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On the geometric structure of the stress and strain tensors, dual variables and objective rates in continuum mechanics

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THE GEOMETRIC STRUCTURE of the stress and strain tensors arising in continuum mechanics is investigated. All tensors are classified into two families, each consists of two subgroups regarded as physically equivalent since they are isometric. Special attention is focussed on the Cauchy stress tensor and it is proved that, corresponding to it, no dual strain measure exists. Some new stress tensors are formulated and the physical meaning of the stress tensor dual to the Almansi strain tensor is made apparent by employing a new decomposition of the Cauchy stress tensor with respect to a Lagrangian basis. It is shown that push-forward/pull-back under the deformation gradient applied to two work conjugate stress and strain tensors do not result in further dual tensors. The rotation field is incorporated as an independent variable by considering simple materials to be constrained Cosserat continua. By the geometric structure of the involved tensors, it is claimed that only the Lie derivative with respect to the flow generated by the rotation group (Green–Naghdi objective rate) can be considered as occurring naturally in solid mechanics and preserving the physical equivalence in rate form.

1. Introduction

IN THE LAST DECADE the interest in the simulation of large solid deformations incorporating finite strains has been immensely growing. The inclusion of finite strain deformations necessitates a geometrically exact description of the strain measures and enforces a new look at the corresponding stress tensors. In fact, the generalization of different small strain theories of solid mechanics such as in the fields of elasticity, visco-elasticity, and elasto-viscoplasticity to include finite strains turn out to be a difficult task. Alone the definition of an appropriate objective stress rate has motivated an intensive research without arriving, according to our opinion, to any satisfactory results (for some more or less heuristic attempts to overcome these difficulties see e.g. [5, 6, 22, 32]). It is our belief that a deep understanding of finite strain theories cannot be achieved without a corresponding understanding of the structure of the strain and stress tensors involved.

This paper is devoted to the study of the geometric structure of the stress and strain tensors and the way the free energy is depending on the latter, defining work conjugate stresses and motivating the definition of objective rates. For a general account we refer of course to standard literature, e.g. [9, 19, 21, 31, 33]. Anyhow, the view on the subject as we intend to give, although classical, provides us, as we believe, with some new results and allows for a deeper understanding of the subject. As far as a Lagrangian formulation is concerned, a purely intrinsic approach was discussed recently by ROUGÉE [24]; a result concerning the dual variable of the inverse tensor of the right Cauchy-Green strain tensor can be found already there.

From the very beginning, it would be constructive to clarify some aspects of the terminology used. First, we will not use the notion of material or spatial tensors since we believe that such a terminology is misleading and lacks any precise physical definition. There is of course something like material (Lagrangian) as well as spatial (Eulerian) descriptions. But it is of crucial importance to understand that the kind of description is completely independent of any stress and strain tensors used. Physical objects such as strain and stress tensors cannot be made dependent on the kind of description or on the coordinate system one is adopting. It is meaningless to consider the Green strain tensor a material tensor, the Almansi strain tensor a spatial one etc. It is of course another story whether a special strain tensor is appropriate for a special kind of description (a precise statement will be given in this paper later on).

Special attention will be focussed on the Cauchy stress tensor and its dual strain measure. In contrast to many statements to be found in the literature concerning the dual variable of the Cauchy stress tensor where it is related either to the Almansi strain tensor or to a logarithmic strain measure (see e.g. [17, 18, 23, 32]), we will show that such a dual variable does not exist. Consequently there will not be any objective rate for the aforementioned stress tensor which can be regarded as a physically useful choice. We believe that by this it is understood why all attempts to define an appropriate objective rate for the Cauchy stress tensor had failed to give realistic responses as applied to different theories of solid mechanics.

Overview and basic results

After a short review of the basic geometric relations needed we define the configuration space with a rotation field incorporated explicitly and derive the corresponding tangent space. The flows generated by the tangent vectors allows for the definition of Lie derivatives with respect to the velocity vector as well as the angular velocity vector. We introduce the operations of *pull-back* and *push-forward* and adapt them to the convected coordinate system used. In the following section different strain measures are defined: most of them are well known in the literature. By analysing their geometric structure, they will be classified into two families for which we adopt, just for convenience, the names strain tensors of the first and of the second type. This classification is carried out completely independent of the kind of description underlying any practical computation for which the termination Lagrangian (material) and Eulerian (spatial) are frequently used. In the following section a similar geometric structure is found for the stress tensors and the same classification is made. A family of new stress tensors is defined and the notion of physically equivalent tensors is introduced. Hereby isometric tensors related by a transformation under the rotation tensor **R** are denoted to by physically equivalent. In other words, mathematical isometry denotes physical equivalence (this statement is of course nothing but the requirement of the invariance of physical relations with respect to the Euclidean group). The subsequent discussion deals with the dependence of the free energy on the strain tensors. We gain the important result that *push-forward/pull-back* transformations of two dual tensors do not preserve the duality. We define a new stress tensor conjugate to the Almansi strains, and the conjugate strains for the Cauchy stress tensor will be discussed where it is proved that such a conjugate variable does not exist. The section closes with a discussion of the material and spatial descriptions where a justification for our classification of tensors is given. It turns out that what we classified as tensors of the first type are well suited for a material description, whereas tensors of the second type are well suited for a spatial one. In the following section the symmetry of the stress tensors is considered and a unified approach to incorporate the rotation field as an independent variable is discussed. It is shown that simple materials can be considered as constrained Cosserat continua. As a by-product of the approach, we obtain for the isotropic case a new variational principle which secures the symmetry of the stretch tensor as well.

We close by a discussion of the objective stress rates, a subject which motivated up to now many controversial debates in the literature. We believe that in the light of the geometric structure of the stress and strain tensors, a very satisfying approach to objective rates can be achieved. We arrive at the result that only one rate, the Lie derivative with respect to the flow generated by the rotation group (the Green–Naghdi rate), can be regarded as arising naturally and as physically consistent.

2. Notion and basic geometric relations

Let $\mathcal{B} \subset \mathcal{R}^3$ be an open set defining a body. The map $\phi(t) : \mathcal{B} \to \mathcal{R}^3$ is an embedding depending on a well chosen parameter $t \in \mathcal{R}$. Hereby $\phi_0 = \phi(t = t_0)$ defines a reference configuration which enables the identification of the material points, so that ϕ_0 is the identity map. With $\phi_0\mathcal{B} = \mathcal{B}_0$ and $\phi(t)\mathcal{B} = \mathcal{B}_t$ one can write

$$\phi(t) : \mathcal{B}_0 \to \mathcal{B}_t \,. \tag{2.1}$$

With $\mathbf{X} \in \mathcal{B}_0$ and $\mathbf{x} \in \mathcal{B}_t$ we get

$$\mathbf{x}(t) = \phi(\mathbf{X}, t) \tag{2.2}$$

and

$$\mathbf{X}(t) = \phi^{-1}(\mathbf{x}, t) \,. \tag{2.3}$$

In this paper we consider convected coordinates which are attached to the body. Given coordinate charts defined on $\mathcal{B}_0$, by the map ϕ corresponding coordinate charts are induced on $\mathcal{B}_t$. On the contrary, the map ϕ^{-1} induces coordinate charts on $\mathcal{B}_0$ if such charts are defined on $\mathcal{B}_t$. The first case is given within a material (Lagrangian) description, where the second is given within a spatial (Eulerian) one. Let now $\{\vartheta^i\}$ be appropriately defined convected coordinate charts, $\mathcal{T}\mathcal{B}_0$ and $\mathcal{T}\mathcal{B}_t$ be the tangent spaces of $\mathcal{B}_0$ and $\mathcal{B}_t$, respectively. We have

$$\mathbf{G}_I = \partial \mathbf{X}/\partial \vartheta^I \quad \text{with} \quad \mathbf{G}_I \in \mathcal{T}\mathcal{B}_0 \tag{2.4}$$

and

$$\mathbf{g}_i = \partial \mathbf{x}/\partial \vartheta^i \quad \text{with} \quad \mathbf{g}_i \in \mathcal{T}\mathcal{B}_t \,. \tag{2.5}$$

The covariant metrics in both configurations are given by the relations

$$G_{IJ} = \mathbf{G}_I \cdot \mathbf{G}_J \quad \text{and} \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \,, \tag{2.6}$$

where a dot denotes scalar product of vectors. Their inverse is denoted by G^{IJ}, g^{ij}.

The tangent of the map ϕ; $\mathcal{T}\phi \equiv \mathbf{F}$ well known as the deformation gradient with

$$\mathcal{T}\phi : \mathcal{T}\mathcal{B}_0 \to \mathcal{T}\mathcal{B}_t \,, \tag{2.7}$$

or

$$\mathbf{F} : \mathbf{G}_I \to \mathbf{g}_i, \qquad \det \mathbf{F} > 0 \,, \tag{2.8}$$

is given as the tensor product

$$\mathbf{F} = \delta^i_J \mathbf{g}_i \otimes \mathbf{G}^J \,.$$

REMARK 2.1 ON NOTATION. In our notation we will use capital indices for components to be raised or lowered using the reference metric and small letters for indices raised or lowered using the actual metric. Using arbitrary coordinates, the deformation gradient reads $\mathbf{F} = F^i_J \mathbf{g}_i \otimes \mathbf{G}^J$ with $F^i_J = \partial\phi^i/\partial X^J$. By the identification of a tensor with its components, the basis can then be dropped (see MARSDEN and HUGHES [19]). Now, in the case of convected coordinates one has $F^i_J = \delta^i_J$ with δ^i_J being the Kronecker delta and the inclusion of the basis vectors in the description of tensors becomes an unrenouncable part of it. A considerable simplification is achieved if we come over to apply the summation convention over same letters, also when they are of different types (capital and small letter indices). In this case the following rule for the Kronecker delta has to hold: $\delta^K_I = \delta^k_i$. Accordingly, it makes sense to write $g_{ij}\mathbf{G}^I \otimes \mathbf{G}^J$ since over the pairs i , I and j, J is to be summed. Further, relations of the type $U_{nI}\delta^N_K = U_{nI}\delta^n_k = U_{kI}$ hold. If ambiguity is excluded, the reader may drop the distinction between capital and small letter indices arriving at a classical notation. The chosen notation has the advantage of allowing for a compact description as well as making many *pull-back/push-forward* transformations very transparent, as will be shown in the subsequent discussion. In addition, we will keep the same notation for tensors when the components are raised or lowered. From the context it should be clear which kind of tensor is used.

According to our remark, we may write

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^I . \tag{2.9}$$

The polar decomposition theorem applies resulting in

$$\mathbf{F} = \mathbf{RU}, \qquad \mathbf{F} = \widehat{\mathbf{U}}\mathbf{R}, \qquad \text{and} \qquad \widehat{\mathbf{U}} = \mathbf{RUR}^T , \tag{2.10}$$

with $\mathbf{U}$ and $\widehat{\mathbf{U}}$ being symmetric and $\mathbf{R} \in SO(3)$.

Equation (2.9) entails the relation

$$\mathbf{g}_i = \mathbf{F}\mathbf{G}_I . \tag{2.11}$$

By that, and in addition to the base system $\mathbf{G}_I$ in the reference configuration, the new base system $\mathbf{g}_i$ in the actual configuration is defined. On the other hand, and by the polar decomposition (2.10), we have

$$\mathbf{g}_i = \mathbf{RU}\mathbf{G}_I \qquad \text{or} \qquad \mathbf{g}_i = \widehat{\mathbf{U}}\mathbf{R}\mathbf{G}_I . \tag{2.12}$$

Two new base systems can now be defined:

$$\widehat{\mathbf{g}}_i := \mathbf{U}\mathbf{G}_I \qquad \text{or} \qquad \widehat{\mathbf{g}}_i := \mathbf{R}^T\mathbf{g}_i \tag{2.13}$$

and

$$\widehat{\mathbf{G}}_I := \mathbf{R}\mathbf{G}_I \qquad \text{or} \qquad \widehat{\mathbf{G}}_I := \widehat{\mathbf{U}}^{-1}\mathbf{g}_i . \tag{2.14}$$

The base $\widehat{\mathbf{g}}_i$ can now be interpreted either as a forward transformation of $\mathbf{G}_I$ by $\mathbf{U}$ or as a back transformation of $\mathbf{g}_i$ by $\mathbf{R}^T$. Correspondingly, the base $\widehat{\mathbf{G}}_I$ can be understood either as a forward transformation of $\mathbf{G}_I$ by $\mathbf{R}$ or as a back transformation of $\mathbf{g}_i$ by $\widehat{\mathbf{U}}^{-1}$. We may say that $\mathbf{G}_I$ defines a reference base system, $\mathbf{g}_i$ the actual one, $\widehat{\mathbf{g}}_i$ a stretched base system and $\widehat{\mathbf{G}}_I$ the rotated one. As an immediate result we have

$$G_{IJ} = \widehat{\mathbf{G}}_I \cdot \widehat{\mathbf{G}}_J \qquad \text{and} \qquad g_{ij} = \widehat{\mathbf{g}}_i \cdot \widehat{\mathbf{g}}_j . \tag{2.15}$$

We note also the important relation

$$\widehat{\mathbf{g}}_i = \mathbf{F}^T\widehat{\mathbf{G}}_I . \tag{2.16}$$

It is easily checked that the following very useful relations hold:

$$
\begin{aligned}
\mathbf{g}_i &= \mathbf{F}\mathbf{G}_I\,, & \mathbf{G}_I &= \mathbf{F}^{-1}\mathbf{g}_i\,,\\
\mathbf{g}^i &= \mathbf{F}^{-T}\mathbf{G}^I\,, & \mathbf{G}^I &= \mathbf{F}^T\mathbf{g}^i\,,\\
\widehat{\mathbf{g}}_i &= \mathbf{F}^T\widehat{\mathbf{G}}_I\,, & \widehat{\mathbf{G}}_I &= \mathbf{F}^{-T}\widehat{\mathbf{g}}_i\,,\\
\widehat{\mathbf{g}}^i &= \mathbf{F}^{-1}\widehat{\mathbf{G}}^I\,, & \widehat{\mathbf{G}}^I &= \mathbf{F}\widehat{\mathbf{g}}^i\,.
\end{aligned}
\tag{2.17}
$$

REMARK 2.2 ON NOTATION. For the sake of brevity two isometric tensors are distinguished by the use of a "hat". By this we are forced to denote many tensors differently than somehow established in the literature (e.g. we are writing $\widehat{\mathbf{U}}$ instead of $\mathbf{V}$). Further, tensors defined with respect to the basis equipped with the reference metric are denoted by capital letters; those taken with respect to the basis furnished with the actual metric are denoted by small letters. By that we believe that, in spite of the large number of tensors used, a maximum of clarity is achieved.

