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On the mechanical energy dissipation in small deformation elasticity and the simple analytic expression for the viscous kink-shaped solitary wave

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WAVE PROPAGATION in the material abruptly changing its elastic moduli with the change of the deformation sign is considered. According to Cz. Eimer, such a material models the behavior of a cracked linear elastic medium under one-dimensional deformation. The solution obtained earlier for the shock propagation is compared now with the, proposed by the present author, simple analytic expression for diffuse shock propagation in Eimer's material endowed with the Voigt viscosity. It is shown that neither the propagation velocity, nor the total mechanic dissipation power depends on the viscosity coefficient. These quantities retain, for arbitrary viscosity, the same values as in the case of discontinuity propagation in a purely elastic material. The slope of stress (strain) wave-profile changes with the viscosity change, tending to infinity if the latter tends to zero. The results obtained prove that an apparent paradox of a well-defined dissipation power, which can be calculated in the framework of purely elastic model within the small deformation approach in spite of the fact, that even the dissipation mechanism had not been defined earlier, can be explained on the basis of the limit transition for the viscoelastic solution.

1. Introduction

RECENTLY the present author considered [1] a simple, purely mechanical, description of one-dimensional discontinuity propagation scheme in the elastic material governed by the homogeneous constitutive stress-strain relation of order one (modeling, according to CZ. EIMER [2–5], a cracked, initially linear, elastic material)([1]).

In the case of one-dimensional problem the nonlinearity reduces to the step-wise change of the elastic modulus at zero strain; it is assumed that its value is higher in the compressed regions than in the extended ones. The linearity of the constitutive relation in the regions of constant strain sign made it possible to obtain some effective discontinuous solutions in the closed analytical form. In all cases considered, the mechanical energy dissipation rate could be readily found, despite the fact that no dissipation mechanisms at the discontinuity had been previously assumed. In the framework of a purely mechanical model this bare fact is visible much clearer than usually when it is served wrapped in thermodynamic formalism obscuring the problem. In the present paper we shall demonstrate that the apparent paradox of the non-vanishing dissipation in non-dissipative material can be explained on the ground of the viscoelastic model as the limiting case. The author believes, that these considerations would help to shed some light on the problem of admissibility of the viscous terms frequently introduced into numerical algorithms for suppressing the numerical instabilities at the shock front.

([1]) Writing the mentioned paper [1] the author was not aware of the results obtained by MASLOV and MOSOLOV [6, 7]. In his general considerations (in the paper [1]), the present author in fact proceeded in the same way as the authors of [7]; particular applications of the theory, however (presented in [1]), are completely different from the examples considered in [7].

2. Results for elastic medium

Let us shortly summarize some results of the paper [1]. We started there from the displacement continuity condition in the following form

$$v_0^R + U\varepsilon_0^R = v_0^L + U\varepsilon_0^L, \tag{2.1}$$

where v denotes the material velocity, ε is the strain and U stands for the velocity of the interface between the regions of positive and negative strain. Upper indices $(\cdot)^R$ and $(\cdot)^L$ denote the right- and left-hand values of a quantity $(\cdot)$, lower index $(\cdot)_0$ denotes the value at the interface.

The next condition — the momentum balance — has the following form:

$$(\sigma_0^R - \sigma_0^L) + \rho U(v_0^R - v_0^L) = 0, \tag{2.2}$$

or

$$(c_2^2\varepsilon_0^R - c_1^2\varepsilon_0^L) + U(v_0^R - v_0^L) = 0, \tag{2.3}$$

where $c_1 = c^L$ and $c_2 = c^R$ are the sound velocities in the compressed and in the extended regions and ρ is the mass density. It was assumed in [1] (we shall adopt this assumption also for this paper) that the compressed zone is situated at the left-hand side of the interface.

Combining Eqs. (2.1) and (2.3) one can easily obtain the following expression for the interface velocity

$$U^2 = \frac{c_2^2\varepsilon_0^R - c_1^2\varepsilon_0^L}{\varepsilon_0^R - \varepsilon_0^L}. \tag{2.4}$$

