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Geometrical aspect of symmetric conservative systems of partial differential equations

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IN MATHEMATICAL PHYSICS, one often encounters systems of the first-order conservation laws which imply the additional conservation law and, as their special cases, symmetric conservative and symmetric hyperbolic systems. In particular, systems of the first-order conservation laws which imply the additional conservation law are interesting from the point of view of phenomenological thermodynamics, where the additional conservation law is interpreted as the entropy law. In this paper the geometrical description of such systems, based on the geometrical approach proposed by PERADZYŃSKI [7, 8] and PIEKARSKI [5] is discussed. It should be stressed that this description is different from that usually applied in the theory of conservation laws (H.H. JOHNSON [14]). The applications of the discussed formalism to the symmetric systems are also mentioned.

1. Introduction

IN MATHEMATICAL PHYSICS, one often encounters systems of the first-order conservation laws which imply the additional conservation law and, as their particular cases, the symmetric conservative systems. Such systems are of interest from the point of view of phenomenological thermodynamics where the additional balance law is often interpreted as the entropy balance [1–4]. In [5], the geometrical approach proposed by PERADZYŃSKI was applied in order to integrate the constraints imposed on a system of conservation laws by the second law of thermodynamics. The application of this approach to dissipative fluid theories will be discussed elsewhere [6].

In this paper, we will discuss a geometrical aspect of symmetric conservative and symmetric hyperbolic systems. The applications of the proposed formalism to symmetric systems and to the systems of conservation laws consistent with the additional conservation law are also mentioned.

In order to maintain the self-consistency of presentation, basic definitions and properties known from literature are presented in Sec. 2. In Secs. 3 and 4 we discuss the coordinate-free formulation of the theories of symmetric conservative and symmetric hyperbolic systems. Such a coordinate-free formulation can be of some use in practical calculations because it shows how do the changes of coordinates work in this case. In Sec. 5 we introduce the alternative geometrical representation of discussed systems of P.D.E. and apply it in order to describe the process of symmetrization. For simplicity, we restrict our discussion to the quasi-linear systems of the first-order equations with vanishing production terms. All our results remain valid also for systems with non-vanishing productions. Throughout this paper, we frequently use the exterior products of scalar and vector-valued differential forms; such product is well-defined only if in all tensor products forming the result the "vector index" is permuted into the same position but, for simplicity, this operation is not denoted explicitly. For similar reasons, when the operation of a total contraction can be done only in a unique manner, we do not write elementary contractions between particular indices explicitly. Except where noted to the contrary, the Einstein summation convention for repeated indices is assumed.

The approach presented here can be generalized to the case of the coordinate-free theory of symmetric conservative systems of P.D.E. for the cross-sections of the arbitrary differentiable fibre bundle. However, such a generalization is outside the scope of this paper.

2. Systems of conservation laws consistent with the additional conservation law

Let us discuss a first-order system of quasilinear partial differential equations of the following form

$$w^k{}_j{}^i(y_{j'})\frac{\partial y_j}{\partial x_i} = 0, \tag{2.1}$$

where (x_i), $i = 1,\ldots,n$ are the independent variables, (y_j), $j = 1,\ldots,m$ are the dependent variables, and $w^k{}_j{}^i$, $k = 1,\ldots,m$ are smooth, real functions.

The independent variables shall be interpreted as a local parametrization of the affine space A, whereas the dependent variables shall be interpreted as the parametrization of the manifold Q [5, 7, 8]. As a consequence, the solution of Eq. (2.1) defines the function f from the affine space A into the manifold Q.

Let T_A be a translation space of A and let $\mathbf{e}_1,\ldots,\mathbf{e}_n$ be a basis in T_A. By T_A^* we shall denote the space dual to T_A, and a basis in T_A^*, dual to the basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$ in T_A, shall be denoted as $F^1,\ldots,F^n$;

$$\langle F^i, e_{i'}\rangle = \delta^i_{i'}, \tag{2.2}$$

where $\delta^i_{i'}$ is Kronecker's symbol and $\langle,\rangle$ denotes the action of a form on a vector.