The configuration space and the pull-back/push-forward operations

With the polar decomposition of $\mathbf{F}$ introduced in Eq. (2.10), the rotation tensor $\mathbf{R}$ is uniquely determined by means of the symetry of $\mathbf{U}$ or that of $\widehat{\mathbf{U}}$ respectively, namely by the condition

$$
\mathbf{R}^T\mathbf{F} = \mathbf{F}^T\mathbf{R}\,. \tag{2.18}
$$

Anyhow, from practical as well as theoretical point of view it is more attractive to treat $\mathbf{R}$ first as an independent variable. The fulfillment of Eq. (2.18) and hence the achievement of the polar decomposition is then carried out as a part of the solution of a given boundary value problem by which the map $\phi(\mathbf{X})$ is determined as well. From theoretical point of view, this statement is equivalent to the understanding that simple materials are treated as a special type of a Cosserat continuum. The discussion of these materials, which we will call restricted Cosserat continua, is postponed until Sec. 6 where a rigorous justification of our approach is given as well. Meanwhile, anyhow, we just need the fact that $\mathbf{R}$ is not given a priori and is treated, as $\phi(\mathbf{X})$, as an unknown field. In this case and for the deformation process to be well defined it becomes necessary to give a precise definition of the configuration space. This is done next. For the mathematical background the reader is referred to ABRAHAM *et al.* [1], CHOCQUET *et al.* [4] or DUBRIN *et al.* [8].

The configuration space, denoted by $\mathcal{C}(\mathcal{B})$, consists of the set of all admissible configurations of the body. It is defined pointwise by the pair

$$
\mathcal{C}(\mathcal{B}) = \{\mathcal{U} = (\phi_t(\mathbf{X}), \mathbf{R}_t) \mid \mathcal{U} : \mathcal{B} \to \mathcal{R}^3 \times SO(3)\}\,. \tag{2.19}
$$

The rotation group can be parametrized with the help of the exponential map:

$$
\mathbf{R} = \exp(\Theta) = \mathbf{1} + \Theta + \frac{\Theta^2}{2!} + \frac{\Theta^3}{3!} + \ldots = \mathbf{1} + \frac{\sin|\theta|}{|\theta|}\Theta + \frac{1-\cos|\theta|}{|\theta|^2}\Theta^2 \tag{2.20}
$$

with the skew-symmetric tensor $\Theta = -\Theta^T \in so(3)$, the tangent space of $SO(3)$ at the identity (see e.g. [1, 8]), and with $\theta \in \mathcal{R}^3$ denoting the corresponding axial vector.

For two elements $\mathcal{U}$, $\mathcal{V}$ of $\mathcal{C}$ with $\mathcal{U} = (\phi_{t_0}(\mathbf{X}), \mathbf{R}_{t_0})$ and $\mathcal{V} = (\phi_{t-t_0}(\mathbf{X}), \mathbf{R}_{t-t_0})$ the group operation is then defined by

$$\begin{aligned}(2.21)\qquad \mathcal{U} + \mathcal{V} &= (\phi_{t_0}(\mathbf{X}), \mathbf{R}_{t_0}) + (\phi_{t-t_0}(\mathbf{X}), \mathbf{R}_{t-t_0}) \\ &= (\phi_{t-t_0} \circ \phi_{t_0}(\mathbf{X}), \mathbf{R}_{t-t_0}\mathbf{R}_{t_0}) = (\phi_t(\mathbf{X}), \mathbf{R}_{t-t_0}\mathbf{R}_{t_0}).\end{aligned}$$

In the last equation use is made of the fact that ϕ_t is a one-parameter diffeomorphism for which the relations $\phi_t \circ \phi_s = \phi_{t+s}$ ($\circ$ denotes the composition of maps) and $\phi_{-t} = \phi_t^{-1}$ hold. The group structure is not Abelian and the neutral element is $(\phi_{t=0}(\mathbf{X}), \mathbf{1})$ where $\phi_{t=0}$ is the identity map.

Of crucial importance for our considerations is the fact that the group $SO(3)$ is a Lie group, i.e. it defines a manifold. Hence, the configuration space itself defines a manifold. On this manifold curves are defined as one-parameter subgroups. A curve $\mathcal{V}(t)$ on $\mathcal{C}$ passing trough $\mathcal{U} = (\phi_{t_0}(\mathbf{X}), \mathbf{R}_{t_0})$ with $\mathcal{V}(t = t_0) = \mathcal{U}$ is then given as

$$(2.22)\qquad \mathcal{V}(t) = [\phi_{t-t_0} \circ \phi_{t_0}(\mathbf{X}), \exp(t\mathbf{\Omega})\mathbf{R}_{t_0}], \qquad \mathbf{\Omega} \in so(3).$$

Its tangent at $\mathcal{U}$ is defined in the usual manner by

$$\begin{aligned}(2.23)\qquad D\mathcal{U} &= \frac{D}{Dt}\mathcal{V}_{|t=t_0} = \frac{D}{Dt}[\phi_t(\mathbf{X}), \exp(t\mathbf{\Omega})\mathbf{R}]_{|t=t_0} \\ &= (\mathbf{v}_{t_0}, \mathbf{\Omega}\mathbf{R}_{t_0}) \quad \text{with} \quad \mathbf{\Omega} \in so(3), \quad \mathbf{v} = \frac{D}{Dt}\phi_t.\end{aligned}$$

From physical point of view $D\mathcal{V}$ defines the way to take variations of a physical quantity (e.g. free energy) depending on the configuration space. Mathematically, $(\mathbf{v}_{t_0}, \mathbf{\Omega}\mathbf{R}_{t_0})$ is an element of the tangent space $T\mathcal{C}(\mathcal{B})$ at the point defined by $(\phi(\mathbf{X}), \mathbf{R})_{t=t_0}$. In other words, a deformation process parametrized by t defines a flow on the configuration space with tangent vectors given by $(\mathbf{v}_t, \mathbf{\Omega}\mathbf{R}_t)$. From the definition of the configuration space we see that *push-forward/pull-back* operations of the tangent space of $\mathcal{B}$ are furnished under the map ϕ as well as under the action of the transformation group $SO(3)$. Moreover, any deformation process induces a tangent vector field on $\mathcal{C}(\mathcal{B})$ with respect to which Lie derivetives are well defined; a fact which we will need in Sec. 7. That is, Lie derivatives can be considered with respect to the tangent vector field given by $\mathbf{v}$ as well as with respect to the tangent vector field defined by $\mathbf{\Omega}\mathbf{R}$. For more details concerning alternative definitions of $\mathcal{C}(\mathcal{B})$, second derivatives, and possible metrics to be defined on $\mathcal{C}(\mathcal{B})$, the reader is referred to SANSOUR and BEDNARCZYK [27].

At this stage it is useful to define the pull-back and push-forward operation under $\mathbf{F}$ for both co- as well as contravariant tensors. Both operations make sense since $\mathbf{F}$ defines a diffeomorphism. Here we specialize them for the coordinate system used and the chosen notation. For clarity we will distinguish between covariant and contravariant tensors. For the covariant tensors $\mathbf{A} = A_{IJ}\mathbf{G}^I \otimes \mathbf{G}^J$, $\mathbf{a} = a_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, the *push-forward/pull-back* operations which are denoted by ϕ_* and ϕ^*, respectively, are defined as follows:

$$(2.24)\qquad \phi_*\mathbf{A} = \mathbf{F}^{T-1}\mathbf{A}\mathbf{F}^{-1} = A_{IJ}\mathbf{g}^i \otimes \mathbf{g}^j,$$

$$(2.25)\qquad \phi^*\mathbf{a} = \mathbf{F}^T\mathbf{a}\mathbf{F} = a_{ij}\mathbf{G}^I \otimes \mathbf{G}^J.$$

Analogously, the *pull-back/push-forward* operations for the contravariant tensors $\mathbf{Z} = Z^{IJ}\mathbf{G}_I \otimes \mathbf{G}_J$, $\mathbf{z} = z^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$ are defined by

$$(2.26)\qquad \phi_*\mathbf{Z} = \mathbf{F}\mathbf{Z}\mathbf{F}^T = Z_{IJ}\mathbf{g}_i \otimes \mathbf{g}_j,$$

$$\phi^* \mathbf{z} = \mathbf{F}^{-1} \mathbf{z} \mathbf{F}^{T-1} = z^{ij} \mathbf{G}_I \otimes \mathbf{G}_J \,. \tag{2.27}$$

An important feature of these transformation is already apparent. The coordinate of a tensor are invariant under these transformation, the basis only is transformed. In other words, given any tensor, new tensors are generated by preserving the coordinates but transforming the basis. It is stressed that this transparent rule is due to the use of convected coordinates. Analogously, and by the definition of our configuration space, *push-forward/pull-back* operations under $\mathbf{R}$ are well defined. We will denote them by $\mathbf{R}_*$ and $\mathbf{R}^*$, respectively. They have already been used by SIMO and MARSDEN [28].

The transformation rules can be generalized in order to encompass transformations due to the action of tensors like $\mathbf{U}$ or $\widehat{\mathbf{U}}$. By the definitions

$$\mathbf{U}_* = \mathbf{R}^* \circ \phi_*, \qquad \mathbf{U}^* = \phi^* \circ \mathbf{R}_*, \qquad \widehat{\mathbf{U}}_* = \phi_* \circ \mathbf{R}^*, \qquad \widehat{\mathbf{U}}^* = \mathbf{R}_* \circ \phi^* \,, \tag{2.28}$$

we arrive at well defined operations which we will call, just for convenience, *push-forward/ pull-back* operations under $\mathbf{U}$ and $\widehat{\mathbf{U}}$, respectively. Note that these transformations are defined by Eqs. (2.28) and not by a flow generated by a tangent vector field. Adding to $\mathbf{A}$ and $\mathbf{a}$ the tensors $\widehat{\mathbf{A}} = A_{IJ} \widehat{\mathbf{G}}^I \otimes \mathbf{G}^J$, $\widehat{\mathbf{a}} = a_{ij} \widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j$ as well as the analogously defined tensors $\widehat{\mathbf{Z}}$ and $\widehat{\mathbf{z}}$, we perform in the following these transformations

$$\mathbf{R}_*(\mathbf{A}) = \mathbf{R} \mathbf{A} \mathbf{R}^T = A_{IJ} \widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J \,, \tag{2.29}$$

$$\mathbf{R}^*(\mathbf{a}) = \mathbf{R}^T \mathbf{a} \mathbf{R} = a_{ij} \widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j \,, \tag{2.30}$$

$$\mathbf{U}_*(\mathbf{A}) = \mathbf{U}^{T-1} \mathbf{A} \mathbf{U}^{-1} = A_{IJ} \widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j \,, \tag{2.31}$$

$$\mathbf{U}^*(\widehat{\mathbf{a}}) = \mathbf{U}^T \widehat{\mathbf{a}} \mathbf{U} = a_{ij} \mathbf{G}^I \otimes \mathbf{G}^J \,. \tag{2.32}$$

Analogously, the *pull-back/push-forward* operations for the contravariant tensors read

$$\mathbf{R}_*(\mathbf{Z}) = \mathbf{R} \mathbf{Z} \mathbf{R}^T = Z^{IJ} \widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J \,, \tag{2.33}$$

$$\mathbf{R}^*(\mathbf{z}) = \mathbf{R}^T \mathbf{z} \mathbf{R} = z^{ij} \widehat{\mathbf{g}}_i \otimes \widehat{\mathbf{g}}_j \,, \tag{2.34}$$

$$\mathbf{U}_*(\mathbf{Z}) = \mathbf{U} \mathbf{Z} \mathbf{U}^T = Z^{IJ} \widehat{\mathbf{g}}_i \otimes \widehat{\mathbf{g}}_j \,, \tag{2.35}$$

$$\mathbf{U}^*(\widehat{\mathbf{z}}) = \mathbf{U}^{-1} \widehat{\mathbf{z}} \mathbf{U}^{-T} = z^{ij} \mathbf{G}_I \otimes \mathbf{G}_J \,. \tag{2.36}$$

Equation (2.16) motivates the definition of a further group of transformations under the action of $\mathbf{F}^T$. We denote it by φ^* and φ_*, respectively. Applied to the tensors $\widehat{\mathbf{A}}$, $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{Z}}$, $\widehat{\mathbf{z}}$ we get

$$\varphi_* \widehat{\mathbf{A}} = \mathbf{F}^{-1} \widehat{\mathbf{A}} \mathbf{F}^{T-1} = A_{IJ} \widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j \,, \tag{2.37}$$

$$\varphi^* \widehat{\mathbf{a}} = \mathbf{F} \widehat{\mathbf{a}} \mathbf{F}^T = a_{ij} \widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J \,, \tag{2.38}$$

$$\varphi_* \widehat{\mathbf{Z}} = \mathbf{F}^T \widehat{\mathbf{Z}} \mathbf{F} = Z^{IJ} \widehat{\mathbf{g}}_i \otimes \widehat{\mathbf{g}}_j \,, \tag{2.39}$$

$$\varphi^* \widehat{\mathbf{z}} = \mathbf{F}^{T-1} \widehat{\mathbf{z}} \mathbf{F}^{-1} = z^{ij} \widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J \,. \tag{2.40}$$

For further purpose we write the metric in absolute notation:

$$\begin{aligned} \mathbf{G} &:= G_{IJ} \mathbf{G}^I \otimes \mathbf{G}^J \,, \\ \widehat{\mathbf{G}} &:= G_{IJ} \mathbf{G}^I \otimes \mathbf{G}^J \,, \\ \mathbf{g} &:= g_{ij} \mathbf{g}^i \otimes \mathbf{g}^j \,, \end{aligned}$$

and

$$\widehat{\mathbf{g}} := g_{ij} \widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j \,,$$

REMARK 2.3. It must be stressed that since any configuration of the body is embedded in an Euclidean space, all these tensors denote one and the same object, namely the identity tensor. That we are distinguishing them in notation is only a matter of convenience and to make subsequent operations more transparent. For instance, the right Cauchy–Green tensor as will be subsequently discussed is defined by: $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. We can also understand $\mathbf{C}$ as the *pull back* of $\mathbf{g}$: $\mathbf{C} = \phi^*\mathbf{g} = \mathbf{F}^T\mathbf{g}\mathbf{F}$. Both definitions of $\mathbf{C}$ are equivalent since $\mathbf{g}$ is the identity. Nevertheless, by understanding $\mathbf{C}$ as a "*pull back*" we immediately see that the components C_{ij} of $\mathbf{C}$ with respect to the reference base are given by the metric g_{ij}.

