}[n(\langle P_{ij}\rangle + \langle U_i\rangle\langle p_j\rangle)] + n\langle p_j\rangle\frac{\partial\langle U_j\rangle}{\partial x_i} = 0,$$

$$\frac{\partial n\langle P_{ij}\rangle}{\partial t} + \langle U_k\rangle\frac{\partial n\langle P_{ij}\rangle}{\partial x_k} + \frac{3k+1}{5k}\left\{\delta_{ij}\frac{\partial}{\partial x_k}\langle P\rangle\langle u_k\rangle + \frac{\partial}{\partial x_i}\langle P\rangle\langle u_j\rangle + \frac{\partial}{\partial x_j}\langle P\rangle\langle u_i\rangle\right\} + \frac{n}{2k}\left(\langle P_{jk}\rangle\frac{\partial\langle U_k\rangle}{\partial x_i} + \langle P_{ik}\rangle\frac{\partial\langle U_k\rangle}{\partial x_j}\right) + \frac{n\langle P_{ij}\rangle}{\tau} = 0, \tag{2.6}$$

$$\langle\bar{U}\rangle = \langle\bar{w}+\bar{V}\rangle, \qquad P = \frac{1}{3}(P_{11}+P_{22}+P_{33}).$$

The latter term in the second equation (2.6) is obtained by integrating the integral of collisions.

Assuming formally τ to be a small parameter in the first approximation from the equation for $\langle P_{ij}\rangle$, one can derive

$$\langle P_{ij}\rangle = -\frac{1}{2k}\langle P\rangle\tau\left(\frac{\partial\langle U_i\rangle}{\partial x_j}+\frac{\partial\langle U_j\rangle}{\partial x_i}\right). \tag{2.7}$$

If we determine the averaged vorticity as

$$\langle\bar{\Omega}\rangle = \mathrm{rot}\langle\bar{U}\rangle = \mathrm{rot}\bar{w} = \mathrm{rot}\, n\langle\bar{p}\rangle,$$

then from the last relations it is seen that the performation of the operation rot from the first equation (2.6) and the substitution into it Eq. (2.7), lead to the equation for $\langle\bar{\Omega}\rangle$ in the simplest approximation of the scalar turbulent viscosity.

Let us show that in the framework of the present model the Kolmagorov–Richardson's cascade is possible. As we are dealing with the point model, it is necessary to obtain the relationship between the momentum and the dimension of vortices. From the invariantness of equations of motion with the Hamiltonian (2.1) with respect to stretching of the impulse space, we have

$$p \sim \frac{3}{r^{2-\beta}}, \quad \beta = 1/k. \tag{2.8}$$

It is seen that in the limiting cases of thin rings and Hill's vortex (2.8), the requested dependence is exactly reproduced.

In virtue of the physical content of the model, the cascade process arises here not on the basis of splitting up the vortices (three-wave processes [22]) but by their stretching in the process of a pair (four-wave) interaction.

In [23] a stationary solution of kinetic equations with four-wave interactions has been constructed,

$$f(\bar{p}) = B|\bar{p}|^s, \quad s = -\frac{1}{2}(3d+m), \tag{2.9}$$

which corresponds to a constant energy flow through the momentum space; here d is the dimension of the momentum space, m is the index of homogeneity of the scattering function $U(\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{p}_4)$ in the collision integral.

Writing down for the scattering function the identity of the form

$$\int d\sigma U_{12} = \int d\bar{p}_3\, d\bar{p}_4\, U(\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{p}_4)\,\delta(\bar{p}_1+\bar{p}_2-\bar{p}_3-\bar{p}_4)\,\delta[T(\bar{p}_1)+T(\bar{p}_2)-T(\bar{p}_3)-T(\bar{p}_4)]$$

and fulfilling the scale transformation $\bar{p}_i \to \lambda\bar{p}_i$ with regard to Eq. (2.8), we obtain in our case

$$m = -8/3+4/3\beta, \quad s = -\frac{1}{2\cdot 3}(19+4\beta). \tag{2.10}$$

By using Eq. (2.9) and (2.10) and passing on to the wave vectors we can write down the expression for the energy spectrum

$$E(k) \sim T(k)f(k)p^2(k)\left|\frac{dp}{dk}\right| \sim k^{-\frac{3}{2(2-\beta)}}, \quad k = |\bar{k}|,$$

then it follows that $E(k) \sim k^{-5/3}$ at $\beta_0 = 11/10$, $\beta_0 \in [1/2, 7/5]$. It has been shown that at $\beta = \beta_0$ the integral of collisions is converged to the solution (2.9)–(2.10).

The authors hope that further studies of the models considered in the paper by the methods of nonequilibrium statistical mechanics will enable one to derive new structural relations for the processes of the turbulent transfer. The model of vortices with dipole interaction can also be useful when generalizing for the three-dimensional case the non-grid algorithms of the type [8].

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Received October 22, 1981.