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Hydrodynamic interactions between falling solid particles of different densities(*)

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THE VELOCITY of sedimentation of a suspension of rigid spheres in a viscous fluid is calculated. Hydrodynamic interactions between a large number of spheres are taken into account for a small volume concentration and homogeneous distribution of the spheres. A probabilistic approach introduced by G. K. Batchelor is used. Results are obtained for spheres of different densities and equal radii.

Obliczono prędkości osadzania się zawiesiny sztywnych kulek w cieczy lepkiej. Uwzględniono oddziaływanie między dużą liczbą kulek przy założeniu małej koncentracji objętościowej i jednorodnego rozkładu kulek. Zastosowano podejście probabilistyczne G. K. Batchelora. Otrzymano wyniki dla kulek o różnej gęstości i stałej średnicy.

Вычислена скорость оседания взвеси жестких шариков в вязкой жидкости. Учтены взаимодействия между большим количеством шариков, при предположении малой объемной концентрации и однородного распределения шариков. Применен пробабилистических подход Г. К. Бэтчелора. Получены результаты для шариков с разными плотностями и постоянным диаметром.

1. Introduction

WE CONSIDER the effects of hydrodynamic interactions upon the sedimentation of spherical particles in a fluid. The particles are assumed to be small enough for the creeping flow equations to be valid. Moreover, the particles are assumed to be solid, so that the no-slip condition applies on their surface.

The problem of sedimentation of identical spheres was treated by BATCHELOR [1] using a probabilistic approach that he developed for that purpose. The sedimentation of drops of different radii, but equal densities, using Batchelor's method, was calculated by HABER and HETSRONI [6].

In the present paper we are concerned with the opposite case of solid spheres of equal radius, say a, but different densities. The full calculations to which this paper refers are part of F. FEUILLEBOIS' thesis [5].

2. The results for two spheres

Before considering the many spheres problem, we expose the results for two spheres, which will be used in the rest of the paper. The problem of creeping flow around two

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spheres can be separated into several problems (i.e. motion parallel to the line of centers of the spheres in the same direction, plus motion in the opposite directions, plus motion perpendicular to the line of centers, ...) due to the linearity of the Stokes equations of fluid motion.

The separated problems have been considered by many authors. We may refer to, for example, WACHOLDER and SATHER [9] who have used these different results to calculate the trajectories of one sphere relative to another. The case that they consider is general, in the sense that the spheres have different radii and different densities. It happens that different types of trajectory occur: either open trajectories where one sphere is coming from infinity, passing near the other sphere, and going to infinity in the other direction; or closed trajectories where the sphere number 1 cannot escape from a vicinity of the sphere number 2, in despite of the fact that both may have different velocities of sedimentation when falling is isolation.

This last situation may occur only for certain ranges of the ratio of densities and of radii (see WACHOLDER and SATHER's paper). For the case of interest to us, i.e. the same radius, the trajectories are all open, which will allow us to calculate the probability distribution function, as explained later. It happens that it is easier for the probabilistic calculation to use the velocities of the spheres expressed in terms of the applied forces, when the applied couples are zero,

$$\begin{aligned} v_{p_1\parallel} &= \frac{1}{6\pi a\mu}[A_{11}F_{1\parallel}+A_{12}F_{2\parallel}], \\ v_{p_1\perp} &= \frac{1}{6\pi a\mu}[B_{11}F_{1\perp}+B_{12}F_{2\perp}], \end{aligned} \tag{2.1}$$

$$\begin{aligned} v_{p_2\parallel} &= \frac{1}{6\pi a\mu}[A_{21}F_{1\parallel}+A_{22}F_{2\parallel}], \\ v_{p_2\perp} &= \frac{1}{6\pi a\mu}[B_{21}F_{1\perp}+B_{22}F_{2\perp}]. \end{aligned} \tag{2.2}$$

The coefficients are then called "mobility coefficients". We have written them in the form developed by BATCHELOR [3]. The index $\parallel$ (resp. $\perp$) denotes the motion parallel (resp. perpendicular) to the line of centers of the spheres.