REMARK 2.4. It should be clear that the boundary value problem can be formulated without a direct reference to the rotation tensor $\mathbf{R}$. In this case the rotation tensor is actually a hidden variable. Further, the definition of the configuration space as given in Eq. (2.19) is well suited for a material description. In the case of spatial description $\phi_t(\mathbf{X})$ and $\mathbf{R}$ should be replaced by $\phi_t^{-1}(\mathbf{x})$ and by $\mathbf{R}^{-1} = \mathbf{R}^T$ (concerning this point see also Sec. 5.4).

3. Strain tensors and their geometric structure

In this section we discuss, in a general setting, the underlying geometric structure of the strain tensors.

By the polar decomposition theorem (2.10), the right and left stretch tensors $\mathbf{U}$, $\widehat{\mathbf{U}}$ are defined as

$$\mathbf{U} = \mathbf{R}^T\mathbf{F}, \qquad \widehat{\mathbf{U}} = \mathbf{F}\mathbf{R}^T, \qquad \mathbf{U} = \mathbf{U}^T, \qquad \widehat{\mathbf{U}} = \widehat{\mathbf{U}}^T . \tag{3.1}$$

We consider next the Cauchy–Green tensors

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 , \tag{3.2}$$

and

$$\widehat{\mathbf{C}} = \mathbf{F}\mathbf{F}^T = \widehat{\mathbf{U}}^2 \tag{3.3}$$

as well as the inverse of the already defined strain tensors:

$$\widehat{\mathbf{u}} := \mathbf{U}^{-1} , \tag{3.4}$$

$$\mathbf{u} := \widehat{\mathbf{U}}^{-1} , \tag{3.5}$$

$$\widehat{\mathbf{c}} := \mathbf{C}^{-1} , \tag{3.6}$$

$$\mathbf{c} := \widehat{\mathbf{C}}^{-1} . \tag{3.7}$$

In order to get strain tensors which vanish at the reference configuration one defines usually

$$\begin{aligned}
\mathbf{H} &:= (\mathbf{U} - \mathbf{G}) , \\
\widehat{\mathbf{H}} &:= (\widehat{\mathbf{U}} - \widehat{\mathbf{G}}) , \\
\mathbf{E} &:= \frac{1}{2}(\mathbf{C} - \mathbf{G}) , \\
\widehat{\mathbf{E}} &:= \frac{1}{2}(\widehat{\mathbf{C}} - \widehat{\mathbf{G}}) , \\
\mathbf{h} &:= (\mathbf{g} - \mathbf{u}) ,
\end{aligned}$$

$$\widehat{\mathbf{h}} := (\widehat{\mathbf{g}} - \widehat{\mathbf{u}}),$$
$$\widehat{\mathbf{e}} := \frac{1}{2}(\widehat{\mathbf{g}} - \widehat{\mathbf{c}}),$$
$$\mathbf{e} := \frac{1}{2}(\mathbf{g} - \mathbf{c}),$$

The tensor $\mathbf{E}$ is known as the Green strain tensor, $\mathbf{e}$ as the Almansi one. $\mathbf{c}$ and $\widehat{\mathbf{c}}$ are named after Finger and $\widehat{\mathbf{E}}$ and $\widehat{\mathbf{e}}$ are known as Karni–Reiner tensors. Anyhow the labels are not unique, see e.e. TRUESDELL and NOLL [31], NOLL [21], PRAGER [23], MC VEAN [20], ERINGEN [9]. The tensors $\mathbf{H}$ and $\mathbf{h}$ are usually called engineering strains.

REMARK 3.1. Although $\mathbf{C}$ and $\widehat{\mathbf{C}}$ are functions of $\mathbf{U}$ and $\widehat{\mathbf{U}}$ respectively, it is useful to work with both independently. First, $\widehat{\mathbf{U}}$ and $\mathbf{U}$ are defined by Eq. (3.1), that is, for their computation a rotation tensor is explicitly needed, whereas $\mathbf{C}$ and $\widehat{\mathbf{C}}$ are just functions of the displacement field up to quadratic terms. Second, whereas the conjugate stress tensors of $\widehat{\mathbf{C}}$ and $\mathbf{C}$ are symmetric (by the symmetry of the Cauchy stress tensor), those of $\mathbf{U}$ and $\widehat{\mathbf{U}}$ are generally not (details are given in the Secs. 5 and 6).

The strain measures exhibit a certain geometric structure which is essential for a correct physical understanding. To get an insight into that structure, it is useful to resolve the strain tensors with respect to certain base systems. To this end let us rewrite some of the mentioned tensors with the help of the relations (2.9)–(2.17) as the tensor product of two vectors.

One may verify that the following relations hold

$$\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^I \tag{3.8}$$

or

$$\mathbf{F} = \widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{g}}_i, \tag{3.9}$$

$$\mathbf{R} = \widehat{\mathbf{G}}_I \otimes \mathbf{G}^I \tag{3.10}$$

or

$$\mathbf{R} = \mathbf{g}_i \otimes \widehat{\mathbf{g}}^i, \tag{3.11}$$

$$\mathbf{U} = \widehat{\mathbf{g}}_i \otimes \mathbf{G}^I, \tag{3.12}$$

$$\widehat{\mathbf{U}} = \mathbf{g}_i \otimes \widehat{\mathbf{G}}^I, \tag{3.13}$$

$$\widehat{\mathbf{u}} = \mathbf{G}_I \otimes \widehat{\mathbf{g}}^i, \tag{3.14}$$

$$\mathbf{u} = \widehat{\mathbf{G}}_I \otimes \mathbf{g}^i. \tag{3.15}$$

Resolving $\widehat{\mathbf{g}}_i$ with respect to the base $\mathbf{G}_J$:

$$\widehat{\mathbf{g}}_i = U_{iJ}\mathbf{G}^J, \tag{3.16}$$

one gets with Eq. (3.12)

$$\mathbf{U} = U_{iJ}\mathbf{G}^I \otimes \mathbf{G}^J. \tag{3.17}$$

With Eqs. (3.17), $(2.10)_3$ and $(2.14)_1$ we have further

$$\widehat{\mathbf{U}} = U_{iJ}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J \tag{3.18}$$

which leads with Eq. (3.13) to

$$\mathbf{g}_i = U_{iJ}\widehat{\mathbf{G}}^J. \tag{3.19}$$

On the other hand, the decomposition of $\mathbf{G}_I$ with respect to the base $\widehat{\mathbf{g}}^j$ together with the relation $\mathbf{G}_I \cdot \widehat{\mathbf{g}}_j = U_{jI}$ shows that

$$(3.20)\qquad \mathbf{G}_I = U_{jI}\widehat{\mathbf{g}}^j \quad \text{and} \quad \widehat{\mathbf{g}}^i = (U^{-1})^{Ji}\mathbf{G}_J \quad \text{with} \quad (U^{-1})^{Ik}U_{kJ} = \delta^I_J\,.$$

Using Eqs. (3.8)–(3.15), (3.19), (3.20) the definitions of the strain measure as well as the symmetry of $\mathbf{U}$ and $\widehat{\mathbf{U}}$, we get the following relationships:

$$(3.21)\qquad \mathbf{U} = \mathbf{R}^T\mathbf{F}, \qquad \mathbf{U} = U_{iJ}\mathbf{G}^I \otimes \mathbf{G}^J\,,$$

$$(3.22)\qquad \widehat{\mathbf{U}} = \mathbf{F}\mathbf{R}^T, \qquad \widehat{\mathbf{U}} = U_{iJ}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J\,,$$

$$(3.23)\qquad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{C} = g_{ij}\mathbf{G}^I \otimes \mathbf{G}^J\,,$$

$$(3.24)\qquad \widehat{\mathbf{C}} = \mathbf{F}\mathbf{F}^T = \widehat{\mathbf{U}}^2, \qquad \widehat{\mathbf{C}} = g_{ij}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J\,,$$

$$(3.25)\qquad \widehat{\mathbf{u}} = \mathbf{F}^{-1}\mathbf{R}, \qquad \widehat{\mathbf{u}} = U_{iJ}\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j\,,$$

$$(3.26)\qquad \mathbf{u} = \mathbf{R}\mathbf{F}^{-1}, \qquad \mathbf{u} = U_{iJ}\mathbf{g}^i \otimes \mathbf{g}^j\,,$$

$$(3.27)\qquad \widehat{\mathbf{c}} = \mathbf{F}^{-1}\mathbf{F}^{T-1}, \qquad \widehat{\mathbf{c}} = G_{IJ}\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j\,,$$

$$(3.28)\qquad \mathbf{c} = \mathbf{F}^{T-1}\mathbf{F}^{-1}, \qquad \mathbf{c} = G_{IJ}\mathbf{g}^i \otimes \mathbf{g}^j\,.$$

Already at this stage a certain geometric structure is apparent. The strain tensors can be classified into two families, each family consists of two subgroups:

1. Covariant tensors taken with respect to a basis endowed with the metric G_{IJ};
a) the base system is $\mathbf{G}_I$,
b) the base system is $\widehat{\mathbf{G}}_I$.
2. Covariant tensors taken with respect to a basis endowed with the metric g_{ij};
a) the base system is $\mathbf{g}_i$,
b) the base system is $\widehat{\mathbf{g}}_i$.

For convenience, we call the first family tensors of the first type, and the second family tensors of the second type. This classification is of great significance when discussing the material and spatial descriptions. Actually it would be motivated to designate the first family Lagrangian tensors, the second one — the Eulerian tensors. Unfortunately, the terminology of Lagrangian and Eulerian tensors to be found in the literature does not fit our classification (tensors taken with respect to $\widehat{\mathbf{G}}_i$ are called also Eulerian). Hence, to avoid ambiguities we stick to the suggested names of first and second type tensors.

For the strain tensors which vanish at the reference configuration we get in analogy with Eqs. (3.21)–(3.28)

$$(3.29)\qquad \mathbf{H} := (\mathbf{U} - \mathbf{G}) = (\mathbf{R}^T\mathbf{F} - \mathbf{G}), \qquad \mathbf{H} = (U_{iJ} - G_{IJ})\mathbf{G}^I \otimes \mathbf{G}^J\,,$$

$$(3.30)\qquad \widehat{\mathbf{H}} := (\widehat{\mathbf{U}} - \widehat{\mathbf{G}}) = (\mathbf{F}\mathbf{R}^T - \widehat{\mathbf{G}}), \qquad \widehat{\mathbf{H}} = (U_{iJ} - G_{IJ})\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J\,,$$

$$(3.31)\qquad \mathbf{E} := \frac{1}{2}(\mathbf{C} - \mathbf{G}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{G}), \qquad \mathbf{E} = \frac{1}{2}(g_{ij} - G_{IJ})\mathbf{G}^I \otimes \mathbf{G}^J\,,$$

$$(3.32)\qquad \widehat{\mathbf{E}} := \frac{1}{2}(\widehat{\mathbf{C}} - \widehat{\mathbf{G}}) = \frac{1}{2}(\widehat{\mathbf{U}}^2 - \widehat{\mathbf{G}}), \qquad \widehat{\mathbf{E}} = \frac{1}{2}(g_{ij} - G_{IJ})\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J\,,$$

$$(3.33)\qquad \mathbf{h} := (\mathbf{g} - \mathbf{u}) = (\mathbf{g} - \mathbf{R}\mathbf{F}^{-1}), \qquad \mathbf{h} = (g_{ij} - U_{iJ})\mathbf{g}^i \otimes \mathbf{g}^j\,,$$

$$(3.34)\qquad \widehat{\mathbf{h}} := (\mathbf{g} - \widehat{\mathbf{u}}) = (\mathbf{g} - \mathbf{F}^{-1}\mathbf{R}), \qquad \widehat{\mathbf{h}} = (g_{ij} - U_{iJ})\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j\,,$$

$$(3.35)\qquad \widehat{\mathbf{e}} := \frac{1}{2}(\widehat{\mathbf{g}} - \widehat{\mathbf{c}}) = \frac{1}{2}(\widehat{\mathbf{g}} - \widehat{\mathbf{u}}^2), \qquad \widehat{\mathbf{e}} = \frac{1}{2}(g_{ij} - G_{IJ})\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j\,,$$

$$\mathbf{e} := \frac{1}{2}(\mathbf{g} - \mathbf{c}) = \frac{1}{2}(\mathbf{g} - \mathbf{u}^2), \qquad \mathbf{e} = \frac{1}{2}(g_{ij} - G_{IJ})\mathbf{g}^i \otimes \mathbf{g}^j . \tag{3.36}$$

