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Nonlinear functionals in existence theorems for reaction-diffusion systems

B. KAŹMIERCZAK (WARSZAWA)

THE NOTION of nonlinear functional is used to prove the existence of solutions for time-independent reaction-diffusion systems, in which nonlocal terms may occur and whose coefficients may depend on first derivatives.

Introduction

THIS WORK was stimulated by the interesting papers [2] and [3] of FITZGIBBON and MORGAN, who used the notion of nonlinear functionals in existence proofs for some time-independent reaction-diffusion systems. Its aim is to generalize, in a way, one of the theorems in [2]. The generalization consists in the fact that we do not assume a separable structure of the functional (as in [2]) and consider a more general type of equations than those discussed in [2] or [3]. The coefficients of elliptic operators may depend on u (and x) whereas the right-hand sides may have nonlocal terms and depend on the first derivatives of u (and x, of course). However, we were forced to impose more complex conditions than those in [2] or [3].

1. Setting of the problem and main assumptions

We consider the following system of elliptic equations:

$$\begin{aligned} -L[u]u_i &= F_i[u] \quad \text{in } D, \\ u_i &= t_i \qquad \text{on } \partial D, \end{aligned} \tag{1.1}$$

where $1 \le i \le m$ and there is no summation over i. In subsequent assumptions $\alpha \in (0,1)$ will be a fixed number.

A. D is a bounded domain in $\mathbb{R}^n$, $n \ge 1$, with boundary of $C^{2+\alpha}$ class. Without losing generality we may assume that $x_j > 0$, $j = 1,\ldots,n$, for $x = (x_1,\ldots,x_n) \in \overline{D}$. ■

B. For $i = 1,\ldots,m$:
1) $t_i \in C^{2+\alpha}(\overline{D}) \cap C^{2+\alpha}(\partial D)$,
2) $F_i : C^2(\overline{D}) \longrightarrow C^\alpha(\overline{D})$,
3) $L[u] = \sum\limits_{j,l=1}^{n} A_{jl}(x,u(x))\partial_{jl}^2 - \mathcal{C}(x,u(x),\partial u(x)) \cdot \nabla$,

where all A_{jl} are of C^1 class in every compact subset of $\overline{D} \times \mathbb{R}^m$, and for every finite $u = (u_1,\ldots,u_m) \in \mathbb{R}^m$, $x \in \overline{D}$ and $\xi \in \mathbb{R}^m$, $\xi \ne 0$, there exists a real $s > 0$ such that

$$\sum_{j,l=1}^{n} A_{jl}(x,u)\xi_j\xi_l > s\xi^2.$$

Here $\mathcal{C} : \overline{D} \times \mathbb{R}^m \times \mathbb{R}^{nm} \to R^n$ and $\mathcal{C}$ is of C^1 class in every compact subset of its arguments.

■

REMARK. In the above notation F_i should be treated as an operator. In particular it could be of the form

$$F_i[u](x) = \mathbf{c}_i(x)(\nabla u_i(x)) + f_i(x, u).$$

■

Let M be a closed unbounded (in general) subset of the space $\mathbb{R}^m$.

C. There exist a $C^2(M, \mathbb{R}^1)$ function H such that

1) $H(u)$ tends to plus infinity for $\|u\|$ tending to infinity, where $\| \ \|$ denotes a norm in $\mathbb{R}^m$;

2) there exist finite $N \in \mathbb{R}^1$ and a bounded function $\lambda^* : \mathbb{R}^1_+ \to \mathbb{R}^1$ such that for all $u \in C^2(\overline{D}, M)$, $\|u\|_{C^0} \leq \underline{u}$:

$$\sum_{\substack{i,j=1,\dots,m \\ p,z=1,\dots,n}} -\{(1-\beta)s\delta_{pz} + \beta A_{pz}(x, u(x))\} H_{ij}(u(x)) u_{i,p}(x) u_{j,z}(x)$$
$$+ \sum_{j=1,\dots,m} \beta H_j(u(x)) F_j[u](x) < \lambda^*(\underline{u}) H(u(x)) + N$$

for all $x \in \overline{D}$, $\beta \in [0,1]$. (δ_{pz} denotes Kronecker's delta, $u_{j,z}$ denotes $\partial_z u_j$ and H_j denotes the partial derivative of H with respect to u_j).