The index p_1 (resp. p_2) stands for particle 1 (resp. particle 2). By symmetry

$$A_{11} = A_{22}, \quad A_{12} = A_{21}, \quad B_{11} = B_{22}, \quad B_{12} = B_{21}. \tag{2.3}$$

The coefficients A_{11}, A_{12} for the motions parallel to the line of centers are found from:

(i) the friction coefficient calculated for both spheres moving in the same direction by STIMSON and JEFFERY [8] the inverse of which is

$$A_{11}+A_{12} = \left[\frac{4}{3}\sinh\alpha\sum_{n=1}^{\infty}\frac{n(n+1)}{(2n-1)(2n+3)}\left\{1-\frac{4\sinh^2\left(n+\frac{1}{2}\right)\alpha-(2n+1)^2\sinh^2\alpha}{2\sinh(2n+1)\alpha+(2n+1)\sinh 2\alpha}\right\}\right]^{-1} \tag{2.4}$$

with α given by

$$\cosh\alpha = \frac{r}{2a} \tag{2.5}$$

and r is the center to center distance;

(ii) the friction coefficient calculated by BRENNER [4] for two spheres moving towards each other (in fact it is calculated for the equivalent problem of a sphere moving towards a plane free surface); the inverse of this coefficient is

$$A_{11}-A_{12} = \left[\frac{4}{3}\sinh\alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left\{\frac{4\cosh^2\left(n+\frac{1}{2}\right)\alpha+(2n+1)^2\sinh^2\alpha}{2\sinh(2n+1)\alpha-(2n+1)\sin 2\alpha}-1\right\}\right]^{-1}. \tag{2.6}$$

The coefficients B_{11}, B_{12} cannot be given in such a closed form. A few values of the friction coefficients have been calculated by O'NEILL and MAJUMDAR [7], with the use of a computer, and the corresponding values of B_{11}, B_{12} are given by BATCHELOR [3].

For intermediate ranges of the separation distance between the two spheres, we had to interpolate between these values. For a vanishingly small gap between the spheres, A_{11}, A_{12} are regular, whereas B_{11}, B_{12} vary fast, with a logarithmic singularity.

For a large separation distance between both spheres, the results from the method of reflexions given by BATCHELOR [3] are

$$\begin{aligned} A_{11} &= 1-\frac{15}{4}\left(\frac{a}{r}\right)^4+O\left(\frac{a}{r}\right)^6, \\ A_{12} &= \frac{3}{2}\frac{a}{r}-\left(\frac{a}{r}\right)^3+O\left(\frac{a}{r}\right)^7, \\ B_{11} &= 1+O\left(\frac{a}{r}\right)^6, \\ B_{12} &= \frac{3}{4}\frac{a}{r}+\frac{1}{2}\left(\frac{a}{r}\right)^3+O\left(\frac{a}{r}\right)^7. \end{aligned} \tag{2.7}$$

3. Statistics for many spheres

For a sphere A falling in isolation, the limit sedimentation velocity is given from the Stokes drag, balanced with weight and buoyancy:

$$\mathbf{v}_{pA_s} = \frac{2a^2(\varrho_{pA}-\varrho)\mathbf{g}}{9\mu}, \tag{3.1}$$

where ϱ_{pA} is the density of sphere A and ϱ is the fluid density; a is the sphere radius, $\mathbf{g}$ is the gravity.

For many spheres, multiple interactions may modify the Stokes drag and hence the limit sedimentation velocity.

These interactions may be either direct actions of other spheres onto the sphere A "test" considered, or indirect actions of other spheres such as back action from another sphere, or chain actions between several spheres.

In this paper we will assume that the spheres are uniformly distributed in the fluid, and that their volume concentration is low, so that no cluster of more than two spheres is present. We will use statistics, assuming that there are large numbers of spheres:

$N_A \gg 1$ spheres A of density ϱ_{pA} and radius a,
$N_B \gg 1$ spheres B of density ϱ_{pB} and the same radius a.

We assume, moreover, that the spheres are in a large container, and that wall effects are negligible, that the spheres rotate freely (i.e. no external couple is applied to them), and that no force other than that due to gravity (buoyancy plus weight) and fluid drag is exerted on the spheres. We neglected here, in particular, any Brownian force.