Adopting now geometric language by considering the *push-forward/pull-back* operations as defined in Eqs. (2.24)–(2.40), one can see from Eqs. (3.21)–(3.36) and by considering the definitions of the basis in Eqs. (2.11), (2.13) and (2.14) that the following two groups of relations hold:

$$\widehat{\mathbf{U}} = \mathbf{R}_*(\mathbf{U}) , \tag{3.37}$$

$$\widehat{\mathbf{H}} = \mathbf{R}_*(\mathbf{H}) , \tag{3.38}$$

$$\mathbf{u} = \mathbf{R}_*(\widehat{\mathbf{u}}) , \tag{3.39}$$

$$\mathbf{h} = \mathbf{R}_*(\widehat{\mathbf{h}}) , \tag{3.40}$$

$$\widehat{\mathbf{C}} = \mathbf{R}_*(\mathbf{C}) , \tag{3.41}$$

$$\widehat{\mathbf{E}} = \mathbf{R}_*(\mathbf{E}) , \tag{3.42}$$

$$\mathbf{c} = \mathbf{R}_*(\widehat{\mathbf{c}}) , \tag{3.43}$$

$$\mathbf{e} = \mathbf{R}_*(\widehat{\mathbf{e}}) , \tag{3.44}$$

and

$$\widehat{\mathbf{u}} = \varphi_*\widehat{\mathbf{U}} , \tag{3.45}$$

$$\mathbf{u} = \phi_*\mathbf{U} , \tag{3.46}$$

$$\mathbf{C} = \phi^*\mathbf{g} = \mathbf{U}^*(\widehat{\mathbf{g}}) , \tag{3.47}$$

$$\widehat{\mathbf{C}} = \varphi^*\widehat{\mathbf{g}} = \widehat{\mathbf{U}}^*(\mathbf{g}) , \tag{3.48}$$

$$\widehat{\mathbf{c}} = \varphi_*\widehat{\mathbf{G}} = \mathbf{U}_*(\mathbf{G}) , \tag{3.49}$$

$$\mathbf{c} = \phi_*\mathbf{G} = \widehat{\mathbf{U}}_*(\widehat{\mathbf{G}}) , \tag{3.50}$$

$$\widehat{\mathbf{e}} = \varphi_*\widehat{\mathbf{E}} = \mathbf{U}_*(\mathbf{E}) , \tag{3.51}$$

$$\mathbf{e} = \phi_*\mathbf{E} = \widehat{\mathbf{U}}_*(\widehat{\mathbf{E}}) . \tag{3.52}$$

First of all we emphasize again that the use of convected coordinates ensures the invariance of the coordinate of a tensor under these transformations. Only the basis is transformed. We have four base systems $\mathbf{G}_I$, $\widehat{\mathbf{G}}_I$, $\mathbf{g}_i$ and $\widehat{\mathbf{g}}_i$, correspondingly four covariant strain tensors with components U_{iJ} and further four tensors with components g_{ij} and G_{IJ}, respectively. One and the same tensor can be differently interpreted, for instance $\widehat{\mathbf{C}}$ can be understood as the *push-forward* of $\mathbf{C}$ by $\mathbf{R}$, as the *pull-back* of $\mathbf{g}$ by $\widehat{\mathbf{U}}$ or as the *pull-back* of $\widehat{\mathbf{g}}$ under $\mathbf{F}^T$. Similar interpretations can be given to other strain tensors which can be read from Eqs. (3.37)–(3.52).

The first group of isometric transformations is in fact the most important one. It plays in the considerations of this paper a fundamental role. Isometric tensors have the same components relative to bases differing by a rotation, thus the metric is preserved. Since any physical relation should be invariant with respect to rigid rotations, we conclude that, physically, isometric tensors are equivalent. Furthermore, the *pull-back/push-forward* transformations under $\mathbf{R}$ will play an essential role in defining objective rates as we will suggest later on in this paper.

The relations (3.37)–(3.52) are of course valid for arbitrary coordinates. Nevertheless, by the use of convected coordinates the operations of *pull-back/push-forward* are applied to physically interesting objects. For instance, geometrically, $\mathbf{C}$ can be understood as the *pull back* of the Riemannian metric $\mathbf{g} : \mathbf{C} = \phi^*\mathbf{g}$. In the case of arbitrary coordinates the components of $\mathbf{g}$ are arbitrary and completely independent of the deformation process, and hence are physically of less interest. The physically interesting object is hidden in the transformation operators of the "pull back" itself. Not so in the case of convected coordinates where the components of $\mathbf{g}$ are g_{ij}, which depend on the deformation process alone, and the pull back operation means that the components of $\mathbf{C}$ are identical with g_{ij}. Thus the transformed object $\mathbf{g}$ is at the same time the physically interesting object.

By the *pull-back/push-forward* operations we were able to avoid inverse operations on the matrix of the components g_{ij}, U_{iJ} or G_{IJ}. Such inverse operations have the theoretical feature that we can write what we called tensors of the second type with respect to bases furnished with the metric of the reference configuration, and tensors of the first type with respect to bases furnished with the actual metric. It is straightforward to see that the following relations also hold:

$$\mathbf{U} = (U^{-1})^{Ij}\widehat{\mathbf{g}}_i \otimes \widehat{\mathbf{g}}_j, \qquad \mathbf{H} = ((U^{-1})^{Ij} - g^{ij})\widehat{\mathbf{g}}_i \otimes \widehat{\mathbf{g}}_j\,, \tag{3.53}$$

$$\widehat{\mathbf{U}} = (U^{-1})^{Ij}\mathbf{g}_i \otimes \mathbf{g}_j, \qquad \widehat{\mathbf{H}} = ((U^{-1})^{Ij} - g^{ij})\mathbf{g}_i \otimes \mathbf{g}_j\,, \tag{3.54}$$

$$\mathbf{C} = G^{IJ}\widehat{\mathbf{g}}_i \otimes \widehat{\mathbf{g}}_j, \qquad \mathbf{E} = 1/2(G^{IJ} - g^{ij})\widehat{\mathbf{g}}_i \otimes \widehat{\mathbf{g}}_j\,, \tag{3.55}$$

$$\widehat{\mathbf{C}} = G^{IJ}\mathbf{g}_i \otimes \mathbf{g}_j, \qquad \widehat{\mathbf{E}} = 1/2(G^{IJ} - g^{ij})\mathbf{g}_i \otimes \mathbf{g}_j\,, \tag{3.56}$$

$$\widehat{\mathbf{u}} = (U^{-1})^{Ij}\mathbf{G}_I \otimes \mathbf{G}_J, \qquad \widehat{\mathbf{h}} = (G^{IJ} - (U^{-1})^{Ij})\mathbf{G}_I \otimes \mathbf{G}_J\,, \tag{3.57}$$

$$\mathbf{u} = (U^{-1})^{Ij}\widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J, \qquad \mathbf{h} = (G^{IJ} - (U^{-1})^{Ij})\widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J\,, \tag{3.58}$$

$$\widehat{\mathbf{c}} = g^{ij}\mathbf{G}_I \otimes \mathbf{G}_J, \qquad \widehat{\mathbf{e}} = 1/2(G^{IJ} - g^{ij})\mathbf{G}_I \otimes \mathbf{G}_J\,, \tag{3.59}$$

$$\mathbf{c} = g^{ij}\widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J, \qquad \mathbf{e} = 1/2(G^{IJ} - g^{ij})\widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J\,. \tag{3.60}$$

Here the strain tensors are treated as contravariant tensors.

From these equations we see that it is always possible to formulate all tensors with respect to basis furnished with the reference metric respectively the actual metric. The price to be paid is given by the inverse operations to be applied on U_{iJ}, G_{IJ} and g_{ij}. This point will later on have considerable consequences by the definition of objective rates. We remark, that the last equations allow for an alternative interpretation of the transformation operations. One may read

$$\mathbf{C} = \varphi_*(\widehat{\mathbf{G}}^{-1}) = \mathbf{U}_*(\mathbf{G}^{-1})\,, \tag{3.61}$$

$$\widehat{\mathbf{C}} = \phi_*(\mathbf{G}^{-1}) = \widehat{\mathbf{U}}_*(\widehat{\mathbf{G}}^{-1})\,, \tag{3.62}$$

$$\widehat{\mathbf{c}} = \phi^*(\mathbf{g}^{-1}) = \widehat{\mathbf{U}}^*(\widehat{\mathbf{g}}^{-1})\,, \tag{3.63}$$

$$\mathbf{c} = \varphi^*(\widehat{\mathbf{g}}^{-1}) = \widehat{\mathbf{U}}^*(\mathbf{g}^{-1})\,. \tag{3.64}$$

4. Stress tensors

By the Cauchy theorem we have:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}\,, \tag{4.1}$$

with $\mathbf{t}$ as the true stress vector, $\mathbf{n}$ as the unit normal vector, and $\boldsymbol{\sigma}$ as the symmetric Cauchy stress tensor. We consider further the decompositions

$$\mathbf{n} = n_i\mathbf{g}^i\,, \tag{4.2}$$

$$\mathbf{n} = N_I \widehat{\mathbf{G}}^I \,, \tag{4.3}$$

and define the following two stress vectors

$$\mathbf{t}^i := \boldsymbol{\sigma} \mathbf{g}^i \,, \tag{4.4}$$

$$\mathbf{T}^I := \boldsymbol{\sigma} \widehat{\mathbf{G}}^I \,. \tag{4.5}$$

From Eqs. (4.1)–(4.3) we have immediately $\mathbf{t} = \mathbf{t}^i n_i = \mathbf{T}^I N_I$. Further the relations hold

$$\boldsymbol{\sigma} = \mathbf{t}^i \otimes \mathbf{g}_i , \qquad \boldsymbol{\sigma} = \mathbf{T}^I \otimes \widehat{\mathbf{G}}_I \,. \tag{4.6}$$

Starting from these relations, different families of stress tensors can be generated. For this purpose we consider the decompositions

$$\mathbf{t}^i := \sigma^{ij} \mathbf{g}_j \,, \tag{4.7}$$

$$= \gamma^{iJ} \widehat{\mathbf{G}}_J \,, \tag{4.8}$$

and

$$\mathbf{T}^I := \Xi^{Ij} \mathbf{g}_j \,, \tag{4.9}$$

$$= \Sigma^{IJ} \widehat{\mathbf{G}}_J \,, \tag{4.10}$$

with the help of which we can write for Eqs. (4.6)

$$\boldsymbol{\sigma} = \sigma^{ij} \mathbf{g}_j \otimes \mathbf{g}_i \,, \tag{4.11}$$

$$= \Sigma^{IJ} \widehat{\mathbf{G}}_J \otimes \widehat{\mathbf{G}}_I \tag{4.12}$$

or

$$\boldsymbol{\sigma} = \gamma^{iJ} \widehat{\mathbf{G}}_J \otimes \mathbf{g}_i \,, \tag{4.13}$$

$$= \Xi^{Ij} \mathbf{g}_j \otimes \widehat{\mathbf{G}}_I \,, \tag{4.14}$$

respectively. The last two equations motivate the definitions

$$\widehat{\boldsymbol{\Gamma}} = \gamma^{iJ} \widehat{\mathbf{G}}_J \otimes \widehat{\mathbf{G}}_I \tag{4.15}$$

and

$$\widehat{\boldsymbol{\Xi}} = \Xi^{Ij} \widehat{\mathbf{G}}_J \otimes \widehat{\mathbf{G}}_I \,, \tag{4.16}$$

resulting in

$$\boldsymbol{\sigma} = \widehat{\boldsymbol{\Gamma}} \widehat{\mathbf{U}}^T \,, \tag{4.17}$$

$$= \widehat{\mathbf{U}} \widehat{\boldsymbol{\Xi}} \,. \tag{4.18}$$

The different decompositions of the Cauchy stress tensor, Eqs. (4.11), (4.12) on the one hand and Eqs. (4.13), (4.14) on the other hand, are neither arbitrary nor superfluous. They are in fact necessary in order to generate all stress tensors needed so as to correspond, in a sense to be made precise in the next section, to the strain tensors introduced in the last section. The physical meaning of the different components is apparent from the definitions (4.7)–(4.10).

It is easily verified that the different stress components are related according to

$$\Sigma^{IJ} = G^{IM} U_{kM} \sigma^{kl} U_{lN} G^{NJ} = \gamma^{jI} U_{jL} G^{LJ} \,. \tag{4.19}$$

In Eq. (4.17) the tensor $\widehat{\Gamma}$ is left multiplied by $\widehat{\mathbf{U}}$, while $\widehat{\Xi}$ is right multiplicated by $\widehat{\mathbf{U}}$, to give σ. Since σ as well as $\widehat{\mathbf{U}}$ are symmetric, it follows from Eqs. (4.17) and (4.18)

(4.20) $$\Xi = \Gamma^T ,$$

respectively

(4.21) $$\Xi^{Ij} = \gamma^{jI} .$$

Hence it suffices to consider only one of both tensors. Here we choose Γ.

For the isometric Cauchy stress tensor $\widehat{\sigma} = \mathbf{R}^*(\sigma)$, also called the rotated stress tensor (MARSDEN and HUGHES [19]), we have immediately from (4.11) and (4.12)

(4.22) $$\widehat{\sigma} = \sigma^{ij}\widehat{\mathbf{g}}_j \otimes \widehat{\mathbf{g}}_i ,$$

(4.23) $$= \Sigma^{IJ}\mathbf{G}_J \otimes \mathbf{G}_I .$$

By systematic applications of the *pull-back* and *push-forward* operations to the already defined stress tensors, different families of new stress tensors are generated. First, we apply the transformation operators on the Cauchy stress tensor and on its isometric tensor:

(4.24) $$\boldsymbol{\Sigma} = \phi^*\sigma = \mathbf{U}^*\widehat{\sigma}, \qquad \boldsymbol{\Sigma} = \sigma^{ij}\mathbf{G}_J \otimes \mathbf{G}_I ,$$

(4.25) $$\widehat{\boldsymbol{\Sigma}} = \varphi^*\widehat{\sigma} = \widehat{\mathbf{U}}^*(\sigma), \qquad \widehat{\boldsymbol{\Sigma}} = \sigma^{ij}\widehat{\mathbf{G}}_J \otimes \widehat{\mathbf{G}}_I ,$$

(4.26) $$\widehat{\boldsymbol{\lambda}} = \varphi_*\sigma = \mathbf{U}_*\widehat{\sigma}, \qquad \widehat{\boldsymbol{\lambda}} = \Sigma^{IJ}\widehat{\mathbf{g}}_j \otimes \widehat{\mathbf{g}}_i ,$$

(4.27) $$\boldsymbol{\lambda} = \phi_*(\widehat{\sigma}) = \widehat{\mathbf{U}}_*(\sigma), \qquad \boldsymbol{\lambda} = \Sigma^{IJ}\mathbf{g}_j \otimes \mathbf{g}_i .$$

Explicitly we have

(4.28) $$\boldsymbol{\Sigma} = \mathbf{F}^{-1}\sigma\mathbf{F}^{T-1} ,$$

(4.29) $$\widehat{\boldsymbol{\Sigma}} = \widehat{\mathbf{U}}^{-1}\sigma\widehat{\mathbf{U}}^{T-1} ,$$

(4.30) $$\boldsymbol{\lambda} = \widehat{\mathbf{U}}\sigma\widehat{\mathbf{U}}^T ,$$

(4.31) $$\widehat{\boldsymbol{\lambda}} = \mathbf{F}^T\sigma\mathbf{F} .$$

Further relations can be read from Eqs. (4.24)–(4.31). Note the following interesting analogy. While $\boldsymbol{\Sigma}$ is the *pull-back* of σ under $\mathbf{F}$, $\boldsymbol{\lambda}$ is the *push-forward* of $\widehat{\sigma}$ under the same map. Correspondingly, while $\widehat{\boldsymbol{\Sigma}}$ is the *pull-back* of $\widehat{\sigma}$ under $\widehat{\mathbf{F}}^T$, $\widehat{\boldsymbol{\lambda}}$ is the *push-forward* of σ under the same map.