The natural base vectors ∂_{x_i}, $i = 1,\ldots,n$ of the coordinate system

$$\mathbb{R}^n \ni (x_1,\ldots,x_n) \to a + x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n \in A \tag{2.3}$$

(where a denotes an arbitrary point of A) can be identified with the corresponding vectors $\mathbf{e}_i$ from the basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$ whereas the natural base forms dx^i of this coordinate system can be identified with forms F^i introduced by the relation (2.2). The derivative f_* of the function f, : $A \to Q$, is given by

$$f_* = \frac{\partial y_j}{\partial x_i}\partial y_j \otimes dx^i = \frac{\partial y_j}{\partial x_i}\partial y_j \otimes F^i, \tag{2.4}$$

where ∂y_j, $j = 1,\ldots,n$ are the natural based vectors of the coordinate system (y_j), $j = 1,\ldots,m$ on Q.

As it has been observed by PERADZYŃSKI [7, 8], the system (2.1) can be written by means of the contractions of f_* with the fields of the two-point tensors w^k,

$$w^k = w^k{}_j{}^i(y_{j'})dy^j \otimes \mathbf{e}_i, \qquad k = 1,\ldots,m \tag{2.5}$$

and if the system (2.1) is determined, then it can be geometrically represented by a corresponding m-dimensional vector subbundle of $T^*(Q)\otimes T_A$ (by $T^*(Q)$ we denote the cotangent bundle of Q).

In order to see that, let us compute such contractions explicitly

$$\begin{aligned}\langle w^k, f_*\rangle &= \langle w^k{}_j{}^i dy^j \otimes \mathbf{e}_i, \frac{\partial y_{j'}}{\partial x_{i'}}\partial y_{j'} \otimes F^{i'}\rangle \\ &= w^k{}_j{}^i \delta^j_{j'}\delta^{i'}_i \frac{\partial y_{j'}}{\partial x_{i'}} = w^k{}_j{}^i\frac{\partial y_j}{\partial x_i} = 0.\end{aligned} \tag{2.6}$$

From geometrical point of view, the fields of the two-point tensors w^k define a set of local, linearly independent cross-sections of the corresponding m-dimensional vector subbundle of $T^*(Q) \otimes T_A$. Of course, different choices of linearly independent cross-sections define different equivalent forms of the discussed system of P.D.E. The fields $w^1, \ldots, w^k$ are the vector-valued differential 1-forms on Q with values in T_A and closed forms define conservation laws; if

$$dw^k = 0, \tag{2.7}$$

then locally

$$w^k = d\mathbf{u}^k \tag{2.8}$$

and

$$\langle w^k, f_* \rangle = \langle d\mathbf{u}^k, f_* \rangle = \frac{\partial v^{k,i}}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \frac{\partial}{\partial x_i}(u^{k,i}(y_{j'})) = 0, \tag{2.9}$$

where $u^{k,i}$ are the components of $\mathbf{u}^k$ in the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$

$$\mathbf{u}^k = u^{k,i}\mathbf{e}_i. \tag{2.10}$$

Hence, by a set of conservation laws we mean such a m-dimensional vector subbund[illegible] of $T^*(Q) \otimes T_A$ which is locally spanned by sets of cross-sections composed of closed 1-f[illegible]rms.

Let λ_k, $k = 1, \ldots, m$ be smooth, real functions defined on Q. Then the vector-valued differential 1-form given by

$$\Omega = \sum_{k=1}^{m} \lambda_k d\mathbf{u}^k \tag{2.11}$$

defines the quasilinear partial differential equation which is implied by the set of conservation laws

$$\langle \Omega, f_* \rangle = \sum_{k=1}^{m} \lambda_k \langle d\mathbf{u}^k, f_* \rangle. \tag{2.12}$$

We shall say that a system of balance laws, defined by a set of 1-forms $d\mathbf{u}^1, \ldots, d\mathbf{u}^m$, implies an additional conservation law if the equation

$$0 = d\Omega = \sum_{k=1}^{m} d\lambda_k \wedge d\mathbf{u}^k \tag{2.13}$$

has a non-trivial solution (by non-triviality we mean here that not all λ_k are constant).