The statistics we use is basically that defined by BATCHELOR [1], except that it is generalized here to the case of two types of spheres. We define the probability of existence of a configuration near the configuration $\mathscr{C}_N$ as

$$(3.2)\qquad P(\mathscr{C}_N)d\mathscr{C}_N = P(\mathbf{x}+\mathbf{r}_{A_1}, \mathbf{x}+\mathbf{r}_{A_2}, \ldots, \mathbf{x}+\mathbf{r}_{A_{N_A}}, \mathbf{x}+\mathbf{r}_{B_1}, \mathbf{x}+\mathbf{r}_{B_2}, \ldots, \mathbf{x}+\mathbf{r}_{B_{N_B}})d^3r_{A_1}d^3r_{A_2}\ldots d^3r_{A_{N_A}}d^3r_{B_1}d^3r_{B_2}\ldots d^3r_{B_{N_B}}.$$

This means the probability that, simultaneously,

the center of sphere A_1 is in the small volume $d^3r_{A_1}$ in the vicinity of $\mathbf{x}+\mathbf{r}_{A_1}$;
the center of sphere A_2 is in the small volume $d^3r_{A_2}$ in the vicinity of $\mathbf{x}+\mathbf{r}_{A_2}$

(... ...);

the center of sphere B_{N_B} is in the small volume $d^3r_{B_{N_B}}$ in the vicinity of $\mathbf{x}+\mathbf{r}_{B_{N_B}}$.
Here

$$N = N_A+N_B.$$

We write the normalization condition

$$(3.3)\qquad \frac{1}{N_A!\,N_B!}\int_v P(\mathscr{C}_N)d\mathscr{C}_N = 1$$

by integrating each $\mathbf{r}$ over the large volume v containing all spheres, and remarking that spheres A (resp. B) can be interchanged without changing the flow field.

We define also the conditional probability ([1])

$$(3.4)\qquad P(\mathscr{C}_N|\mathbf{x}_0) = \frac{P(\mathscr{C}_{N+1})}{P(\mathbf{x}_0)},$$

where $P(\mathbf{x}_0)$ is the probability to find a sphere about $\mathbf{x}_0$; this probability is uniform by assumption.

The average sedimentation velocity of a "test" sphere, say "A" sphere, located at $\mathbf{x}_0$, is defined as

$$(3.5)\qquad \overline{\mathbf{v}_{pA}(\mathbf{x}_0)} = \frac{1}{N_A!\,N_B!}\int_v \mathbf{v}_{pA}(\mathbf{x}_0, \mathscr{C}_N)P(\mathscr{C}_N|\mathbf{x}_0)d\mathscr{C}_N.$$

Our purpose is to calculate this average velocity.

([1]) The P are densities of probability but we will call them only "probabilities" for simplification.

4. The reduction to two spheres

The formula (3.5) is quite general, and in order to simplify it and take on further the calculation, we may use the assumption that only two spheres may be in the same vicinity.

Considering only two spheres, the test A sphere at $\mathbf{x}_0$ plus a second sphere (either A or B) at $\mathbf{x}_0+\mathbf{r}$, one would obtain, in place of Eq. (3.5)

$$\overline{\mathbf{v}_{pA}(\mathbf{x}_0)} \overset{?}{=} \int_v \mathbf{v}_{pA}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}) P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0) d^3r. \tag{4.1}$$

The formula has a question mark because we may ask whether the integral written in this way is convergent.

In fact it is not, as can be seen from the mobility coefficients for large r, Eq. (2.7), which contain terms in $1/r$, $1/r^3$, and also from the fact that

$$P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0) \to P(\mathbf{x}_0+\mathbf{r}) = \text{const} \quad \text{for} \quad r \to \infty, \tag{4.2}$$

i.e. that spheres do not interact when well separated. The remedy to this situation has been given by BATCHELOR [1]. The idea is that the influence of other spheres of the suspension should be taken into account (two spheres are not enough) by specifying the average velocity at any point of the suspension.