■

The next assumption describes the relation between λ^*, L, D and s (comp. B). First of all, one can notice that every $A_{jj}(x, u) > s$.

D. Let $u \in C^2(\overline{D}, M)$, $\|u\|_{C^0(\overline{D})} < \underline{u}$ and $\phi \in C^2(\overline{D}, [0,1])$. Let

$$\mathcal{L}^\phi[u] := \phi L[u] + (1-\phi)s\Delta.$$

We assume that for all the possible solutions U of the linear scalar problem

$$\begin{aligned} -\mathcal{L}^\phi(x, u(x))\xi &= \lambda\phi(x)\zeta + f \quad &&\text{in } D, \\ \zeta &= g &&\text{on } \partial D, \end{aligned} \tag{1.2}$$

with $\lambda \leq \lambda^*(\underline{u})$, there exists a finite constant K independent of ϕ, u, f and g such that

$$\|U\|_{C^0(\overline{D})} < K\|g\|_{C^0(\overline{D})} + \|f\|_{C^0(\overline{D})}.$$

■

REMARK. By standard results on eigenvalues of elliptic operators it is possible to give explicit conditions implying D.

D*. There exist $\delta > 0$, $J \in \{1, \dots, n\}$, $\eta \in \mathbb{R}^1_+$ and continuous functions $c : \mathbb{R}^m \to \mathbb{R}^1$ and $z : \mathbb{R}^m \to \mathbb{R}^1$ such that for all non-negative $\underline{u}$ and $u \in M$, $\|u\|_{\mathbb{R}^m} < \underline{u}$ we have:

1) $\mathcal{C}_J(x, u, p) < c(u)$ for all $x \in \overline{D}$ and $p \in \mathbb{R}^{nm}$;

2) $\eta > \sup_{x \in D} x_J$;

3) $s(z(u))^2 - c(u)z(u) > 1$;

4) $\sup_{x \in D} \{\exp(z(u)\eta) - \exp(z(u)x_J)\}\lambda^*(\underline{u}) < 1 - \delta$.

■

LEMMA. D^* implies D. Besides, for every u and ϕ (such as in D) we have $\lambda^*(\underline{u}) \leq \lambda_{(\phi,u)}$, where $\lambda_{(\phi,u)}$ is the smallest eigenvalue of the operator $(-\mathcal{L}^\phi[\mathrm{u}])$ (defined in D). ■

Proof. For a given $u \in C^2(\overline{D}, M)$ with $\|\mathrm{u}\|_{C^0(\overline{D})} < \underline{u}$ let $M_{(u)} := M \cap \{\sigma \in R^m : \|\sigma\| \leq \underline{u}\}$ and $c_{(u)} = \sup_{\sigma \in M_{(u)}} c(\sigma)$. If $c_{(u)} = c(\sigma^*)$, then let $z_{(u)} = z(\sigma^*)$. Now we can proceed (with slight modifications) as in the proof of Theorem 8.8 in [4]. Thus, first let us note that $\mathcal{L}^\phi(x, u(x))$ can be written as $\sum a_{jl}(x)\partial^2_{jl}$ and $a_{JJ}(x) > s$. Without losing generality we may take $J = 1$. Let us define

$$h(x) = \|g\|_{C^0(\overline{D})} + [\exp(z(\sigma^*)\eta) - \exp(z(\sigma^*)x_1)]\|f^*\|_{C^0(\overline{D})},$$

where $f^* = f + \lambda\phi U$. Then, according to $D^*.1$, 3 we have

$$\begin{aligned}-\mathcal{L}^\phi[u]h &= \exp(z(\sigma^*)x_1)[a_{11}(x)(z(\sigma^*))^2 - z(\sigma^*)\mathcal{C}_1(x, u(x), \partial u(x))]\|f^*\|_{C^0(\overline{D})}\\ &\geq \exp(z(\sigma^*)x_1)[s(z(\sigma^*))^2 - z(\sigma^*)c(u(x)))]\|f^*\|_{C^0(\overline{D})}\\ &\geq exp(z(\sigma^*)x_1)[s(z(\sigma^*))^2 - z(\sigma^*)c(\sigma^*)]\|f^*\|_{C^0(\overline{D})} \geq exp(z(\sigma^*)x_1)\|f^*\|_{C^0(\overline{D})}.\end{aligned}$$