Actually, Eqs. (4.12)–(4.23) motivate the notation $\widehat{\Lambda}$ for σ and Λ for $\widehat{\sigma}$. Hence, according to our classifications, the Cauchy stress tensor as well as its isometric tensor may be considered tensors of the first as well as of the second type, depending on the stress components used. This statement reflects an important aspect of this paper. In fact, in formulating the Cauchy stress tensor in a natural way with respect to (in our classification) Lagrangian bases we are going beyond the well established treatment of the Cauchy stress tensor opening new doors to define new stress tensors which turn out to be essential for a complete understanding of the structure of the stress and strain tensors. This somehow surprising characteristic peculiarity of the Cauchy stress tensor is confirmed further when discussing the corresponding dual variables in Sec. 5.

With the application of the different transformations on $\widehat{\Gamma}$ we obtain further the stress tensors

(4.32) $$\Gamma = \mathbf{R}^*(\widehat{\Gamma}), \qquad \Gamma = \gamma^{iJ}\mathbf{G}_J \otimes \mathbf{G}_I ,$$

$$\gamma = \widehat{\mathbf{U}}_*(\widehat{\Gamma}), \qquad \gamma = \gamma^{iJ}\mathbf{g}_j \otimes \mathbf{g}_i\,, \tag{4.33}$$

$$\widehat{\gamma} = \varphi_*\widehat{\Gamma}, \qquad \widehat{\gamma} = \gamma^{iJ}\widehat{\mathbf{g}}_j \otimes \widehat{\mathbf{g}}_i\,. \tag{4.34}$$

It is easily checked that the following results also hold

$$\gamma = \widehat{\mathbf{U}}\sigma\,, \tag{4.35}$$

$$\widehat{\gamma} = \mathbf{U}\widehat{\sigma}\,, \tag{4.36}$$

$$\widehat{\Gamma} = \widehat{\mathbf{U}}\widehat{\Sigma}\,, \tag{4.37}$$

$$\Gamma = \mathbf{U}\Sigma\,. \tag{4.38}$$

Altogether we have three contravariant components σ^{ij}, Σ^{ij} and γ^{ij} defining ten stress tensors of four different bases.

In analogy to the expressions we get for the strains in Eqs. (3.53)–(3.60), we may write the stress tensors of the second type λ and $\widehat{\lambda}$ with respect to bases furnished with the reference metric

$$\lambda = U_{iK}\Sigma^{KL}U_{jL}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J \quad \text{and} \quad \widehat{\lambda} = U_{iK}\Sigma^{KL}U_{jL}\mathbf{G}^I \otimes \mathbf{G}^J\,. \tag{4.39}$$

Alternatively, owing to Eq. (4.19) and the relation $g_{ij} = U_{iM}G^{MN}U_{jN}$ we arrive at the interesting expression

$$\lambda = \sigma_{ij}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J \quad \text{and} \quad \widehat{\lambda} = \sigma_{ij}\mathbf{G}^I \otimes \mathbf{G}^J\,, \tag{4.40}$$

with $\sigma_{ij} = g_{ik}\sigma^{kl}g_{lj}$. Similar expressions can be given for tensors of the first type as well: e.g. one has $\Sigma = \Sigma_{ij}\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j$ etc.

REMARK 4.1. Alternatively, one can multiply these stress tensors by $J = \det \mathbf{F}$ getting the so-called weighted stress tensors which are frequently used in the literature. The most known of them are $\tau = J\sigma$ as the Kirchhoff stress tensor, $\mathbf{S} = J\Sigma$ as the second Piola–Kirchhoff stress tensor, and $\mathbf{r} = J\Gamma$ as the Biot–Lurie stress tensor.

REMARK 4.2. By the symmetry of the Cauchy stress tensor the components σ^{ij} and Σ^{ij} are symmetric. This entails that the stress tensors $\widehat{\sigma}$, λ, $\widehat{\lambda}$, $\widehat{\Sigma}$ and Σ are symmetric (see (4.24)–(4.27)). The tensors γ, γ, $\widehat{\Gamma}$ and Γ are in general not symmetric. As it will be discussed in Sec. 6, it turns out that in the special case of isotropy when $\widehat{\Gamma}$ and $\widehat{\mathbf{U}}$ are coaxial, the symmetry of σ entails (because of Eq. (4.17)) that of $\widehat{\Gamma}$ and, hence, that of $\widehat{\gamma}$, γ and Γ.

5. The Helmholz free energy and the concept of dual variables

For the purposes of this study, it is completely sufficient to consider the simplest possible theory governed by a free energy function. The results are not affected if the theory is generalized to include thermal effects or further dissipative processes. First we define what to understand under a conjugate pair and give for all strain tensors already defined the corresponding conjugate stress tensor. In the following subsection the question of the dual variable of the Cauchy stress tensor is discussed and it is proved that, in general, such a dual variable does not exist. The last subsection is concerned with the conditions to be fulfilled in order for the *push-forward* of a stress or strain tensor under $\mathbf{F}$ to achieve a dual variable.

5.1. Energetically consistent dual variables

We assume the Helmholz free energy Ψ to depend in a suitable way (frame-indifferent) on the deformation gradient only. Its material time derivative

$$\rho\dot{\Psi} = \boldsymbol{\sigma} : \mathbf{d} \tag{5.1}$$

defines then the rate of specific work, with

$$\mathbf{d} = \frac{1}{2}(\mathbf{l} + \mathbf{l}^T), \qquad \mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{5.2}$$

and (:) denoting double contraction ($\boldsymbol{\sigma} : \mathbf{d} = \operatorname{tr}(\boldsymbol{\sigma}\mathbf{d}^T)$). From Eqs. (2.9) and $(5.2)_2$ one gets

$$\mathbf{l} = \dot{\mathbf{g}}_i \otimes \mathbf{g}^i, \tag{5.3}$$

and thus for **d**

$$\mathbf{d} = d_{ij}\mathbf{g}^j \otimes \mathbf{g}^i = \frac{1}{2}\dot{g}_{ij}\mathbf{g}^j \otimes \mathbf{g}^i, \tag{5.4}$$

and for $\dot{\Psi}$

$$\rho\dot{\Psi} = \frac{1}{2}\sigma^{ij}\dot{g}_{ij}. \tag{5.5}$$

The equation for $\dot{\Psi}$ can be written in a slightly modified form if one considers the difference of two metrics, and not the metric itself, as the physically interesting object. Thus we write

$$d_{ij} = \frac{1}{2}\frac{D(g_{ij} - G_{IJ})}{Dt}. \tag{5.6}$$

This modification is clearly admissible since velocities make sense for material points only and for such a point the placement **X** and the metric G_{IJ} are time-independent. Since σ^{ij} and g_{ij} are the components of $\boldsymbol{\Sigma}$ and **C**, respectively, with respect to time-independent bases, it follows immediately

$$\rho\dot{\Psi} = \frac{1}{2}\boldsymbol{\Sigma} : \dot{\mathbf{C}} = \boldsymbol{\Sigma} : \dot{\mathbf{E}}. \tag{5.7}$$

Now, $\widehat{\boldsymbol{\Sigma}}$ and $\widehat{\mathbf{C}}$ have the same components with respect to a rotating, and hence time-dependent bases. It is interesting that this kind of time dependence (with the metric being preserved) does not affect the expressions for $\dot{\Psi}$, and that Eq. (5.7) is equivalent to

$$\rho\dot{\Psi} = \frac{1}{2}\widehat{\boldsymbol{\Sigma}} : \dot{\widehat{\mathbf{C}}} = \widehat{\boldsymbol{\Sigma}} : \dot{\widehat{\mathbf{E}}}. \tag{5.8}$$

To see this we define

$$\boldsymbol{\Omega} := \dot{\mathbf{R}}\mathbf{R}^T, \tag{5.9}$$

and by $\mathbf{R}\mathbf{R}^T = \mathbf{1}$ we have

$$\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T. \tag{5.10}$$

Equations (3.24) and $(2.14)_1$ yield

$$\dot{\widehat{\mathbf{C}}} = \dot{g}_{ij}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J + \boldsymbol{\Omega}\widehat{\mathbf{C}} + \widehat{\mathbf{C}}\boldsymbol{\Omega}^T \tag{5.11}$$

and

$$\begin{aligned}\rho\dot{\Psi} &= \frac{1}{2}\widehat{\boldsymbol{\Sigma}} : (\dot{g}_{ij}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J + \boldsymbol{\Omega}\widehat{\mathbf{C}} + \widehat{\mathbf{C}}\boldsymbol{\Omega}^T) \\ &= \frac{1}{2}\sigma^{ij}\dot{g}_{ij} + \widehat{\boldsymbol{\Sigma}}\widehat{\mathbf{C}} : \boldsymbol{\Omega} \\ &= \frac{1}{2}\sigma^{ij}\dot{g}_{ij} \,.\end{aligned}$$

Here we made use of $\widehat{\boldsymbol{\Sigma}}\widehat{\mathbf{C}} = \boldsymbol{\sigma}$ and of the symmetry of $\boldsymbol{\sigma}$.

To look for expressions equivalent to Eq. (5.1) but formulated in terms of the other strain tensors, the free energy is considered as a function of these strain tensors and, via the latter, as a function of the deformation gradient. According to the second law of thermodynamics (TRUESDELL and NOLL [31], WANG and TRUESDELL [33]) we have

$$\boldsymbol{\sigma} = \rho\frac{\partial\boldsymbol{\Psi}}{\partial\mathbf{F}}\mathbf{F}^T \,, \tag{5.12}$$

where ρ denotes the actual mass density. For simplicity, the arguments will not be written down explicitly. It is always understood that within a material description all fields are defined over the Lagrangian coordinate $\mathbf{X}$, while within a spatial one all fields are defined over the Eulerian coordinate $\mathbf{x}$. Let $\boldsymbol{\varepsilon}$ denote a suitable strain measure. By the chain rule we have

$$\boldsymbol{\sigma} = \rho\frac{\partial\boldsymbol{\Psi}}{\partial\boldsymbol{\varepsilon}}\frac{\partial\boldsymbol{\varepsilon}}{\partial\mathbf{F}}\mathbf{F}^T \,. \tag{5.13}$$

Hence, the derivatives of the different strain measures relative to $\mathbf{F}$ are needed. In order to avoid a large amount of tensor operations we will take these derivatives with respect to Cartesian tensor components. We get

$$\partial C_{ij}/\partial F_{rq} = F_{rj}\delta_{iq} + F_{ri}\delta_{jq} \,, \tag{5.14}$$

$$\partial \widehat{C}_{ij}/\partial F_{rq} = F_{jq}\delta_{ri} + F_{iq}\delta_{rj} \,, \tag{5.15}$$

$$\partial C^{-1}_{ij}/\partial F_{rq} = -(F^{-1}_{ir}F^{-1}_{qk}F^{-1}_{jk} + F^{-1}_{jr}F^{-1}_{ik}F^{-1}_{qk}) \,, \tag{5.16}$$

$$\partial \widehat{C}^{-1}_{ij}/\partial F_{rq} = -(F^{-1}_{kr}F^{-1}_{qi}F^{-1}_{kj} + F^{-1}_{ki}F^{-1}_{kr}F^{-1}_{qj}) \,, \tag{5.17}$$

$$\partial U_{ij}/\partial F_{rq} = R_{ri}\delta_{qj} \,, \tag{5.18}$$

$$\partial \widehat{U}_{ij}/\partial F_{rq} = R_{jq}\delta_{ri} \,, \tag{5.19}$$

$$\partial U^{-1}_{ij}/\partial F_{rq} = -F^{-1}_{ir}F^{-1}_{qk}R_{kj} \,, \tag{5.20}$$

$$\partial \widehat{U}^{-1}_{ij}/\partial F_{rq} = -F^{-1}_{kr}F^{-1}_{qj}R_{ik} \,, \tag{5.21}$$

$$\partial \widehat{U}^{2}_{ij}/\partial F_{rq} = R_{iq}\widehat{U}_{rj} + R_{jq}\widehat{U}_{ri} \,. \tag{5.22}$$

Hereby we made use of the definitions of the strain tensors (3.21)–(3.28).

REMARK 5.1. In deriving Eqs. (5.18)–(5.22) we made use of the considerations of Sec. 2. That is, the rotation tensor is treated as an independent variable which does not depend directly on $\mathbf{F}$. This fact will be verified in the next section when the symmetry conditions of the stress tensors are discussed. Further in Eq. (5.22) the symmetry of $\widehat{\mathbf{U}}$ has been exploited.