We take the frame of reference such that this average velocity is identically zero:

$$\begin{aligned} \frac{1}{N_A!N_B!} \int_v \mathbf{v}_\infty(\mathbf{x}_0, \mathscr{C}_N) P(\mathscr{C}_N) d\mathscr{C}_N = 0, \\ \frac{1}{N_A!N_B!} \int_v \nabla^2 \mathbf{v}_\infty(\mathbf{x}_0, \mathscr{C}_N) P(\mathscr{C}_N) d\mathscr{C}_N = 0, \end{aligned} \tag{4.3}$$

where $\mathbf{v}_\infty(\mathbf{x}_0, \mathscr{C}_N)$ is the velocity at a point $\mathbf{x}_0$ located either in the fluid or in a particle, due to all N spheres in the $\mathscr{C}_N$ configuration. We write first the expression (3.5) to be calculated in the form

$$\overline{\mathbf{v}_{pA}(\mathbf{x}_0)} = \frac{1}{N_A!N_B!} \int_v \left\{ \mathbf{v}_{pA_s} + \mathbf{v}_\infty(\mathbf{x}_0, \mathscr{C}_N) + \frac{a^2}{6}[\nabla^2 \mathbf{v}_\infty(\mathbf{x}, \mathscr{C}_N)]_{\mathbf{x}-\mathbf{x}_0} + \mathbf{w}(\mathbf{x}_0, \mathscr{C}_N) \right\} P(\mathscr{C}_N|\mathbf{x}_0) d\mathscr{C}_N. \tag{4.4}$$

We have written it this way because for two spheres the quantity between braces reduces then to the Faxen formula (containing all terms in $1/r$, $1/r^3$), plus an interaction term which is $O(1/r^4)$.

Combining Eqs. (4.3) with (4.4), we may obtain terms of the form

$$\mathbf{v}_\infty(\mathbf{x}_0, \mathscr{C}_N)[P(\mathscr{C}_N|\mathbf{x}_0) - P(\mathscr{C}_N)]$$

under the integral sign.

The quantity between brackets vanishes for spheres far apart, and permits the simplified expression for two spheres

$$\mathbf{v}_\infty(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})[P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0) - P(\mathbf{x}_0+\mathbf{r})]$$

to be integrable.

The expression for the average velocity of sedimentation is then calculated from Eqs. (4.3) and (4.4) to be

$$\overline{\mathbf{v}_{pA}(\mathbf{x}_0)} = \mathbf{v}_{pA_s} + \bar{\mathbf{v}}' + \bar{\mathbf{v}}'' + \bar{\mathbf{w}} \tag{4.5}$$

with

$$\begin{aligned}
\bar{\mathbf{v}}' &= \int\limits_{v,\mathbf{x}_0 \in \text{fluid}} \mathbf{v}_\infty(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}_A)[P(\mathbf{x}_0+\mathbf{r}_A|\mathbf{x}_0) - P(\mathbf{x}_0+\mathbf{r}_A)]d^3r_A \\
&\quad + \int\limits_{v;\mathbf{x}_0 \in \text{fluid}} \mathbf{v}_\infty(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}_B)\,[P(\mathbf{x}_0+\mathbf{r}_B|\mathbf{x}_0) - P(\mathbf{x}_0+\mathbf{r}_B)]d^3r_B - c_A\bar{\mathbf{v}}_{pA} - c_B\bar{\mathbf{v}}_{pB}; \\
\bar{\mathbf{v}}'' &= \int\limits_{v,\mathbf{x}_0 \in \text{fluid}} \frac{a^2}{6}[\nabla^2 v_\infty(\mathbf{x}, \mathbf{x}+\mathbf{r}_A)]_{\mathbf{x}=\mathbf{x}_0}[P(\mathbf{x}_0+\mathbf{r}_A|\mathbf{x}_0) - P(\mathbf{x}_0+\mathbf{r}_A)]d^3r_A \\
&\quad + \int\limits_{v;\mathbf{x}_0 \in \text{fluid}} \frac{a^2}{6}[\nabla^2\mathbf{v}_\infty(\mathbf{x}, \mathbf{x}+\mathbf{r}_B)]_{\mathbf{x}=\mathbf{x}_0}[P(\mathbf{x}_0+\mathbf{r}_B|\mathbf{x}_0) - P(\mathbf{x}_0+\mathbf{r}_B)]d^3r_B + \frac{1}{2}c_A\mathbf{v}_{pA_s} + \frac{1}{2}c_B\mathbf{v}_{pB_s}.
\end{aligned} \tag{4.6}$$

The second particle is either a particle A (at $\mathbf{x}_0+\mathbf{r}_A$) or a particle B (at $\mathbf{x}_0+\mathbf{r}_B$).