Now, we take $\nu = U - h$. Then $\nu = g - h \leq 0$ on ∂D and $\mathcal{L}^\phi[u]\nu \geq f^* + \|f^*\|_{C^0(\overline{D})} \geq 0$ in $\overline{D}$. Hence from the maximum principle (see for example Theorem 8.1 in [4]) we infer that $\nu \leq 0$, i.e. $U \leq h$ in $\overline{D}$. Similarly, if $\nu = U + h$, then in the same way we can prove that $\nu \geq 0$, i.e. $U \geq -h$ in $\overline{D}$. Thus, we obtain an implicit bound for U

$$\|U\|_{C^0(\overline{D})} \leq \|g\|_{C^0(\overline{D})} + W\|f^*\|_{C^0(\overline{D})},$$

where $W = \sup_{x\in D}[exp(z(\sigma^*)\eta) - exp(z(\sigma^*)x_1)]$. Thus

$$\|U\|_{C^0(\overline{D})} \leq \|g\|_{C^0(\overline{D})} + W\|f\|_{C^0(\overline{D})} + W\lambda\|U\|_{C^0(\overline{D})}.$$

Hence, for λ satisfying $D^*.4$ we obtain the inequality

$$\|U\|_{C^0(\overline{D})} \leq (\|g\|_{C^0(\overline{D})} + W\|f\|_{C^0(\overline{D})})\delta^{-1}.$$

To prove the second part of the Lemma suppose that, for some r and u, we have $\lambda = \lambda^*(\underline{u}) = \lambda_{(\phi u)}$. But then, according to the first part of the Lemma, we would be able to obtain *a priori* estimates in $C^0(\overline{D})$, which is impossible. ■

As in [2], solutions of (1.1) will be approximated by solutions of the "bounded" problems. Namely, let $k^0_r \in C^2(\mathbb{R}^m, [0,1])$, $k^1_r \in C^2(\mathbb{R}^{nm}, [0,1])$, where $r > 0$ and

$$k^0_r(u) = \begin{cases} 1 & \text{for } \|u\| < r,\\ 0 & \text{for } \|u\| > 2r,\end{cases} \quad k^1_r(p) = \begin{cases} 1 & \text{for } \|p\| < r,\\ 0 & \text{for } \|p\| > 2r,\end{cases}$$

($\|p\|$ denotes a $\mathbb{R}^{nm}$ norm of p). We choose k^0_r, k^1_r in such a way that their first and second derivatives tend to 0 in C^0 norm for r tending to plus infinity. Let

$$K_r[u](x) := k^0_r(u(x))k^1_r(\partial u(x)).$$

Now, for $\nu \in C^2(\overline{D}, \mathbb{R}^m)$ let

$$\mathcal{L}_r[\nu] := K_r[\nu]L[\nu] + (1 - K_r[\nu])s\Delta - K_r[\nu]\mathcal{C}[\nu]\cdot\nabla, \tag{1.3}$$

where for the sake of simplicity we have denoted $\mathcal{C}(x,\nu,\partial\nu)$ by $\mathcal{C}[\nu]$. Let $w = T\nu$ be the unique $C^{2+\alpha}(\overline{D})$ solution of the problem:

$$\begin{aligned} -\mathcal{L}_r[\nu]w_i &= K_r[\nu]F_i[\nu] && \text{in } D,\\ w_i &= t_i && \text{on } \partial D, \end{aligned}$$

$1 \le i \le m$. For every finite positive r the operator T is a bounded linear operator from $C^2(\overline{D})$ to $C^{2+\alpha}(\overline{D})$. Thus, it is continuous in $C^2(\overline{D})$ and it maps all of $C^2(\overline{D})$ onto a bounded subset of $C^{2+a}(\overline{D})$. Due to the Schauder fixed point theorem there exist a $C^2(\overline{D})$ solution of the system

$$(1.4)_r \qquad \begin{aligned} -\mathcal{L}_r[u]u_i &= K_r[u]F_i[u] && \text{in } D,\\ u_i &= t_i && \text{on } \partial D. \end{aligned}$$

In the subsequent section our aim is to prove that for r sufficiently large the solution of $(1.4)_r$ is also a solution of the initial system (1.1) with values in M. To do this we impose two other assumptions. They concern the possibility of *a priori* estimates and invariance of M.