Then locally

$$\Omega = d\mathbf{s}, \tag{2.14}$$

and the conservation law defined by $\langle d\mathbf{s}, f_* \rangle$ is usually called the entropy balance whereas the functions λ_k, $k = 1, \ldots, m$ are called Lagrange–Liu multipliers [1]. In the rest of this paper, the 1-form $\mathbf{s}$ shall be called the entropy 0-form, and the expressions of the type $\langle d\mathbf{s}, f_* \rangle$ shall be alternatively denoted as $\operatorname{div} \mathbf{s}$. Of course, if $\mathbf{s}$ satisfies the condition

$$d\mathbf{s} = \sum_{k=1}^{m} \lambda_k d\mathbf{u}^k, \tag{2.15}$$

then

$$\mathbf{s}' := \mathbf{s} + \sum_{k=1}^{m} \alpha_k \mathbf{u}^k, \tag{2.16}$$

where $\alpha_1, \ldots, \alpha_m$ are arbitrary real constants, satisfies

$$d\mathbf{s}' = \sum_{k=1}^{m} (\lambda_k + \alpha_k) d\mathbf{u}^k. \tag{2.17}$$

In other words, the solution of the equation for the entropy 0-form is determined up to the linear combination of the conservation laws.

3. Symmetric conservative systems

The condition

$$d\Omega = d\Big(\sum_{k=1}^{m} \lambda_k d\mathbf{u}^k\Big) = \sum_{k=1}^{m} d\lambda_k \wedge d\mathbf{u}^k = 0 \tag{3.1}$$

which states that the 1-form Ω is closed, means, at the same time, that the 1-form Φ given by

$$\Phi = \sum_{k=1}^{m} \mathbf{u}^k \otimes d\lambda_k \tag{3.2}$$

is closed.

If the condition (3.1) is satisfied, then locally both forms are given by differentials

$$\Omega = d\mathbf{s}, \qquad \Phi = d\underline{\mathcal{F}}, \tag{3.3}$$

which are related by the identity

$$\underline{\mathcal{F}} = \sum_{k=1}^{m} \lambda_k \mathbf{u}^k - \mathbf{s}. \tag{3.4}$$

Let us assume that the functions $\lambda_1, \ldots, \lambda_m$ form a local coordinate system on Q. In these coordinates, the differential $d\underline{\mathcal{F}}$ is given by

$$d\underline{\mathcal{F}} = \sum_{k=1}^{m} \frac{\partial \underline{\mathcal{F}}}{\partial \lambda_k} \otimes d\lambda_k. \tag{3.5}$$

After comparing Eq. (3.5) with Eqs. (3.2) and $(3.3)_2$ we see that

$$\mathbf{u}^k = \frac{\partial \underline{\mathcal{F}}}{\partial \lambda_k}. \tag{3.6}$$

As a consequence, the discussed system of conservation laws takes the symmetric conservative form

$$\operatorname{div} \frac{\partial \underline{\mathcal{F}}}{\partial \lambda_k} = \frac{\partial}{\partial x_i} \frac{\partial \mathcal{F}^i}{\partial \lambda_k} = 0, \qquad k = 1, \ldots, m, \tag{3.7}$$

where $\mathcal{F}^i$ are the components of $\underline{\mathcal{F}}$ in the basis $\mathbf{e}, \ldots, \mathbf{e}_n$

$$\underline{\mathcal{F}} = \mathcal{F}^i \mathbf{e}_i. \tag{3.8}$$

From the above remarks it follows that the system of conservation laws which implies an additional conservation law can be transformed into the symmetric conservative form if and only if the Lagrange–Liu multipliers form a local coordinate system on Q.

In order to write the symmetric conservative system in the coordinate-free form, we shall represent Eq. (3.6) as a contraction of the differential $d\underline{\mathcal{F}}$ with the field of natural base vectors of the coordinate system (λ_k), $k = 1, \ldots, m$

$$d\underline{\mathcal{F}} \odot \partial_{\lambda_k} = \left(\frac{\partial \underline{\mathcal{F}}}{\partial \lambda_j} \otimes d\lambda_j\right) \odot \partial_{\lambda_k} = \frac{\partial \underline{\mathcal{F}}}{\partial \lambda_j} \delta^j_k = \frac{\partial \underline{\mathcal{F}}}{\partial \lambda_k}. \tag{3.9}$$

Let us consider a class of affine transformations of coordinates on Q

$$\widetilde{\lambda}_{k'} = A_{k'k}\lambda_k + c_{k'}, \qquad k, k' = 1, \ldots, m, \tag{3.10}$$

where $A_{k'k}$ is a nondegenerate real matrix and $c_{k'}$ are real numbers. The set of vector fields corresponding to natural base vectors of the coordinates of the type (3.10) form a m-dimensional commutative Lie algebra $\mathcal{L}$ (multiplication is defined as a commutator of vector fields).