The integrated terms in Eq. (4.6) contain c_A, c_B which are the volume concentrations of particles A (resp. B). These terms originate from integration of Eq. (4.3) for situations where the point $\mathbf{x}_0$ is located in the particles (A or B). The expression for $\bar{\mathbf{w}}$ results from straightforward integration of the interaction term $\mathbf{w}$ in Eq. (4.4) which decreases fast enough for $\mathbf{w}P$ to be integrable

$$\bar{\mathbf{w}} = \int\limits_v \mathbf{w}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}_A)P(\mathbf{x}_0+\mathbf{r}_A|\mathbf{x}_0)d^3r_A + \int\limits_v \mathbf{w}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}_B)P(\mathbf{x}_0+\mathbf{r}_B|\mathbf{x}_0)d^3r_B. \tag{4.7}$$

5. Calculation of the conditional probability

The conditional probability to find a sphere A at $\mathbf{x}_0+\mathbf{r}_A$ when a sphere test A is fixed at $\mathbf{x}_0$ is written using the uniformity assumption, plus the condition that both spheres do not overlap:

$$P(\mathbf{x}_0+\mathbf{r}_A|\mathbf{x}_0) = \begin{cases} 0 & \text{if} \quad r_A < 2a, \\ n_A & \text{if} \quad r_A > 2a, \end{cases} \tag{5.1}$$

where n_A is the number of spheres A per unit volume of the mixture. The conditional probability to find a sphere B at $\mathbf{x}_0+\mathbf{r}_B$ when a sphere A test is fixed at $\mathbf{x}_0$ is more complicated, as both spheres have a relative motion which may influence this probability:

$$P(\mathbf{x}_0+\mathbf{r}_B|\mathbf{x}_0) = \begin{cases} 0 & \text{if} \quad r_B < 2a, \\ n_B p(\mathbf{r}_B) & \text{if} \quad r_B > 2a. \end{cases} \tag{5.2}$$

Here we introduce $p(\mathbf{r}_B)$ which has to be calculated from the hydrodynamic interactions between both spheres in relative motion. For convenience, we will drop the subscript B in this calculation.

The equation satisfied by the conditional probability is a Fokker–Planck type equation, describing the continuity of the cloud of all possible positions of the center of the B sphere in the surroundings of the A sphere. For $p(\mathbf{r})$ this is written as

$$\frac{\partial p}{\partial t}+\nabla\cdot(p\cdot\mathbf{u}) = 0. \tag{5.3}$$

The same equation was used by BATCHELOR and GREEN [2] in the related problem of two spheres in a pure straining motion flow. Here $\boldsymbol{\mu}$ is the relative velocity of both spheres due to gravity effects. From Eqs. (2.1) and (2.2)

$$\mathbf{u} = \frac{1}{6\pi a\mu}\left[(A_{11}-A_{12})\frac{\mathbf{rr}}{r^2}+(B_{11}-B_{12})\left(\mathbf{1}-\frac{\mathbf{rr}}{r^2}\right)\right]\cdot(\mathbf{F}_B-\mathbf{F}_A), \tag{5.4}$$

where $\mathbf{1}$ is the identity tensor; $\mathbf{F}_A$, $\mathbf{F}_B$, are the forces (weight plus buoyancy) applied to A, B, respectively. The boundary condition to be applied to Eq. (5.3) with Eq. (5.4) is a condition at infinity, i.e. for spheres far apart.

We see here that it is important that sphere B may effectively come from and go to infinity relative to sphere A, as announced at the beginning of the paper. If it were not so, the boundary condition to apply would pose a serious problems.

The boundary condition has already been given in Eq. (4.2), on physical grounds.

We will show in the next paragraph that this result can be obtained rigorously, yielding

$$r\to\infty,\qquad p(\mathbf{r})\to 1. \tag{5.5}$$

The integration of Eq. (5.3) with Eqs. (5.4) and (5.5) can be performed with the same kind of analysis as the one used by BATCHELOR and GREEN [2] for the case of two spheres in a pure straining motion flow, although the expression for $\mathbf{u}$ is here different. The analytical results is a function of the nondimensional distance r/a:

$$p(\mathbf{r}) = q(r) = Q\left(\frac{r}{a}\right) = \frac{1}{A_{11}-A_{12}}\exp\int\limits_{\frac{r}{a}}^{\infty}\frac{2(A_{11}-A_{12}-B_{11}+B_{12})}{R(A_{11}-A_{12})}\,dR, \tag{5.6}$$

where A_{11}, A_{12}, B_{11}, B_{12} are functions of this nondimensional distance also. For large r/a, this expression can be developed using Eq. (2.7)

$$Q\left(\frac{r}{a}\right) = 1+\frac{15}{8}\left(\frac{a}{r}\right)^4+\frac{9}{4}\left(\frac{a}{r}\right)^5+O\left(\frac{a}{r}\right)^6. \tag{5.7}$$