E. There exists a continuous function $Q : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ independent of r, such that if $u_r \in C^2(\overline{D})$ satisfies $(1.4)_r$ and $\|u_r(x)\| < P$ for $x \in \overline{D}$, then $\|\partial u_r(x)\| + \|\partial^2 u_r(x)\| < Q(P)$ for all $x \in \overline{D}$. ($\|\ \|$ denote norms in $\mathbb{R}^m$, $\mathbb{R}^{nm}$ and $\mathbb{R}^{n^2 m}$, respectively). ∎

REMARK. If F has no nonlocal terms, then E follows for example (under some additional conditions) from sections VIII.1-4 in [5]. ∎

F. If $t = (t_1,\dots,t_m) \in C^{2+\alpha}(\overline{D}, M)$ and u_r satisfies $(1.4)_r$, then $u_r(x) \in M$ for all $x \in \overline{D}$. ∎

2. Existence theorem

Now, we are in a position to prove our existence theorem.

THEOREM. *Assume* A, B, C, D, E *and* F. *Then, there exist at least one* $C^{2+\alpha}(\overline{D}, M)$ *solution of* (1.1). ∎

Proof. The proof of the theorem will consist in showing that the solutions of $(1.4)_r$ are bounded in C^0 norm uniformly in r. To prove this fact let u_r be a smooth solution of $(1.4)_r$ and let $H_r(x) := H(u_r(x))$. By a straightforward calculation we conclude that H_r satisfies, according to C and (1.3), the following identity:

$$\begin{aligned} -\mathcal{L}_r(x,u_r(x))H_r &= \lambda^*(\underline{u}_r)K_r[u_r](x)H_r + N + B_r(x) && \text{in } D,\\ H_r &= H(t) && \text{on } \partial D, \end{aligned}$$

where λ^* is defined in C and D, $\underline{u} = \|u_r\|_{C^0(\overline{D})}$ and $B_r(x) \le 0$ for $x \in \overline{D}$. Now, let $\overline{H}_{re}$ denote the unique smooth solution of the linear problem:

$$\begin{aligned} -\mathcal{L}_r(x,u_r(x))z &= \lambda^*(\underline{u}_r)K_r[u_r](x)z + N && \text{in } D,\\ z &= h_e && \text{on } \partial D, \end{aligned}$$

where $h_e \in C^{2+\alpha}(\overline{D})$ and $H(t(x)) + e \geq h_e(x) \geq H(t(x))$ for all $x \in \overline{D}$ and some $e > 0$, which can be taken arbitrarily small. According to Lemma, the functions $\overline{H}_{re}$ are bounded in $C^0(\overline{D})$ norm uniformly in r.

Let $R_{re} := \overline{H}_{re} - H_r$. Then R_{re} satisfies the identity:

$$\begin{aligned} -\mathcal{L}_r(x, u_r(x))R_{re} &= \lambda^*(\underline{u}_r)K_r[u_r](x)R_{re} - B_r(x) \quad && \text{in } D, \\ R_{re} &= \underline{h}_e && \text{on } \partial D, \end{aligned}$$

where $C^2(\overline{D}) \ni \underline{h}_e \to 0$ as $e \to 0$ uniformly on ∂D. According to the second part of Lemma 1.2 and Theorem 4.4 in [1] we can find a function $\underline{R}_r \geq 0$ satisfying the same equation but with zero boundary condition. If $S_{re} := R_{re} - \underline{R}_r$, then

$$\begin{aligned} -\mathcal{L}_r(x, u_r(x))S_{re} &= \lambda^*(\underline{u}_r)K_r[u_r](x)S_{re} \quad && \text{in } D, \\ S_{re} &= \underline{h}_e && \text{on } \partial D. \end{aligned}$$