Let $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be any set of vector fields forming the basis of this algebra. The explicit form of vector fields $\mathbf{b}_1, \ldots, \mathbf{b}_m$ is given by

$$\mathbf{b}_k = C^j_k \partial_{\lambda_j}, \qquad j = 1, \ldots, m, \tag{3.11}$$

where C^j_k is a constant and nondegenerate matrix. The coordinate lines of coordinate systems of the form (3.10) are then the integral curves of the vector fields of the form (3.11). Our symmetric conservative system can now be written as

$$\operatorname{div}(d\underline{\mathcal{F}} \odot \mathbf{b}_k) = 0, \qquad k = 1, \ldots, m. \tag{3.12}$$

The coordinate systems of the form (3.10) correspond to Lagrange–Liu multipliers. These multipliers are always defined up to additive constants but this arbitrariness leaves the vector fields from $\mathcal{L}$ invariant. The symmetric conservative system which in a particular system of coordinates takes the form (3.7) implies the additional balance law given by

$$\frac{\partial}{\partial x_i}\left[\lambda_k \frac{\partial \mathcal{F}^i}{\partial \lambda_k} - \mathcal{F}^i\right] = 0. \tag{3.13}$$

However, such an additional balance law depends on the choice of a coordinate system of the form (3.10). The symmetric conservative system written in the coordinate-free form (3.12) does not imply any particular additional conservation law but the whole family of conservation laws.

Hence, as a coordinate-free definition of a symmetric conservative system we can take a pair $(\mathcal{L}, \underline{\mathcal{F}})$ composed of an m-dimensional commutative Lie algebra $\mathcal{L}$ of vector fields together with a vector-valued 0-form $\underline{\mathcal{F}}$ defined on Q and with values in T_A. In literature, one sometimes discusses symmetric conservative systems generated by a single scalar potential [9–11]. From a geometrical point of view, the natural question concerns the relation between the coordinate-free definition of a symmetric conservative system as a pair $(\mathcal{L}, \underline{\mathcal{F}})$ and the scalar potential function. The Lie algebra $\mathcal{L}$ defines a distinguished covariant derivative, which is uniquely determined by a condition that vector fields from $\mathcal{L}$ are absolutely parallel. This covariant derivative can be generalized to the cross-sections of $T_A \otimes T(Q)$ ($T(Q)$ means here the tangent bundle of Q; in order to define this covariant derivative more precisely, we have to reduce the bundle $T_A \otimes T(Q)$ to make it associate with tensor bundles on Q).

Let L be a cross-section of $T_A \otimes T(Q)$ and let us assume that the covariant derivative of L vanishes. Let ξ be a real function on Q.

The vector-valued 0-form $\underline{\mathcal{F}}$ which occurs in the coordinate-free definition of a symmetric conservative system can be now defined as the contraction of L with the differential $d\xi$ of ξ:

$$\underline{\mathcal{F}} := L \odot d\xi\,. \tag{3.14}$$

Hence, as a coordinate-free definition of a symmetric conservative system generated by a single scalar potential we can take a triple $(\mathcal{L}, L, \xi)$ where $\mathcal{L}$ is the already described Lie algebra of vector fields, L is a cross-section of $T_A \otimes T(Q)$ which is absolutely parallel with respect to the connection determined by $\mathcal{L}$ and ξ is a real function on Q.

Let (λ_k), $k = 1, \ldots, m$ be a such coordinate system on Q that its natural base vectors ∂_{λ_k}, $k = 1, \ldots, m$ belong to $\mathcal{L}$. Then L can be written in the form

$$L = \sum_{i,k} L^{ik}\mathbf{e}_i \otimes \partial_{\lambda_k} = \sum_i \mathbf{e}_i \otimes \Big(\sum_k L^{ik}\partial_{\lambda_k}\Big) = \sum_i \mathbf{e}_i \otimes \mathbf{d}_i\,, \tag{3.15}$$

where L^{ik} is a constant matrix, $(\mathbf{e}_i)$, $i = 1, \ldots, n$ is a basis in T_A and

$$\mathbf{d}_i := \sum_k L^{ik}\partial_{\lambda_k} \tag{3.16}$$

are vector fields from $\mathcal{L}$. In order to write the discussed system explicitly, we have to insert Eq. (3.14) into Eq. (3.12). The class of symmetric conservative systems generated by a single scalar potential, introduced above, is a bit more general than that discussed in literature. A detailed discussion of such systems is outside the scope of this paper and shall be not given here.