For other values of r/a, we have calculated numerically the integral in Eq. (5.6). The resulting function is shown in Fig. 1. For closed spheres, Q is going to infinity, suggesting a strong attraction potential. The result is due to the friction coefficient which becomes (paradoxically) infinite in Stokes flow when particle B is moving towards particle A, and very close to it. But then other physical phenomena may be important and, in particular, the Brownian motion should be non-negligible as shown recently by Batchelor (paper to be published in J. Fluid. Mech.).

For spheres far apart, we will give some more details. We have not used until now the normalization condition for the conditional probability. This condition may be written as

$$\int\limits_{v-v_{ex}} P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0)\,d^3r = N_B, \tag{5.8}$$

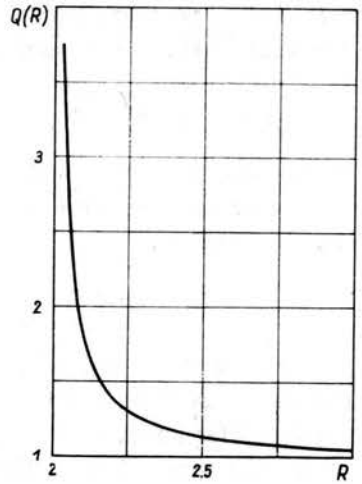

FIG. 1

where v_{ex} is the volume excluded by the presence of the A sphere. The normalization condition of the single probability for the sphere B is written by stating that the same number N_B of spheres B is still present in v:

$$\int_v P(\mathbf{x}_0+\mathbf{r})d^3r = N_B. \tag{5.9}$$

In fact

$$P(\mathbf{x}_0+\mathbf{r}) = n_B, \tag{5.10}$$

by the uniformity assumption.

Let us write the conditional probability for $r > 2a$ in the form

$$P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0) = K\cdot n_B\cdot Q\left(\frac{r}{a}\right), \tag{5.11}$$

where $Q(r/a)$ is the function found precedently, and K has to be found.

Substracting Eq. (5.9) from Eq. (5.8) and using Eqs. (5.10) and (5.11) we get

$$\int_2^\infty [Q(R)-1]R^2dR = \frac{N_B-N_BK+8c_BK}{3c_B\cdot K} \simeq \frac{v}{\frac{4}{3}\pi a^3}\left[\frac{1}{K}-1\right]. \tag{5.12}$$

The integral on the left hand side has been calculated $\simeq 3.6$. On the right hand side, $v/\frac{4}{3}\pi a^3$ is very large. This shows that $K \simeq 1$ and $K \to 1$ for $v \to \infty$. Thus Eq. (5.5) is demonstrated. Note that if we had supposed a priori that $K = 1$, then the first part of Eq. (5.12) would have given the necessary value 8/3 for the integral of the left hand side. But the preceding development shows that this integral should not have any given particular value.

6. Result for the average velocity

The average velocity of sedimentation of the test A sphere is calculated from Eqs. (4.5), (4.6) and (4.7), using the values calculated for the probability, the classical expression of the flow field $\mathbf{v}_\infty$ due to a single sphere in Stokes flow, and the expression for $\mathbf{w}$ from the two-spheres problem.