By inserting these derivatives in Eq. (5.13) we arrive at the following relations:

$$\sigma = 2\rho\mathbf{F}\frac{\partial\Psi}{\partial\mathbf{C}}\mathbf{F}^T = \rho\mathbf{F}\frac{\partial\Psi}{\partial\mathbf{E}}\mathbf{F}^T\,, \tag{5.23}$$

$$= 2\rho\frac{\partial\Psi}{\partial\widehat{\mathbf{C}}}\widehat{\mathbf{C}} = \rho\frac{\partial\Psi}{\partial\widehat{\mathbf{E}}}\widehat{\mathbf{C}}\,, \tag{5.24}$$

$$= 2\rho\widehat{\mathbf{U}}\frac{\partial\Psi}{\partial\widehat{\mathbf{U}}^2}\widehat{\mathbf{U}} = \rho\widehat{\mathbf{U}}\frac{\partial\Psi}{\partial\widehat{\mathbf{E}}}\widehat{\mathbf{U}}\,, \tag{5.25}$$

$$= \rho\mathbf{R}\frac{\partial\Psi}{\partial\mathbf{U}}\mathbf{F}^T = \rho\mathbf{R}\frac{\partial\Psi}{\partial\mathbf{H}}\mathbf{F}^T\,, \tag{5.26}$$

$$= \rho\widehat{\mathbf{U}}\frac{\partial\Psi}{\partial\widehat{\mathbf{U}}} = \rho\widehat{\mathbf{U}}\frac{\partial\Psi}{\partial\widehat{\mathbf{H}}}\,, \tag{5.27}$$

$$= -2\rho\mathbf{F}^{-T}\frac{\partial\Psi}{\partial\widehat{\mathbf{c}}}\mathbf{F}^{-1} = \rho\mathbf{F}^{-T}\frac{\partial\Psi}{\partial\widehat{\mathbf{e}}}\mathbf{F}^{-1}\,, \tag{5.28}$$

$$= -2\rho\widehat{\mathbf{C}}^{-1}\frac{\partial\Psi}{\partial\mathbf{c}} = \rho\widehat{\mathbf{C}}^{-1}\frac{\partial\Psi}{\partial\mathbf{e}}\,, \tag{5.29}$$

$$= -2\rho\mathbf{u}\frac{\partial\Psi}{\partial\mathbf{u}^2}\mathbf{u} = \rho\mathbf{u}\frac{\partial\Psi}{\partial\mathbf{e}}\mathbf{u}\,, \tag{5.30}$$

$$= -\rho\mathbf{R}\frac{\partial\Psi}{\partial\widehat{\mathbf{u}}}\mathbf{F}^{-1} = \rho\mathbf{R}\frac{\partial\Psi}{\partial\widehat{\mathbf{h}}}\mathbf{F}^{-1}\,, \tag{5.31}$$

$$= -\rho\frac{\partial\Psi}{\partial\mathbf{u}}\mathbf{u} = \rho\frac{\partial\Psi}{\partial\mathbf{h}}\mathbf{u}\,. \tag{5.32}$$

The comparison with Eqs. (4.28)–(4.38) shows further that the following relations hold:

$$\boldsymbol{\Sigma} = 2\rho\frac{\partial\Psi}{\partial\mathbf{C}} = \rho\frac{\partial\Psi}{\partial\mathbf{E}}\,, \tag{5.33}$$

$$\widehat{\boldsymbol{\Sigma}} = 2\rho\frac{\partial\Psi}{\partial\widehat{\mathbf{C}}} = \rho\frac{\partial\Psi}{\partial\widehat{\mathbf{E}}}\,, \tag{5.34}$$

$$\boldsymbol{\lambda} = -2\rho\frac{\partial\Psi}{\partial\mathbf{c}} = \rho\frac{\partial\Psi}{\partial\mathbf{e}}\,, \tag{5.35}$$

$$\widehat{\boldsymbol{\lambda}} = -2\rho\frac{\partial\Psi}{\partial\widehat{\mathbf{c}}} = \rho\frac{\partial\Psi}{\partial\widehat{\mathbf{e}}}\,, \tag{5.36}$$

$$\operatorname{Sym}\boldsymbol{\Gamma} = \rho\frac{\partial\Psi}{\partial\mathbf{U}} = \rho\frac{\partial\Psi}{\partial\mathbf{H}}\,, \tag{5.37}$$

$$\operatorname{Sym}\widehat{\boldsymbol{\Gamma}} = \rho\frac{\partial\Psi}{\partial\widehat{\mathbf{U}}} = \rho\frac{\partial\Psi}{\partial\widehat{\mathbf{H}}}\,, \tag{5.38}$$

$$\operatorname{Sym}\boldsymbol{\gamma} = -\rho\frac{\partial\Psi}{\partial\mathbf{u}} = \rho\frac{\partial\Psi}{\partial\mathbf{h}}\,, \tag{5.39}$$

$$\operatorname{Sym}\widehat{\boldsymbol{\gamma}} = -\rho\frac{\partial\Psi}{\partial\widehat{\mathbf{u}}} = \rho\frac{\partial\Psi}{\partial\widehat{\mathbf{h}}}\,, \tag{5.33}$$

with Sym denoting the symmetric part of a tensor.

These equations serve as a definition for the dual variables: *The derivative of the free energy function with respect to a strain tensor delivers its conjugate stress tensor.* This definition is completely equivalent to that used by HILL [15], which rests on the material derivetive of the free energy function. The latter is equal to the stress tensor times the material derivative of the dual strain tensor. In other words, equivalently to Eqs. (5.7) and (5.8) we can write

$$\rho\dot{\Psi} = \widehat{\Gamma} : \dot{\widehat{\mathbf{U}}} = \widehat{\Gamma} : \dot{\widehat{\mathbf{H}}} , \tag{5.41}$$

$$= \Gamma : \dot{\mathbf{U}} = \Gamma : \dot{\mathbf{H}} , \tag{5.42}$$

$$= -1/2\boldsymbol{\lambda} : \dot{\mathbf{c}} = \boldsymbol{\lambda} : \dot{\mathbf{e}} , \tag{5.43}$$

$$= -1/2\widehat{\boldsymbol{\lambda}} : \dot{\widehat{\mathbf{c}}} = \widehat{\boldsymbol{\lambda}} : \dot{\widehat{\mathbf{e}}} , \tag{5.44}$$

$$= -\boldsymbol{\gamma} : \dot{\mathbf{u}} = \boldsymbol{\gamma} : \dot{\mathbf{h}} , \tag{5.45}$$

$$= -\widehat{\boldsymbol{\gamma}} : \dot{\widehat{\mathbf{u}}} = \widehat{\boldsymbol{\gamma}} : \dot{\widehat{\mathbf{h}}} . \tag{5.46}$$

Thus we have $\boldsymbol{\Sigma}$ and $\mathbf{C}$ or $\mathbf{E}$, $\widehat{\boldsymbol{\Sigma}}$ and $\widehat{\mathbf{C}}$ or $\widehat{\mathbf{E}}$, Γ and $\mathbf{U}$ or $\mathbf{H}$, $\widehat{\Gamma}$ and $\widehat{\mathbf{U}}$ or $\widehat{\mathbf{H}}$ etc. as dual variables. Note that only the symmetric parts of Γ, $\widehat{\Gamma}$, $\widehat{\boldsymbol{\gamma}}$ and $\boldsymbol{\gamma}$ are expressed by a free energy function. Their skew-symmetric parts are reactive stresses to be determined by the symmetry conditions of $\mathbf{U}$ and $\widehat{\mathbf{U}}$, respectively. More about this in the next section.

We have also the important fact that the Almansi strain tensor has not the Cauchy tensor as a conjugate variable. In other words, the *push-forward* of two dual variables by ϕ_* does not define further dual variables, since the metric is not preserved.

We note the following analogy. The Green strain tensor $\mathbf{E}$ is the dual variable of $\boldsymbol{\Sigma}$, the *pull-back* of the Cauchy stress tensor ($\boldsymbol{\Sigma} = \phi^*\boldsymbol{\sigma}$). On the contrary, the Almansi strain tensor $\mathbf{e}$, the *push-forward* of the Green strain tensor ($\mathbf{e} = \phi_*\mathbf{E}$), is the dual variable of $\boldsymbol{\lambda}$, the *push-forward* of the rotated Cauchy stress tensor ($\boldsymbol{\lambda} = \phi_*\widehat{\boldsymbol{\sigma}}$).

REMARK 5.1. In MARSDEN and HUGHES [19] the so-called spatial relations are generated from the material relations by *pushing-forward* the latter under $\mathbf{F}$. An extensive application of this technique can be found in SIMO [30]. As shown in SANSOUR [25], this technique does not provide us with new physical relations suitable for a spatial formulation. Moreover, the free energy is always understood as a function of the strain measures for which the material relation is formulated. A famous example is the Doyle–Ericksen formula. The *push-forward* of the relation $\boldsymbol{\Sigma} = \rho\partial\Psi(\mathbf{E})/\partial\mathbf{E}$ leads to the Doyle–Ericksen formula: $\boldsymbol{\sigma} = \rho\partial\Psi(\phi^*\mathbf{e})/\partial\mathbf{e}$ (originally formulated in terms of $\mathbf{g}$: $\boldsymbol{\sigma} = 2\rho\partial\Psi(\phi^*\mathbf{g})/\partial\mathbf{g}$). Physically, the Doyle–Ericksen formula is completely equivalent to $\boldsymbol{\sigma} = \mathbf{F}\boldsymbol{\Sigma}\mathbf{F}^T$, since the free energy is considered a function of $\mathbf{E}$ and not of $\mathbf{e}$; accordingly, we will not adopt this technique here.

5.2. The dual variable of the Cauchy stress tensor

Now, what about $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$? To find their conjugate variable we rewrite Eq. (5.2) using Eq. $(2.10)_1$

$$\mathbf{d} = \frac{1}{2}(\dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}^{-T}\dot{\mathbf{F}}^T) = \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{T-1}\dot{\mathbf{U}}^T)\mathbf{R}^T . \tag{5.47}$$

By the definition

$$\widehat{\mathbf{d}} := \mathbf{R}^T\mathbf{d}\mathbf{R} = \dot{g}_{ij}\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j \tag{5.48}$$

we have $\rho\dot{\Psi} = \widehat{\boldsymbol{\sigma}} : \widehat{\mathbf{d}}$ and

$$(5.49) \qquad \widehat{\mathbf{d}} = \frac{1}{2}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{T-1}\dot{\mathbf{U}}^T).$$

Let $\boldsymbol{\varepsilon}$ and $\widehat{\boldsymbol{\varepsilon}}$ be the dual strain tensors of $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$, respectively. We have now $\dot{\boldsymbol{\varepsilon}} = \mathbf{d}$, $\dot{\widehat{\boldsymbol{\varepsilon}}} = \widehat{\mathbf{d}}$, and

$$(5.50) \qquad \boldsymbol{\varepsilon} = \mathbf{R}\widehat{\boldsymbol{\varepsilon}}\mathbf{R}^T,$$

$$(5.51) \qquad \dot{\widehat{\boldsymbol{\varepsilon}}} = \frac{1}{2}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{T-1}\dot{\mathbf{U}}^T).$$

Thus the dual variables of $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\sigma}}$ are given by solving the differential equation (5.51). If $\mathbf{U}$ is a diagonal tensor, that is $\mathbf{R} = \mathbf{1}$ and $\mathbf{U}$, $\widehat{\mathbf{U}}$ coincide with $\mathbf{F}$, then $\widehat{\boldsymbol{\varepsilon}}$ is immediately given to $\ln\mathbf{U}$, which is defined in the usual sense $(\ln\mathbf{U})_{ij} = U_i\delta_{ij}$, no sum over i). This tensor is known as the Hencky strain (HENCKY [13], HILL [15], for a generalized definition of logarithmic strain measure we refer to HOGER [16]). On the other hand, the existence of two representations of the Cauchy stress tensor and of its isometric (rotated) tensor suggests the existence of two corresponding representations of their duals. That is, one may write for $\widehat{\boldsymbol{\varepsilon}}$ once $\widehat{\boldsymbol{\varepsilon}} = Y_{IJ}\mathbf{G}^I \otimes \mathbf{G}^J$ and once $\widehat{\boldsymbol{\varepsilon}} = \varepsilon_{ij}\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j$. In terms of Y_{IJ}, the dual strains of the stresses Σ^{IJ}, the differential equation to be solved reads

$$(5.52) \qquad \dot{Y}_{IJ} = \frac{1}{2}(G_{IK}(U^{-1})^{Kn}\dot{U}_{nJ} + G_{JK}(U^{-1})^{Kn}\dot{U}_{nI}).$$

ε_{ij}, the dual strains to the stresses σ^{ij}, are then given by the equation

$$(5.53) \qquad \varepsilon_{ij} = U_{iS}G^{SM}Y_{MN}G^{NR}U_{jR}$$

which stands in analogy to Eq. $(4.19)_1$.

In the following we prove that for an arbitrary deformation the tensor $\widehat{\boldsymbol{\varepsilon}}$ and correspondingly the tensor $\boldsymbol{\varepsilon}$ does not exist.

PROPOSITION. *The Cauchy stress tensor does not have any dual variable.*

Proof. If suffices to prove that Eq. (5.52) in the general case does not have solutions.

From Eq. (5.52) it is clear that Y_{IJ} can be considered as a function of U_{iJ}. Hence, using

$$(5.54) \qquad \dot{Y}_{IJ} = \frac{\partial Y_{IJ}}{\partial U_{kL}}\dot{U}_{kL},$$

(5.52) can be converted in a partial differential equation resulting in

$$(5.55) \qquad \frac{\partial Y_{IJ}}{\partial U_{kL}} = \frac{1}{2}(\delta_I^L G_{JM}(U^{-1})^{Mk} + \delta_J^L G_{IM}(U^{-1})^{Mk}).$$

The integrability condition of the last equation reads

$$(5.56) \qquad \frac{\partial Y_{IJ}}{\partial U_{rS}\partial U_{kL}} = \frac{\partial Y_{ij}}{\partial U_{kL}\partial U_{rS}}.$$

With

$$(5.57) \qquad \frac{\partial (U^{-1})^{Ji}}{\partial U_{rS}} = -(U^{-1})^{Jr}(U^{-1})^{Si}$$

one gets at the end for (5.56) the integrability condition

$$(U^{-1})^{Mr}(U^{-1})^{Sk}(\delta_I^L G_{JM} + \delta_J^L G_{IM}) - (U^{-1})^{Mk}(U^{-1})^{Lr}(\delta_I^S G_{JM} + \delta_J^S G_{IM}) = 0 . \tag{5.58}$$

Since the components of U, as far as the symmetry is not violated, can be chosen arbitrary, the last equation cannot hold. As an example one chooses $r = 1$, $i = 1$, $s = 1$, $k = 1$, $l = 2$, $j = 2$. The condition then reads

$$G_{11}(U^{-1})^{11}(U^{-1})^{11} - G_{22}(U^{-1})^{21}(U^{-1})^{12} = 0 \tag{5.59}$$

which is in fact a restriction on the admissible deformations. As previously noted, the case when only diagonal terms are present is much simpler and Eq. (5.52) is easily integrated.

This statement is in fact far reaching and will have considerable influence on the definition of objective rates in the subsequent discussions. Moreover, and since the result is surprising because of the very appealing physical meaning of the Cauchy stress tensor, many questions arise concerning the definition of the physical quantity termed stress. We believe that such questions cannot be answered without appealing to the micro-mechanical foundations of solid mechanics, aspects which lie beyond the scope of this paper.