As $S_{re} \in C^2\overline{D}$ and $\underline{h}_e \leq e$ uniformly in on ∂D, then due to Lemma we conclude that $H_r(x) \leq \overline{H}_r(x) + Ke$ for some finite positive K and all $e \to 0$. Thus $H(u_r(x)) \leq \overline{H}_r(x)$, where by $\overline{H}_r$ we have denoted the unique in $W^{2,q}(D)$, $q > r$, limit of H_{re}. It is bounded in C^1 norm uniformly in r. According to C we conclude that u_r is bounded in $C^0(\overline{D})$ norm uniformly in r by some finite constant P. Due to E this implies boundedness of u_r in C^1 and C^2 norms. Thus, for sufficiently large r, u_r is a solution of the problem (1.1). The Theorem is proved.

■

REMARK. It is easy to note that when coefficients of L do not depend on u then we can get rid of the auxiliary Laplacean and take $\beta = 1$ in C and D with $\mathcal{L}_r[\nu] := L - K_r[\nu]\mathcal{C}[\nu] \cdot \nabla$ in (1.3).

■

3. An example

Let us consider the following system of equations (comp. [2] p. 36):

$$-d_1L(x, u(x))u_1(x) = u_3(x) - u_1(x)u_2(x) + u_1(x) \int_D K_1(x, u(y))dy,$$

$$-d_2L(x, u(x))u_2(x) = u_3(x) - u_1(x)u_2(x) + u_2(x) \int_D K_2(x, u(y))dy,$$

$$-d_3L(x, u(x))u_3(x) = u_1(x)u_2(x) - u_3(x) + u_1(x) \int_D K_1(x, u(y))dy,$$

$$(u_1, u_2, u_3)(x) = (t_1, t_2, t_3)(x) \quad \text{for } x \in \partial D,$$

where d_1, d_2, d_3 are positive constants and L is the same as in B.3 Dividing the i-th equation by d_i we obtain the system of the form (1.1). First of all, we note that by taking $c(u) \equiv 0$ we can deduce the existence of λ^* fulfilling the assumption D. Now, let

$$M := \{u \in \mathbb{R}^3 : u_i \geq 0, i = 1, 2, 3\}.$$

We assume that

I. K_1, K_2, K_3 are smooth in every compact subset of $\overline{D} \times M$.

■

II. For $i = 1, 2, 3$, all $x \in \overline{D}$ and all $u \in C^0(\overline{D}, M)$

$$\int_D K_i(x, u(y))dy \leq \min\{d_1, d_2, 2d_3\}\lambda^*.$$

■

First of all we note that, if $t(x) \in M$ for all $x \in \overline{D}$, then F is fulfilled. The proof is almost exactly the same as that in [2] (Lemma 2.1). Due to the classical result of Ladyzhenskaya and Ural'tseva (Sec. VIII.1-4 in [5]) the assumption E is fulfilled. By taking

$$H(u) = d_1u_1 + d_2u_2 + 2d_3u_3$$

(as in [2]) one can verify check that C is fulfilled due to II, B and Lemma 1.2. Thus the system possesses at least one solution of class $C^{2+\alpha}(\overline{D}, M)$.

References

1. H. AMANN, *Fixed point equations and nonlinear eigenvalue problem in ordered Banach spaces*, SIAM Rev., **18**, 620–709, 1976.
2. W.E. FITZGIBBON and J. MORGAN, *Steady state solutions for certain reaction-diffusion systems*, Nonlinear Analysis, **15**, 27–37, 1990.
3. W.E. FITZGIBBON and J. MORGAN, *Existence of solutions for a class of weakly coupled semilinear elliptic systems*, J. Diff. Eqns., 7, 351–368, 1989.
4. J. SMOLLER, *Shock waves and reaction-diffusion equations*, Springer-Verlag, 1983.
5. O. A. LADYZHENSKAYA and N.N. URAL'TSEVA, *Linear and quasi-linear elliptic equations*, Academic Press, New York 1968.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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