The result is

$$\bar{\mathbf{v}}_{pA} = \mathbf{v}_{pA_s}(1-6.7c_A-2.7c_B)+0.1\mathbf{v}_{pB_s}c_B. \tag{6.1}$$

The particle A is thus slowed down by other particles A and B. But an imprecision on our calculated coefficients is to be taken into account. An error calculation (FEUILLEBOIS' thesis [5]) shows that the precision on the probability Q is of the order

$$\frac{\Delta Q}{Q} = 2.7\frac{\Delta(B_{11}-B_{12})}{B_{11}-B_{12}}. \tag{6.2}$$

By interpolation between tabulated values of B_{11}, B_{12}, we could get only a 10^{-2} precision on $B_{11}-B_{12}$. The resulting error for integrals on Q may then be large, e.g.

$$\Delta\int\limits_2^\infty [Q(R)-1]R^2dR = 0.3. \tag{6.3}$$

New more recent and more precise results for B_{11}, B_{12} by Jeffrey (to be published in J. Fluid. Mech.) have allowed Batchelor and Wen to get more accurate results (to be published in J. Fluid. Mech.).

Formula (6.1) for two types of spheres of different densities can be compared with BATCHELOR'S [1] formula for one type of spheres:

$$\bar{\mathbf{v}}_p = \mathbf{v}_{p_s}(1-6.55c). \tag{6.4}$$

The difference between our coefficient (6.7) (for $c_B = 0$ in Eq. (6.1)) and the coefficient 6.55 of this formula is probably due to the precision problem on $B_{11}-B_{12}$.

7. The problem of nearly identical spheres

Another serious problem arises: if we let $\varrho_{pA} = \varrho_{pB}$ (both spheres of the same density), thus $\mathbf{v}_{p_s} = \mathbf{v}_{pA_s} = \mathbf{v}_{pB_s}$ and $c = c_A+c_B$, we do not obtain the same limit value for $\bar{\mathbf{v}}_p$ as when we take $c_B = 0$.

In fact the physical phenomena leading to the coefficients (6.7) and (2.7) in Eq. (6.1) are of a different nature.

The first coefficient is the one for one type of spheres, and is obtained from a probability distribution function (5.1) uniform by assumption.

However, the second coefficient is the one for coupled A and B spheres ($\varrho_{pA} \neq \varrho_{pB}$), and is obtained from the probability distribution function calculated from hydrodynamic interactions. Now one may wonder what the probability distribution function (and hence the friction coefficient) may be for nearly identical spheres such as $\varrho_{pA} \approx \varrho_{pB}$. We may put the problem in another way: consider the characteristic time of separation of two spheres

$$t_s \sim \frac{a}{u} \sim \frac{\nu}{a \cdot g} \cdot \frac{\varrho}{\varrho_{pB} - \varrho_{pA}}, \tag{7.1}$$

where ν is the kinematic viscosity of the fluid. Consider a characteristic time for the experiment where particles fall in a box of the height l:

$$t_e \sim \frac{l}{v_{pA_s}} \sim \frac{\nu}{a^2 g} \cdot \frac{\varrho}{\varrho_{pA} - \varrho} \cdot l. \tag{7.2}$$

Defining the small number

$$\varepsilon = \frac{\varrho_{pB} - \varrho_{pA}}{\varrho_{pA} - \varrho}, \tag{7.3}$$

we see that

$$t_e \sim \varepsilon \cdot \frac{l}{a} t_s. \tag{7.4}$$

Three possible cases may occur:

1) $$l \gg \frac{a}{\varepsilon}, \quad v \gg \left(\frac{a}{\varepsilon}\right)^3, \quad t_e \gg t_s.$$

The box is large enough and we wait long enough for all possible locations of the sphere B with respect to the sphere A to be attained. The conditional probability is then $n_B Q\left(\frac{r}{a}\right)$:

2) $$l \ll \frac{a}{\varepsilon}, \quad v \ll \left(\frac{a}{\varepsilon}\right)^3, \quad t_e = t_s.$$

The volume of the experiment is too small to observe any relative motion of the two spheres. Then the assumption of uniformity takes over and the conditional probability is n_B:

3) $$l \sim \frac{a}{\varepsilon}, \quad v \sim \left(\frac{a}{\varepsilon}\right)^3, \quad t_e \sim t_s.$$

This is the most general case. During the time t_e of the experiment, the sphere B takes only a limited fraction of its possible positions on the trajectory relative to sphere A. Thus the condition at infinity (5.5) that we used to integrate the equation for the probability distribution function is no more valid here. A special treatment of this case should be considered.

We can remark that a related problem would arise for cases involving spheres of different radii, when the relative trajectories are closed (WALCHOLDER and SATHER [9]).

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