5.3. Conditions of duality under ϕ_*

Before leaving this section we discuss briefly the following question. Given a stress tensor **Z** and its dual strain tensor **Y** defined in the reference configuration, that is $\mathbf{Z} = Z^{ij}\mathbf{G}_I \otimes \mathbf{G}_J$, $\mathbf{Y} = Y_{ij}\mathbf{G}^I \otimes \mathbf{G}^J$, and $\dot{\Psi} = Z^{ij} \cdot \dot{Y}_{ij}$. What are then the corresponding dual variables for $\phi_*\mathbf{Z}$ and $\phi_*\mathbf{Y}$, respectively? The existence of such dual variables is connected with the fulfillment of certain conditions to be formulated next.

The *push-forward* operation ϕ_* can be understood as a *push-forward* by **U** followed by a *push-forward* by **R**. Since transformations due to **R** preserve the duality, it suffices to consider transformations due to **U**. Let $\boldsymbol{\eta}$ be the stress tensor dual to $\mathbf{U}_*(\mathbf{Y})$ and $\boldsymbol{\beta}$ be the strain tensor dual to $\mathbf{U}_*(\mathbf{Z})$. By the invariance of $\dot{\Psi}$ with respect to change of dual variables we have

$$\rho\dot{\Psi} = \mathbf{Z} : \dot{\mathbf{Y}} = \mathbf{U}_*(\mathbf{Z}) : \dot{\boldsymbol{\beta}} , \tag{5.60}$$

$$= \boldsymbol{\eta} : (\mathbf{U}_*(\mathbf{Y}))^{\cdot} . \tag{5.61}$$

For the elaboration of the time derivative of a tensor of the form $\beta_{ij}\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j$ we calculate first the time derivatives of the bases $\widehat{\mathbf{g}}^i$. Using Eq. $(3.20)_2$ and by understanding $(U^{-1})^{Ji}$ as functions of U_{iJ} we get

$$\dot{\widehat{\mathbf{g}}}^i = \frac{\partial (U^{-1})^{Ji}}{\partial U_{rS}}\dot{U}_{rS}\mathbf{G}_J = -(U^{-1})^{Jr}(U^{-1})^{Si}\dot{U}_{rS}\mathbf{G}_J = -(U^{-1})^{Si}\dot{U}_{rS}\mathbf{g}^r \tag{5.62}$$

and hence

$$\dot{\boldsymbol{\beta}} = (\dot{\beta}_{ij} - \beta_{mj}(U^{-1})^{Sm}\dot{U}_{iS} - \beta_{im}(U^{-1})^{Sm}\dot{U}_{jS})\widehat{\mathbf{g}}^i \otimes \widehat{\mathbf{g}}^j . \tag{5.63}$$

With the last relation at hand, we can consider the two cases formulated in (5.60) and (5.61).

CASE 1. The dual variable of $\mathbf{U}_*(\mathbf{Z})$

From Eqs. (5.60) and (5.63) we infer that for any $\mathbf{Y}$ being an appropriate function of $\mathbf{U}$ the following relation

$$\dot{Y}_{ij} = \dot{\beta}_{ij} - \beta_{mj}(U^{-1})^{Sm}\dot{U}_{iS} - \beta_{im}(U^{-1})^{Sm}\dot{U}_{jS} \tag{5.64}$$

has to hold. $\boldsymbol{\beta}$ is then given by solving this equation. As already seen in the case of the Cauchy stress tensor ($\mathbf{Y} = \mathbf{U}^2$) a solution, and hence $\boldsymbol{\beta}$ must not exist. The case $\mathbf{Y} = \mathbf{U}$ has been already considered. Equation (5.60) is up to the sign identical with Eq. $(5.56)_1$ and $\boldsymbol{\beta}$ turns out to be $-\widehat{\mathbf{u}}$, that is $\beta_{ij} = -U_{iJ}$. Equation (5.64) is cleary fulfilled.

CASE 2. The dual variable of $\mathbf{U}_*(Y)$

In this case $\boldsymbol{\beta} = \mathbf{U}_*(\mathbf{Y})$ is known, that is $\beta_{ij} = Y_{ij}$, and we are looking for the corresponding stress tensor $\boldsymbol{\eta}$. From Eq. (5.61) we infer that the equation

$$Z^{ij} \cdot \dot{Y}_{ij} = \eta^{ij}(\dot{Y}_{ij} - Y_{mj}(U^{-1})^{Sm}\dot{U}_{iS} - Y_{im}(U^{-1})^{Sm}\dot{U}_{jS}) \tag{5.65}$$

has to hold. For $\mathbf{Y}$ being an appropriate function of $\mathbf{U}$ the time derivatives can be eliminated to give

$$Z^{ij}\frac{\partial Y_{ij}}{\partial U_{rS}} = \eta^{ij}\left(\frac{\partial Y_{ij}}{\partial U_{rS}} - Y_{mj}(U^{-1})^{Sm}\delta_i^r - Y_{im}(U^{-1})^{Sm}\delta_j^r\right). \tag{5.66}$$

The condition of existence of $\boldsymbol{\eta}$ reduces now to that of the invertibility of the bracketed expression on the right-hand side of Eq. (5.66). A condition which is easier to fulfil than that of Case 1.

5.4. Material and spatial descriptions

In this subsection we give a justification for the classification of tensors we were employing. As already remarked in the introduction, any strain measure can be used irrespective of the kind of description one is adopting. Nevertheless, it is our concern to demonstrate that what we called tensors of the first type are well suited for a material (Lagrangian) description, whereas tensors of the second type are well suited for a spatial (Eulerian) description.

The free energy function depends on the strain measures through their invariants and the invariants of their tensor products with possible structural tensors describing specific anisotropies of the body. Since the inclusion of anisotropies does not affect in any way our considerations, we keep it out focusing on isotropic free energy functions. We choose $\widehat{\mathbf{C}}$ as a representative of the tensors of the first type and $\mathbf{c}$ as a representative of those of the second type.

The free energy is in both cases a function of the invariants

$$\operatorname{tr}\widehat{\mathbf{C}} = g_{ij}G^{IJ}, \qquad \operatorname{tr}\widehat{\mathbf{C}}^2 = g_{ij}g_{kl}G^{JK}G^{IL}, \qquad \operatorname{tr}\widehat{\mathbf{C}}^3 = g_{ij}g_{kl}g_{mn}G^{JK}G^{ML}G^{IN} \tag{5.67}$$

and

$$\operatorname{tr}\mathbf{c} = g_{ij}G^{IJ}, \qquad \operatorname{tr}\mathbf{c}^2 = G_{IJ}G_{KL}g^{jk}g^{il}, \qquad \operatorname{tr}\mathbf{c}^3 = G_{IJ}G_{KL}G_{MN}g^{jk}g^{ml}g^{in}, \tag{5.68}$$

respectively. We recall again that isometric tensors have the same invariants reflecting their physical equivalence.

Now, within the material (Lagrangian) description the reference configuration is given, thus G^{IJ} are a priori known, whereas g_{ij} are depending on the displacement field. g^{ij} are then given as the inverse matrix of g_{ij} and the expressions for them are in general extremely involved. Contrasting this, the actual configuration $\mathbf{x}$ and hence the actual metric g_{ij} are a priori known within a spatial (Eulerian) description. G_{IJ}, and hence G^{IJ}, are now depending on the displacement field with the latter necessitating inverse operations.

Taking a look at Eqs. (5.67) and (5.68) together with the preceding remarks we conclude that what we called tensors of the first type are well suited for a material descriptions, and what we called tensors of the second type are well suited for a spatial description since inverse operations are avoided. Thus it is a matter of practicality to choose tensors of the second type within a spatial description and tensors of the first type within a material one, by no means the matter of theoretical considerations.

6. The independent rotation field and the symmetry of the stress tensors

In the last section, explicitly in deriving (5.18)–(5.22), as well as in Sec. 2 when defining flows on the configuration space, we made use of the fact that the rotation tensor **R** can be considered as an independent variable. This fact turns out to be essential in analyzing the structures of stress tensors as given in Eqs. (5.33)–(5.40). In this section we give a justification of our approach. A corner-stone in our considerations is that the classical continuum theory of simple materials can be achieved as a limit of a micropolar continuum (Cosserat continuum) with independent rotations. The limit case is characterized by, first, the fact that the micro-rotations coincide with the macro-rotations furnished by the polar decomposition of the deformation gradient, and second, the fact that the free energy function is assumed to depend on the stretch tensor alone.

In addition, the question of the symmetry of the involved stress tensors is considered in this section as well. Here we consider the symmetry of the stress tensors generated using the components γ_{iJ}. The symmetry of those stress tensors generated using the components σ_{ij} or $\boldsymbol{\Sigma}_{IJ}$ has been already discussed in Remark 4.2.

6.1. The restricted Cosserat continuum

The equilibrium equations of simple materials read

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}\,, \tag{6.1}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T\,, \tag{6.2}$$

where div denote the divergence operation with respect to the actual configuration. With the help of Eqs. (4.28) and (4.38) the equilibrium equations can be recast in terms of $\boldsymbol{\Gamma}$ to give

$$\operatorname{Div}(\mathbf{R}\boldsymbol{\Gamma}) + \mathbf{f} = 0\,, \tag{6.3}$$

$$\mathbf{U}\boldsymbol{\Gamma}^T = \boldsymbol{\Gamma}\mathbf{U}^T, \qquad \mathbf{U} = \mathbf{U}^T\,. \tag{6.4}$$

Here Div denote the divergence operation with respect to the reference configuration.

We show now that these equations are recorvered by considering a restricted Cosserat continuum. Let $\mathbf{Q} \in SO(3)$ denote the independent rotation field of the Cosserat continuum. By the relation $\mathbf{Q}\mathbf{Q}^T = 1$ we have $\mathbf{Q}^T\dot{\mathbf{Q}} \in so(3)$ as well as $\mathbf{Q}^T\mathbf{Q}_{,i} \in so(3)$, $so(3)$ being the tangent space of $SO(3)$ at the identity consisting of the skew-symmetric tensors. Denoting the axial vectors of the skew-symmetric tensors $\mathbf{Q}^T\dot{\mathbf{Q}}$, $\mathbf{Q}^T\mathbf{Q}_{,i}$ by ω and

$\mathbf{k}_i$ we have

$$\mathbf{Q}^T\dot{\mathbf{Q}} = -\epsilon\boldsymbol{\omega}, \qquad \mathbf{Q}^T\mathbf{Q}_{,i} = -\epsilon\mathbf{k}_i\,, \tag{6.5}$$

where ϵ denotes the three-dimensional Levi–Civita permutation tensor.

The strain measures of the Cosserat continuum are then defined to be (ERINGEN [10])

$$\mathbf{U} = \mathbf{Q}^T\mathbf{F}, \qquad \mathbf{K} = -\mathbf{k}_i \otimes \mathbf{G}^I\,. \tag{6.6}$$

Correspondingly, the free energy is now a function of $\mathbf{U}$ and $\mathbf{K}$. $\mathbf{U}$ is not symmetric any more and the stress and couple tensors are defined to be $\boldsymbol{\Gamma} = \partial\psi/\partial\mathbf{U}$ and $\mathbf{M} = \partial\psi/\partial\mathbf{K}$. According to the first law of thermodynamics we have (inertia terms are suppressed)

$$\int_{\mathcal{B}} \rho\dot{\psi}(\mathbf{U},\mathbf{K})d\mathcal{V} = \int_{\mathcal{B}} (\mathbf{f}\cdot\mathbf{v} + \mathbf{p}\cdot\boldsymbol{\omega})\,d\mathcal{V} + \int_{\partial\mathcal{B}_\sigma} (\mathbf{t}\cdot\mathbf{v} + \mathbf{m}\cdot\boldsymbol{\omega})d\mathcal{A}\,, \tag{6.7}$$

where $\mathbf{f}$, $\mathbf{p}$ denote the force and the couple acting within the body region denoted by $\mathcal{B}$, $\mathbf{t}$, $\mathbf{m}$ are the corresponding boundary terms, $\mathbf{v}$ the velocity field, and $d\mathcal{V}$, $d\mathcal{A}$ the volume and the surface elements. Using classical methods, the integral statement can be transformed to differential equations governing the behaviour of the Cosserat continuum, details which we omit here since they are not important for our purposes. We emphasize again that in Eq. (6.7), both the displacements and the rotations have to be considered as independent fields. To pass to the classical simple continuum we restrict Eq. (6.7) to the following conditions: $\psi = \psi(\mathbf{U})$, $\mathbf{p} = \mathbf{O}$, $\mathbf{m} = \mathbf{O}$, $\mathbf{U} = \mathbf{U}^T$. Whereas the first three conditions can be directly fulfilled, the last one, the symmetry condition, is new. To account for, we include it with the help of a Lagrange multiplier in the functional (6.7).

PROPOSITION. *The functional*

$$\int_{\mathcal{B}} \rho\dot{\psi}(\mathbf{U})\,d\mathcal{V} - \int_{\mathcal{B}} \frac{1}{2}\boldsymbol{\eta} : (\dot{\mathbf{U}} - \dot{\mathbf{U}}^T)\,d\mathcal{V} + \int_{\mathcal{B}} \frac{1}{2}\dot{\boldsymbol{\eta}} : (\mathbf{U} - \mathbf{U}^T)\,d\mathcal{V} - \int_{\mathcal{B}} \mathbf{f}\cdot\mathbf{v}\,d\mathcal{V} - \int_{\partial\mathcal{B}_\sigma} \mathbf{t}\cdot\mathbf{v}\,d\mathcal{A} = 0\,. \tag{6.8}$$

yields the equilibrium equations (6.3) and (6.4) as Euler–Lagrange equations.