4. Symmetrical hyperbolicity of symmetric conservative systems

In order to define symmetrical hyperbolicity of symmetric conservative systems in a coordinate-free manner, we have to introduce first the notion of a chronological structure which is very similar to the chronological structure on the Galilean space-time [12]. Let ψ be a non-zero form from T_A^*; $0 \neq \psi \in T_A^*$. By the inertial basis corresponding to ψ we shall mean such basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in T_A that

$$\begin{aligned} \langle \psi, \mathbf{e}_1\rangle &= 1 \\ \langle \psi, \mathbf{e}_l\rangle &= 0, \qquad l = 2, \ldots, n, \end{aligned} \tag{4.1}$$

and the form ψ itself shall be called the chronological form [5, 12]. Each inertial basis defines a class of coordinate systems on A of the form

$$\mathbb{R}^n \ni (t, z_l) \to a + t\mathbf{e}_1 + z_l\mathbf{e}_l \in A\,, \tag{4.2}$$

where a is an arbitrary point of A. Such coordinate systems shall be called the coordinate systems corresponding to ψ. As it can be easily checked, the transformation rule between different inertial bases corresponding to the same chronological form is

$$\begin{aligned} \mathbf{e}'_1 &= \mathbf{e}_1 + \beta_l\mathbf{e}_l, \qquad & l &= 2, \ldots, n, \\ \mathbf{e}'_l &= B_l^{l'}\mathbf{e}_{l'}, \qquad & l' &= 2, \ldots, n, \end{aligned} \tag{4.3}$$

where $B_l^{l'}$ is an arbitrary non-singular $(n-1) \times (n-1)$ matrix, and β_l are arbitrary real numbers.

Let L be a cross-section of $T_A \otimes T(Q)$ and let us assume that the covariant derivative of L vanishes. Let ξ be a real function on Q.

The vector-valued 0-form $\underline{\mathcal{F}}$ which occurs in the coordinate-free definition of a symmetric conservative system can be now defined as the contraction of L with the differential $d\xi$ of ξ:

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Hence, as a coordinate-free definition of a symmetric conservative system generated by a single scalar potential we can take a triple $(\mathcal{L}, L, \xi)$ where $\mathcal{L}$ is the already described Lie algebra of vector fields, L is a cross-section of $T_A \otimes T(Q)$ which is absolutely parallel with respect to the connection determined by $\mathcal{L}$ and ξ is a real function on Q.

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$$L = \sum_{i,k} L^{ik}\mathbf{e}_i \otimes \partial_{\lambda_k} = \sum_i \mathbf{e}_i \otimes \Big(\sum_k L^{ik}\partial_{\lambda_k}\Big) = \sum_i \mathbf{e}_i \otimes \mathbf{d}_i\,, \tag{3.15}$$

where L^{ik} is a constant matrix, $(\mathbf{e}_i)$, $i = 1, \dots, n$ is a basis in T_A and

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are vector fields from $\mathcal{L}$. In order to write the discussed system explicitly, we have to insert Eq. (3.14) into Eq. (3.12). The class of symmetric conservative systems generated by a single scalar potential, introduced above, is a bit more general than that discussed in literature. A detailed discussion of such systems is outside the scope of this paper and shall be not given here.

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and the form ψ itself shall be called the chronological form [5, 12]. Each inertial basis defines a class of coordinate systems on A of the form

$$\mathbb{R}^n \ni (t, z_l) \to a + t\mathbf{e}_1 + z_l\mathbf{e}_l \in A\,, \tag{4.2}$$

where a is an arbitrary point of A. Such coordinate systems shall be called the coordinate systems corresponding to ψ. As it can be easily checked, the transformation rule between different inertial bases corresponding to the same chronological form is

$$\begin{aligned} \mathbf{e}'_1 &= \mathbf{e}_1 + \beta_l\mathbf{e}_l, \qquad & l &= 2, \dots, n, \\ \mathbf{e}'_l &= B_l^{l'}\mathbf{e}_{l'}, \qquad & l' &= 2, \dots, n, \end{aligned} \tag{4.3}$$

where $B_l^{l'}$ is an arbitrary non-singular $(n-1) \times (n-1)$ matrix, and β_l are arbitrary real numbers.