In the framework of the purely mechanical theory we define the dissipation rate D in any material region as the difference between the power of external forces and the energy growth rate:

$$\begin{aligned} D = {} & -\sigma(a)v(a) + \sigma(b)v(b) \\ & -\frac{d}{dt}\left[\int\limits_a^{Y(t)} \frac{1}{2}(\rho c^2\varepsilon^2 = \rho v^2)dx = \int\limits_{Y(t)}^{b} \frac{1}{2}(\rho c^2\varepsilon^2 + \rho v^2)dx\right], \end{aligned} \tag{2.5}$$

where $Y(t)$ is the current position of the interface, $a < Y(t) < b$. As long as the material outside the interface behaves as purely elastic, the exact position of a and b is not important here. Performing differentiation and tending with a and b to $Y(t)$ one obtains:

$$D = \rho\left\{ - c_1^2\varepsilon_0^L v_0^L + c_2^2\varepsilon_0^R v_0^R - \frac{1}{2}U[(c_1^2(\varepsilon_0^L)^2 + (v_0^L)^2) - (c_2^2(\varepsilon_0^R)^2 + (v_0^R)^2)]\right\}. \tag{2.6}$$

The last result combined with Eqs. (2.1), (2.3) and (2.4) yield the following relation for dissipation rate D in terms of material constants and strain values at both sides of the interface (compare [1]):

$$D = \frac{1}{2}\rho\varepsilon_0^R\varepsilon_0^L(c_2^2 - c_1^2)\sqrt{(c_2^2\varepsilon_0^R - c_1^2\varepsilon_0^L)/(\varepsilon_0^R - \varepsilon_0^L)}. \tag{2.7}$$

Relations (2.1) and (2.2) are valid for any material independently of its viscous or plastic properties, and the possible presence or absence of discontinuity, while relations

(2.4) and (2.7) were derived under the assumption of purely elastic behavior of material outside the discontinuity surface.

3. Viscoelastic solution

In this section we shall forget about the results mentioned above and we shall attempt to find a smooth solution describing the propagation of kink-shaped solitary wave in an Eimer type viscoelastic medium.

Beginning from this point we shall consider Voigt-type viscoelastic material with the following constitutive relation for a one-dimensional case:

$$\sigma = dw(\varepsilon)/d\varepsilon + \overline{\mu}\dot{\varepsilon}\,, \tag{3.1}$$

where $w = \rho c^2\varepsilon^2/2$ is the volumetric elastic energy density, $\overline{\mu}$ denotes effective one-dimensional viscosity. We assume that the value of c (sound velocity) is not constant. In [1] we assumed according to EIMER ([2–5]), that c was a function of the sign of the stress, or — what meant the same — of the sign of the strain. For the case of viscoelastic material the situation is not so simple and, in order to generalize Eimer's model on the viscoelastic material, we should answer the question: what should be equal to zero at the moment of the crack opening: stress or strain? We shall leave this question open for the future discussion assuming here that the value of c changes only with the change of the strain sign. We can also expect the change of viscosity due to the crack opening, we shall assume however for the sake of simplicity, that this change can be neglected, i.e. that $\overline{\mu} = \text{const}$.

In the case of viscoelastic materials we shall look for the continuous solutions only, thus it will be convenient to rewrwite relation (2.5) in the following form:

$$D = \int_a^b \left[\frac{\partial}{\partial x}(\sigma v) - \rho\frac{\partial}{\partial t}\left(\frac{v^2}{2} + \frac{c^2\varepsilon^2}{2}\right)\right]dx \\ -\frac{1}{2}U\rho((v^{L^2} - v^{R^2}) + (c_1^2\varepsilon^{L^2} - c_2^2\varepsilon^{R^2}))\,. \tag{3.2}$$

It is not difficult to observe that the second right-hand term vanishes due to continuity of velocity and vanishing strain values at the interface. The last fact is a consequence of the velocity continuity assumed and, consequently, by virtue of Eq. (2.1), also the strain continuity at the interface between the compressed and the extended regions. Performing differentiations under the integration sign, making use of the equation of motion

$$\frac{\partial\sigma}{\partial x} = \rho\dot{v}\,, \tag{3.3}$$

and recalling that $\partial v/\partial x = \dot{\varepsilon}$, one readily obtains the expected familiar result:

$$D = \int_a^b \overline{\mu}\dot{\varepsilon}^2\,dx\,. \tag{3.4}$$

We shall look for the continuous smooth displacement function $u(x,t)$, defined in the infinite region, satisfying the following equation of motion:

$$c^2\frac{\partial^2 u}{\partial x^2} + \frac{\overline{\mu}}{\rho}\frac{\partial^3 u}{\partial x^2\partial t} = \frac{\partial^2 u}{\partial t^2} \tag{3.5}$$