Hence, the symmetrical hyperbolicity of a symmetric conservative system can be invariantly described in the terms of three objects; the Lie algebra $\mathcal{L}$, the vector-valued 0-form $\underline{\mathcal{F}}$ and the chronological form ψ.

5. Geometrical representation of symmetric systems

In this section we shall give a geometrical representation of symmetric systems.

Let us consider a system of P.D.E. which is defined by a set of vector-valued 1-forms w^k, $k = 1, \ldots, m$ in the manner described in Sec. 2. Let $\mathcal{R}^k$, $k = 1, \ldots, m$ be fields of forms on Q with the additional property that for each $q \in Q$ the forms $\mathcal{R}^k(q)$ are linearly independent.

Let us define a cross-section κ of $T^*(Q) \otimes T^*(Q) \otimes T_A$ by the formula

$$\kappa := \sum_{k=1}^{m} \mathcal{R}^k \otimes w^k . \tag{5.1}$$

The discussed system of P.D.E. can be equivalently written in terms of the contraction of κ with f_* (we assume that the index corresponding to $\mathcal{R}^k$ remains uncontracted and the contraction is taken with respect to these remaining indices for which it is well-defined); such a contraction gives

$$\langle \kappa, f_* \rangle = \left\langle \sum_{k=1}^{m} \mathcal{R}^k \otimes w^k, f_* \right\rangle = \sum_{k=1}^{n} \mathcal{R}^k \langle w^k, f_* \rangle = 0 , \tag{5.2}$$

what is equivalent to

$$\bigwedge_{k=1,\ldots,m} \langle w^k, f_* \rangle = w^k{}_j{}^i \frac{\partial y_j}{\partial x_i} = 0 . \tag{5.3}$$

It can easily be checked that such cross-sections of $T^*(Q) \otimes T^*(Q) \otimes T_A$ which are "symmetric in the first two indices" define symmetric systems of P.D.E.

In other words, the symmetric systems can be identified with the cross-sections of $[\mathrm{Sym}(T^*(Q) \otimes T^*(Q))] \otimes T_A$, where Sym denotes the operation of symmetrization.

We shall say that a system of P.D.E. which is invariantly defined as a m-dimensional vector subbundle S of $T^*(Q) \otimes T_A$ can be symmetrized if, for the system $w^1, \ldots, w^m$ of linearly independent cross-sections of S, there exists a set of linearly independent cross-sections $\mathcal{R}^1, \ldots, \mathcal{R}^m$ of $T^*(Q)$ which are such that

$$\sum_{k=1}^{m} \mathcal{R}^k \otimes w^k \in C\{[\mathrm{Sym}(T^*(Q) \otimes T^*(Q))] \otimes T_A\} , \tag{5.4}$$

where $C\{[\mathrm{Sym}(T^*(Q) \otimes T^*(Q))] \otimes T_A\}$ denotes the set of cross-sections of $[\mathrm{Sym}(T^*(Q) \otimes T^*(Q))] \otimes T_A$.

The relation (5.4) can be alternatively written in terms of the exterior product

$$\sum_{k=1}^{m} \mathcal{R}^k \wedge w^k = 0 \tag{5.5}$$

(see the notational convention for the exterior product of scalar and vector-valued forms, mentioned in Introduction).

In order to illustrate the meaning of this definition explicitly, let us write $\mathcal{R}^k$, $k = 1,\dots,m$ and w^k, $k = 1,\dots,m$ in coordinates

$$\mathcal{R}^k = \mathcal{R}^k_j dy^j, \qquad w^k = {w^k}_j{}^i dy^j \otimes \mathbf{e}_i \,. \tag{5.6}$$

Then Eq. (5.4) means that the "field of two-point tensors" given by

$$\sum_{k=1}^{m} \mathcal{R}^k \otimes w^k = \mathcal{R}^k_{j'} {w^k}_j{}^i dy^{j'} \otimes dy^j \otimes \mathbf{e}_i \tag{5.7}$$

is symmetric in the first two indices and the discussed system of P.D.E. takes the following symmetrical form:

$$\langle \mathcal{R}^k_{j'} {w^k}_j{}^i dy^{j'} \otimes dy^j \otimes \mathbf{e}_i, f_* \rangle = \mathcal{R}^k_{j'} {w^k}_j{}^i \frac{\partial y_j}{\partial x_i} = 0 \,. \tag{5.8}$$

Of course, symmetrizability of a given system of partial differential equations does not depend on the choice of the forms $w^1,\dots,w^m$.

Let $\Omega^1,\dots,\Omega^m$ be the another set of linearly independent cross-sections of the vector bundle S. Then

$$w^k = M^k_I \Omega^I, \qquad \det M^k_I \neq 0 \tag{5.9}$$

and

$$\sum_{k=1}^{m} \mathcal{R}^k \otimes w^k = \sum_{k=1}^{m} \mathcal{R}^k \otimes M^k_I \Omega^I = \sum_{I=1}^{m} \mathcal{R}^k M^k_I \otimes \Omega^I \tag{5.10}$$

what shows that the transformation rule for "symmetrizing forms" is

$$\mathcal{R}^k \to \tilde{\mathcal{R}}^I = \sum_{k=1}^{m} \mathcal{R}^k M^k_I \,. \tag{5.11}$$

Another obvious observation is that the solution of Eq. (5.5) for the forms $\mathcal{R}^1,\dots,\mathcal{R}^m$ is defined up to multiplication by a non-vanishing real function.

Every field of "symmetrizing forms" $\mathcal{R}^1,\dots,\mathcal{R}^m$ defines a covariant derivative on Q; such covariant derivative is uniquely determined by a condition that the fields $\mathcal{R}^1,\dots,\mathcal{R}^m$ are absolutely parallel. In general, the torsion corresponding to this covariant derivative does not vanish. The torsion of this connection vanishes if and only if the fields of forms $\mathcal{R}^1,\dots,\mathcal{R}^m$ are given by differentials of real functions

$$\mathcal{R}^k = d\gamma_k, \qquad k = 1,\dots,m \,, \tag{5.12}$$

and the functions $\gamma_1,\dots,\gamma_m$ form a coordinate system on Q. For convenience, we shall say that fields of forms $\mathcal{R}^1,\dots,\mathcal{R}^m$ are "aholonomic" if this torsion does not vanish and that fields of forms $\mathcal{R}^1,\dots,\mathcal{R}^m$ are "holonomic" if the corresponding torsion vanishes.

In the theory of symmetric systems, one often symmetrizes system of P.D.E. by multiplying it by a matrix of the second derivatives of a certain function [13]. Our geometrical picture immediately suggests the following

OBSERVATION. The determined system of the first-order quasi-linear P.D.E. can be symmetrized by holonomic fields of 1-forms if and only if in a certain coordinate system it can be symmetrized by a matrix of the second derivatives. The corresponding coordinate system is not determined uniquely.

Proof. Let the discussed quasi-linear system be symmetrizable by a holonomic system of 1-forms $d\gamma_1, \ldots, d\gamma_m$. We will show that in a certain class of coordinate systems our system of P.D.E. can be symmetrized by the matrix of second derivatives. Let us write the relation

$$\sum_{k=1}^{m} d\gamma_k \wedge w^k = 0 \tag{5.13}$$

in a certain coordinate system (η_j), $j = 1, \ldots, m$ on Q:

$$\sum_{k=1}^{m} \frac{\partial \gamma_k}{\partial \eta_j} d\eta_j \wedge w^k = 0 \tag{5.14}$$

(for convenience, the terms $w^1, \ldots, w^m$ remain in the coordinate-free form). In particular, we can choose the coordinate system (η_j), $j = 1, \ldots, m$ in such a way that $\partial\gamma_k/\partial\eta_j$ is a symmetric and constant nondegenerate matrix

$$\frac{\partial \gamma_k}{\partial \eta_j} = E_{kj}, \qquad E_{kj} = E_{jk}, \qquad \det E_{kj} \neq 0\,. \tag{5.15}$$

The coordinate systems (γ_k), $k = 1, \ldots, m$ and (η_j), $j = 1, \ldots, m$ are then related by the affine transformation

$$\gamma_k = E_{kj}\eta_j + g_k\,, \tag{5.16}$$

where g_k, $k = 1, \ldots, m$ are arbitrary real numbers. Let us define the real function $\mu : Q \to \mathbb{R}$ which in the coordinates (η_j), $j = 1, \ldots, m$ is given by

$$\mu := \frac{1}{2} E_{kj}\eta_k\eta_j + g_j\eta_j\,. \tag{5.17}$$

Then

$$\gamma_k = \frac{\partial \mu}{\partial \eta_k} \tag{5.18}$$

and

$$\frac{\partial^2 \mu}{\partial \eta_j \partial \eta_k} = \frac{\partial \gamma_k}{\partial \eta_j} = E_{kj}\,, \tag{5.19}$$

which shows that our system of P.D.E. can be symmetrized by the matrix of the second derivatives of the function μ (the derivatives are computed in the coordinate system (η_j), $j = 1, \ldots, m$).