and giving rise to the bounded quasi-stationary stress and strain kink-shaped distribution which moves from the left to right-hand side with the constant velocity U. We shall assume, the same as earlier, that $\varepsilon^L_{-\infty} < 0 < \varepsilon^R_{\infty}$, where lower indices $(\cdot)_{-\infty}$ and $(\cdot)_{\infty}$ denote values at minus and plus infinity. We shall confine our considerations to the class of solutions with bounded stress values, thus we have to assume continuity of velocity at the interface. The displacement u and velocity v continuity requirements together with Eqs. (2.1), (2.2), (these conditions must be fulfilled, of course, also for the continuous solutions) as well as with Eq. (3.1) and with the assumption $\overline{\mu}$ = const., yield at once the following set of continuity conditions at the interface:

$$\text{(3.6)}\qquad \begin{aligned} &\text{a)}\quad u_0^L = u_0^R, \quad \text{b)}\quad v_0^L = v_0^R = v_0, \quad \text{c)}\,\varepsilon_0^L = \varepsilon_0^R = 0\,, \\ &\text{d)}\quad \sigma_0^L = \sigma_0^R, \quad \text{e)}\quad \left(\frac{\partial^2 u}{\partial t \partial x}\right)_0^L = \left(\frac{\partial^2 u}{\partial t \partial x}\right)_0^R . \end{aligned}$$

For stationary motion one can assume, without any loss of generality, that at the interface $x - Ut = 0$.

We shall look now for the displacement fields of the following form:

$$u(x,t) = f(x - Ut) + v_0 t + x_0 \tag{3.7}$$

satisfying Eq. (3.5), conditions (3.6) and the prescribed boundary value conditions at infinities: $\varepsilon(\infty) = \varepsilon^R_{\infty}, \varepsilon(-\infty) = \varepsilon^L_{-\infty}$, where $\varepsilon^L_{-\infty}$ and ε^R_{∞} can be arbitrarily taken, provided the mentioned earlier inequality $\varepsilon^L_{-\infty} < 0 < \varepsilon^R_{\infty}$ is satisfied ([2]).

Using the standard considerations one arrives at the following expression:

$$\text{(3.8)}\qquad u(x,t) = \begin{cases} \varepsilon^L_{\infty}(1 - \exp[\alpha^L(x - Ut)] + \alpha^L(x - Ut))/\alpha^L + v_0 t + u_0 \\ \qquad\qquad\qquad\qquad \text{for} \quad x - Ut < 0\,, \\ \varepsilon^R_{\infty}(1 - \exp[\alpha^R(x - Ut)] + \alpha^R(x - Ut))/\alpha^R + v_0 t + u_0 \\ \qquad\qquad\qquad\qquad \text{for} \quad x - Ut > 0\,. \end{cases}$$

Function $u(x,t)$, given by Eqs. (3.8), supplies proper asymptotic strain values both at plus and minus infinity and meets all continuity conditions (3.6) except, maybe, condition (e) of the velocity gradient continuity. To satisfy this condition, the following relation should be fulfilled:

$$\varepsilon^L_{-\infty}\alpha^L = \varepsilon^R_{\infty}\alpha^R . \tag{3.9_1}$$

Expression (3.7) must be, of course, a solution of Eq. (3.5), and to this end the following two relations must be satisfied:

$$c_1^2 - \frac{\overline{\mu}}{\rho}\alpha^L U - U^2 = 0\,, \tag{3.9_2}$$

$$c_2^2 - \frac{\overline{\mu}}{\rho}\alpha^R U - U^2 = 0\,. \tag{3.9_3}$$

([2]) Since we are looking for the bounded asymptotic values of strain in infinities, and since the velocity and the strain values are, for the field described by Eq. (3.7), mutually connected by the relation $v = -U\varepsilon + v_0$, the velocity gradients in infinities should be equal to zero; thus, prescribing the strain values at infinities we prescribe in fact the stress values (or, equivalently, velocities).

Three relations $(3.9)_1$ $(3.9)_2$ and $(3.9)_3$ constitute the system of three algebraic equations for the three unknown parameters: U, α^R and α^L, which should be expressed in terms of the boundary values and material constants.