In turn, let us assume that in the coordinates (η_j), $j = 1, \ldots, m$ the discussed system of P.D.E. can be symmetrized by the matrix of the second derivatives of the function $H, H : Q \to \mathbb{R}$:

$$\frac{\partial^2 H}{\partial \eta_k \partial \eta_j} d\eta_j \wedge w^k = 0\,. \tag{5.20}$$

Then

$$\det \frac{\partial^2 H}{\partial \eta_k \partial \eta_j} \neq 0 \tag{5.21}$$

and

$$\mathcal{R}^k := \frac{\partial^2 H}{\partial \eta_k \partial \eta_j} d\eta_j = d\left(\frac{\partial H}{\partial \eta_k}\right) \tag{5.22}$$

define the holonomic system of forms which symmetrizes our system of partial differential equations. The forms (5.22) are the differentials of the coordinate lines of the coordinate system (γ_k), $k = 1, \ldots, m$ which is defined by

$$\gamma_k := \frac{\partial H}{\partial \eta_k}. \tag{5.23}$$

Let us now consider a system of balance laws defined by the set of 1-forms $d\mathbf{u}^1, \ldots, d\mathbf{u}^m$ in the manner discussed in Sec. 2. If such a system can be symmetrized, then the corresponding fields of the symmetrizing 1-forms are either aholonomic or holonomic. In the latter case, the identity

$$\sum_{k=1}^{m} d\gamma_k \otimes d\mathbf{u}^k \in C\{[\mathrm{Sym}(T^*(Q) \otimes T^*(Q))] \otimes T_A\} \tag{5.24}$$

implies

$$\sum_{k=1}^{m} d\gamma_k \wedge d\mathbf{u}^k = 0, \tag{5.25}$$

what means that this system is symmetric conservative in the sense described in Sec. 3. The condition (5.25) can be written also in the coordinate-free manner; the symmetrizing 1-forms are then dual to the basis $\mathbf{b}^1, \ldots, \mathbf{b}^m$ of the algebra $\mathcal{L}$ (see Sec. 3). The further discussion depends on whether the considered system of conservation laws can be transformed to the Cauchy form. If this is not the case, then the only thing we know is that in certain coordinates the system can be symmetrized by a matrix of the second derivatives. In turn, if the discussed system of P.D.E. can be transformed to the Cauchy form in the direction of the chronological form $\psi \in T_A^*$, $\psi \neq 0$, then $\mathbf{u}^k \odot \psi$, $k = 1, \ldots, m$ form a coordinate system on Q (see Sec. 2) and the differentials $d\gamma_k$, $k = 1, \ldots, m$ can be written in these coordinates:

$$d\gamma_k = \frac{\partial \gamma_k}{\partial(\mathbf{u}^{k'} \odot \psi)} d(\mathbf{u}^{k'} \odot \psi). \tag{5.26}$$

On the other hand, any symmetric conservative system must satisfy

$$d\mathbf{s} = \sum_{k=1}^{m} \gamma_k d\mathbf{u}^k, \tag{5.27}$$

what contracted with ψ gives

$$d(\mathbf{s} \odot \psi) = \sum_{k=1}^{m} \gamma_k d(\mathbf{u}^k \odot \psi) \tag{5.28}$$

and hence [5]

$$\frac{\partial(\mathbf{s} \odot \psi)}{\partial(\mathbf{u}^k \odot \psi)} = \gamma_k. \tag{5.29}$$

Combining (5.29) with (5.26) we see that in this case the discussed system of P.D.E. can be symmetrized by the matrix of the second derivatives

$$\frac{\partial^2(\mathbf{s} \odot \psi)}{\partial(\mathbf{u}^k \odot \psi)\partial(\mathbf{u}^{k'} \odot \psi)}, \tag{5.30}$$

which is formed by differentiating the projection of the entropy 0-form **s** on the chronological form ψ; the derivatives are computed in the coordinate system $(\mathbf{u}^k \odot \psi)$, $k = 1, \ldots, m$.

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