From Eqs. (3.9) one can readily obtain the following relation for propagation velocity:

$$U^2 = \frac{c_2^2\varepsilon_{\infty}^R - c_1^2\varepsilon_{-\infty}^L}{\varepsilon_{\infty}^R - \varepsilon_{-\infty}^L}. \tag{3.10}$$

Substitution of Eq. (3.10) into Eqs. $(3.9)_2$ and $(3.9)_3$ leads to the following relations for α^L and α^R:

$$\begin{aligned} \alpha^L &= \varepsilon_{\infty}^R\rho(c_1^2 - c_2^2)/\overline{\mu}\sqrt{(c_2^2\varepsilon_{\infty}^R - c_1^2\varepsilon_{-\infty}^L)(\varepsilon_{\infty}^R - \varepsilon_{\infty}^L)}, \\ \alpha^R &= \varepsilon_{-\infty}^L\rho(c_1^2 - c_2^2)/\overline{\mu}\sqrt{(c_2^2\varepsilon_{\infty}^R - c_1^2\varepsilon_{-\infty}^L)(\varepsilon_{\infty}^R - \varepsilon_{-\infty}^L)}. \end{aligned} \tag{3.11}$$

Differentiation of expression (3.7) yield the following expression for $\dot{\varepsilon}$:

$$\dot{\varepsilon}(x,t) = \begin{cases} \alpha^L U\varepsilon_{\infty}^L \exp[\alpha^L(x - Ut)] & \text{for } x - Ut < 0, \\ \alpha^R U\varepsilon_{\infty}^R \exp[\alpha^R(x - Ut)] & \text{for } x - Ut > 0. \end{cases} \tag{3.12}$$

Substitution of Eqs. (3.12) into Eq. (3.4) yields the following integral expression

$$D = \overline{\mu}U^2\left(\varepsilon_{-\infty}^{L^2}\alpha^{L^2}\int\limits_{-\infty}^{0}\exp(2\alpha^L\xi)d\xi + \varepsilon_{\infty}^{R^2}\alpha^{R^2}\int\limits_{0}^{\infty}\exp(2\alpha^R\xi)d\xi\right). \tag{3.13}$$

Performing integration an making use of expressions (3.10) and (3.11) we arrive at the final expression for D

$$D = \frac{1}{2}\rho\varepsilon_{\infty}^R\varepsilon_{-\infty}^L(c_2^2 - c_1^2)\sqrt{(c_2^2\varepsilon_{\infty}^R - c_1^2\varepsilon_{-\infty}^L)/(\varepsilon_{\infty}^R - \varepsilon_{-\infty}^L)}. \tag{3.14}$$

4. Discussion

Relations (3.10) and (3.14) indicate that, at least for the material under consideration, neither the propagation velocity nor the total dissipation of the stationary kink-shaped wave depends on the viscosity; moreover, the dissipation and velocity values are exactly the same as in the case of a discontinuous solution for the elastic material.

The only difference between formulae (3.10) and (3.14) versus their counterparts (2.4) and (2.7) consists in the presence of strain values in infinities instead of the values at both sides of the interface. It is evident however from relations (3.11) and the expression (3.7) that with the viscosity coefficient tending to zero, the values of strains and velocities in every point, no matter how close to the interface, tend to the asymptotic values at infinity.

The only important wave-motion parameter which depends on the viscosity coefficient is the strain (stress) wave profile slope.

It should be underlined here that, contrary to the case of $\overline{\mu} \to 0$, for which the limit transition in the solution turned out to be obvious, there is no immediate limit transition for $c_1 \to c_2$. Linear parabolic equation of motion for the Voigt viscoelastic material does not admit any bounded solitary wave-type solution. Probably some non-stationary viscous terms should be added to the expression (3.7) in order to obtain correctly both limit transitions; this problem, however, is outside the scope of the present paper.

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References

1. A. BLINOWSKI, *Small amplitude wave propagation in Eimer's cracked material (The instructive case of irremovable nonlinearity)*, Arch. Mech., **44**, 3, 331–344, 1992.
2. Cz. EIMER, *Elasticity of cracked medium*, Arch. Mech., **30**, 6, pp. 827–836, 1978.
3. Cz. EIMER, *Bulk constitutive relations for cracked materials*, Arch. Mech., **31**, 4, pp. 519–533, 1979.
4. Cz. EIMER, *Bulk properties of cracked medium*, [in:] Continuum Models of Discrete Systems, pp. 819–835, University of Waterloo Press, 1980.
5. Cz. EIMER, *On the behavior of elastic materials with scattered extending cracks*, Arch. Mech., **33**, 1, pp. 21–36, 1981.
6. V.P. MASLOV, P.P. MOSOLOV, *The oscillations of the rods with two elastic moduli* [in Russian], Uspekhy Mat. Nauk., **36**, 3, pp. 240–241, 1981.
7. V.P. MASLOV, P.P. MOSOLOV, *The general theory of the solutions of the equations of motion of the elastic medium with changing elastic moduli* [in Russian], Prikladnaya Matematika i Mekhanika, **49**, 3, pp. 419–437, 1985.

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