Proof. First we recall that

$$\dot{\mathbf{F}} = \mathbf{v}_{,I} \otimes \mathbf{G}^I\,. \tag{6.9}$$

Using Eq. $(6.6)_1$ and noticing that $\partial\psi/\partial\mathbf{U} = \operatorname{sym}\boldsymbol{\Gamma}$ by the symmetry of $\mathbf{U}$ we get

$$\int_{\mathcal{B}} \operatorname{Sym}\boldsymbol{\Gamma} : (\dot{\mathbf{Q}}^T\mathbf{F} + \mathbf{Q}^T\dot{\mathbf{F}})\,d\mathcal{V} + \int_{\mathcal{B}} \operatorname{Skew}\boldsymbol{\eta} : (\dot{\mathbf{Q}}^T\mathbf{F} + \mathbf{Q}^T\dot{\mathbf{F}})\,d\mathcal{V} + \int_{\mathcal{B}} \frac{1}{2}\dot{\boldsymbol{\eta}} : (\mathbf{Q}^T\mathbf{F} - \mathbf{F}^T\mathbf{Q})\,d\mathcal{V} - \int_{\mathcal{B}} \mathbf{f}\cdot\mathbf{v}\,d\mathcal{V} - \int_{\partial\mathcal{B}_\sigma} \mathbf{t}\cdot\mathbf{v}\,d\mathcal{A} = 0\,, \tag{6.10}$$

which may be rewritten in the form

$$\int_{\mathcal{B}} (\operatorname{Sym}\boldsymbol{\Gamma} + \operatorname{Skew}\boldsymbol{\eta})\mathbf{U}^T : \dot{\mathbf{Q}}^T\mathbf{Q}\,d\mathcal{V} + \int_{\mathcal{B}} \frac{1}{2}\mathbf{Q}\boldsymbol{\Gamma} : \dot{\mathbf{F}}\,d\mathcal{V} + \int_{\mathcal{B}} \frac{1}{2}\dot{\boldsymbol{\eta}} : (\mathbf{Q}^T\mathbf{F} - \mathbf{F}^T\mathbf{Q})\,d\mathcal{V} - \int_{\mathcal{B}} \mathbf{f}\cdot\mathbf{v}\,d\mathcal{V} - \int_{\partial\mathcal{B}_\sigma} \mathbf{t}\cdot\mathbf{v}\,d\mathcal{A} = 0\,. \tag{6.11}$$

From Eq. (6.9) and using the fact that all velocities are arbitrary, and by identifying the skew-symmetric part of η with that of Γ we end up with the local statements

$$\operatorname{Div}(\mathbf{Q}\Gamma) + \mathbf{f} = \mathbf{0}, \tag{6.12}$$

$$\Gamma\mathbf{U}^T = \text{symmetric}, \tag{6.13}$$

$$\mathbf{Q}^T\mathbf{F} - \mathbf{F}^T\mathbf{Q} = \mathbf{0} \tag{6.14}$$

and the corresponding boundary conditions. By Eq. (6.14) we have then $\mathbf{Q} = \mathbf{R}$ (remember, $\mathbf{R}$ is determined by Eq. (2.18). That is, the micro-rotations coincide with the macro-rotations furnished by the polar decomposition of $\mathbf{F}$. Hence the statement of the proposition.

We conclude:

1. Also in a classical theory the rotation field can be considered as an independent variable provided the symmetry of the stretch tensor is secured. The elaborated rotation field coincides then with that furnished by the polar decomposition theorem.

2. The stress tensor Γ and, consequently, the corresponding family of stress tensors ($\widehat{\Gamma}$, $\widehat{\gamma}$, γ) are non-symmetric in general. Their symmetric part is expressed by the free energy function while the skew-symmetric part describes reactive stresses.

The functional (6.8) is equivalent to formulations to be found in the literature constructed directly as weak forms of Eqs. (6.3) and (6.4) without referring to a Cosserat continuum (see ATLURI [2], BUFLER [3]).

The above treatment is given for a general non-isotropic body. The special case of isotropy allows for a simpler treatment of the problem without regarding to a Lagrange multiplier. Moreover, the stress tensor turns out to be symmetric. As a by-product, we obtain a new variational statement. These aspects will be considered next.

6.2. The case of isotropy

In the following we assume that the free energy corresponding to the Cosserat continuum is given as an isotropic function of the stretch tensor alone and that no external couples are acting on the body.

PROPOSITION. *Consider the functional* (6.7). *Let* $\psi(\mathbf{U}) = \psi(\mathrm{I}, \mathrm{II}, \mathrm{III})$ *be any function of the invariants of* $\mathbf{U}$ *defined by*

$$\mathrm{I} = \operatorname{tr}\mathbf{U}, \qquad \mathrm{II} = \operatorname{tr}\mathbf{U}^2, \qquad \mathrm{III} = \operatorname{tr}\mathbf{U}^3. \tag{6.15}$$

Let further $\mathbf{m} = \mathbf{0}$, $\mathbf{p} = \mathbf{0}$. *The stress tensor* Γ *is then symmetric and the Cosserat continuum reduces to an isotropic non-polar one.*

Proof. The functional (6.7) has now the special form

$$\int_{\mathcal{B}} \rho\dot{\psi}(\mathrm{I}, \mathrm{II}, \mathrm{III})\, d\mathcal{V} - \int_{\mathcal{B}} \mathbf{f}\cdot\mathbf{v}\, d\mathcal{V} - \int_{\partial\mathcal{B}_\sigma} \mathbf{t}\cdot\mathbf{v}\, d\mathcal{A} = 0. \tag{6.16}$$

It is clear that the Euler–Lagrange equations of this functional coincide with both equations (6.12), (6.13) with the fundamental difference that no symmetry conditions on $\mathbf{U}$ are a priori employed. The equations read

$$\operatorname{Div}\left(\mathbf{Q}\frac{\partial\psi(\mathrm{I}, \mathrm{II}, \mathrm{III})}{\partial\mathbf{U}}\right) + \mathbf{f} = \mathbf{0}, \tag{6.17}$$

$$\frac{\partial\psi(\mathrm{I},\mathrm{II},\mathrm{III})}{\partial\mathbf{U}}\mathbf{U}^T = \text{symmetric}. \tag{6.18}$$

Making use of

$$\frac{\partial\mathrm{I}}{\partial\mathbf{U}} = \mathbf{1}, \tag{6.19}$$

$$\frac{\partial\mathrm{II}}{\partial\mathbf{U}} = 2\mathbf{U}^T, \tag{6.20}$$

$$\frac{\partial\mathrm{III}}{\partial\mathbf{U}} = 3\mathbf{U}^{2T}, \tag{6.21}$$

we may obtain

$$\frac{\partial\psi(\mathrm{I},\mathrm{II},\mathrm{III})}{\partial\mathbf{U}} = \frac{\partial\psi}{\partial\mathrm{I}}\frac{\partial\mathrm{I}}{\partial\mathbf{U}} + \frac{\partial\psi}{\partial\mathrm{II}}\frac{\partial\mathrm{II}}{\partial\mathbf{U}} + \frac{\partial\psi}{\partial\mathrm{III}}\frac{\partial\mathrm{III}}{\partial\mathbf{U}} = \alpha_0\mathbf{1} + \alpha_1\mathbf{U} + \alpha_2\mathbf{U}^T, \tag{6.22}$$

where α_0, α_1 and α_2 are functions of the invariants I, II, III. Inserting Eq. (6.22) in Eq. (6.18) we obtain

$$\alpha_0(\mathbf{U} - \mathbf{U}^T) + \alpha_1(\mathbf{U}^2 - \mathbf{U}^{2^T}) + \alpha_2(\mathbf{U}^3 - \mathbf{U}^{3^T}) = 0 \tag{6.23}$$

resulting in

$$\mathbf{U} = \mathbf{U}^T. \tag{6.24}$$

That is, the stretch tensor is symmetric, hence the symmetry of the stress tensor $\boldsymbol{\Gamma}$ as claimed in the proposition.

REMARK 6.1. Since $\mathbf{U}$ turns out to be symmetric, $\psi(\mathrm{I},\mathrm{II},\mathrm{III})$ is a completely represented isotropic function of $\mathbf{U}$. This does not hold for a non-symmetric $\mathbf{U}$. Anyhow, the above proposition holds true if one is regarding ψ from the very beginning as a complete isotropic function of a non-symmetric $\mathbf{U}$ and not only of the three invariants given by (6.15). A proof can be found in SANSOUR and BEDNARCZYK [27].

7. Objective rates and the case of anisotropy

In this section we discuss different types of objective rates, and look for a natural definition of an objective time derivative.

Most of the rate-forms of the constitutive relations known in the literature relate an objective rate of the Cauchy tensor to the symmetric part of the velocity gradient $\mathbf{d}$. Widely used are the Lie derivative, also known as the Oldroyd rate, and the Zaremba–Jaumann rate. Applied to $\boldsymbol{\sigma}$ and $\mathbf{e}$, the Lie derivative with respect to the velocity vector $\mathbf{v}$ reads

$$L_{\mathbf{v}}\boldsymbol{\sigma} = \phi_*\left(\frac{d}{dt}\phi^*\boldsymbol{\sigma}\right) = \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^T, \tag{7.1}$$

$$L_{\mathbf{v}}\mathbf{e} = \phi_*\left(\frac{d}{dt}\phi^*\mathbf{e}\right) = \dot{\mathbf{e}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^T. \tag{7.2}$$

The familiar Zaremba–Jaumann objective-rate will be denoted by $\mathcal{J}$ and is to be understood as a differential operator. It reads

$$\mathcal{J}(\boldsymbol{\sigma}) = \dot{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}^T, \tag{7.3}$$

$$\mathcal{J}(\mathbf{e}) = \dot{\mathbf{e}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W}^T\,, \tag{7.4}$$

where $\mathbf{W} = \frac{1}{2}(\mathbf{l} - \mathbf{l}^T)$.

Since $\mathbf{d} = L_{\mathbf{v}}\mathbf{g} = L_{\mathbf{v}}\mathbf{e}$ we get

$$\rho\dot{\Psi} = \frac{1}{2}\boldsymbol{\sigma} : L_{\mathbf{v}}\mathbf{g} = \boldsymbol{\sigma} : L_{\mathbf{v}}\mathbf{e}\,. \tag{7.5}$$

In the literature there are some attempts to consider those strain tensors as dual to $\boldsymbol{\sigma}$ for which a relation of the type (7.5) exists (see e.g. HAUPT and TSAKMAKIS [12], LEHMAN *et al.* [18]). Anyhow, having the results of Sec. 5 in mind we can see that these definitions are by no means natural or useful. The fact that a relation of the type (7.5) exists, can not automatically produce integrability of a constitutive equation between $L_{\mathbf{v}}\boldsymbol{\sigma}$ and $L_{\mathbf{v}}\mathbf{e}$. These aspects have been discussed in detail by SANSOUR and BEDNARCZYK [26] where it was demonstrated that, for integrability, rate-type constitutive equations in terms of the Lie derivative has to fulfil certain restrictions; in terms of the Zaremba–Jaumann rate, integrability is excluded at all. Further, relations of the type (7.5) are not unique in the sense that the Cauchy stress tensor can appear in conjunction with different strain tensors depending on the chosen objective rate.

A further physically motivated restriction on objective stress rates is given by considering the decomposition of tensors in a deviatoric and a spherical part. It is physically reasonable to claim that the objective rate of a deviator is also a deviator.

We claim now that an objective rate should arise naturally, be integrable for a constant tangent operator, and be a deviator if applied to a deviator. In the following we show that such an objective rate in fact does exist for all stress and strain tensors (with the exception of the Cauchy stress tensor and its isometric tensor) with time derivative being non-objective.

It is obvious that the same rates had to be chosen for the strain tensors and their stress duals. We begin with the tensors of the first type and observe (see Eqs. (3.21)–(3.36), (4.24)–(4.34)) that the time derivative of the tensors $\mathbf{C}$, $\mathbf{E}$, $\mathbf{U}$, $\mathbf{H}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{\Gamma}$ are trivially objective. The tensors $\widehat{\mathbf{C}}$, $\widehat{\mathbf{E}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{H}}$, $\widehat{\boldsymbol{\Sigma}}$ and $\widehat{\boldsymbol{\Gamma}}$ are taken relative to the rotated basis $\widehat{\mathbf{G}}_I$. The following derivative, applied to a tensor $\widehat{\mathbf{C}}$, is now motivated:

$$L_{\omega}\widehat{\mathbf{C}} = \mathbf{R}_*\left(\frac{d}{dt}\mathbf{R}^*(\widehat{\mathbf{C}})\right) = \dot{\widehat{\mathbf{C}}} - \boldsymbol{\Omega}\widehat{\mathbf{C}} - \widehat{\mathbf{C}}\boldsymbol{\Omega}^T\,, \tag{7.6}$$

where we made use of Eq. (5.9). It coincides with the Lie derivative with respect to the tangent vector field $\boldsymbol{\Omega}\mathbf{R}$ ($\boldsymbol{\omega}$ denotes the axial vector of $\boldsymbol{\Omega}$). The existence of such a tangent vector field, and hence the existence of a corresponding flow, was discussed in Sec. 2.

This derivative, which was proposed by GREEN and NAGHDI [11] in connection with the Cauchy stress tensor itself (see also DIENES [6]), defines in a completely natural way objective rates, since it eliminates that part given by the rotation of the basis. We have:

$$L_{\omega}\widehat{\mathbf{C}} = \dot{g}_{ij}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J, \qquad L_{\omega}\widehat{\boldsymbol{\Sigma}} = \dot{\sigma}^{ij}\widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J \qquad \text{etc.} \tag{7.7}$$

We turn now to discuss the tensors of the second type. Understanding them as the inverse of tensors of the first type we see immediately, that the time derivatives of the tensors $\widehat{\mathbf{c}} = \mathbf{C}^{-1}$, $\widehat{\mathbf{e}} = \frac{1}{2}(\widehat{\mathbf{g}} - \widehat{\mathbf{c}})$, $\widehat{\mathbf{u}} = \mathbf{U}^{-1}$, and $\widehat{\mathbf{h}} = \widehat{\mathbf{g}} - \widehat{\mathbf{u}}$ are trivially objective. The same is also true for their stress duals $\widehat{\boldsymbol{\lambda}}$ and $\widehat{\boldsymbol{\gamma}}$.

If we state that one and the same rate had to be chosen for a tensor and its inverse, we see that the derivative L_ω is now a natural choice also for the tensors $\mathbf{c} = \widehat{\mathbf{C}}^{-1}$, $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{c})$, $\mathbf{u} = \widehat{\mathbf{U}}^{-1}$, and $\mathbf{h} = \mathbf{g} - \mathbf{u}$. Thus this rate is also the natural choice for the stress duals $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$. It is easily verified if one observes the structure of the tensors as stated at the end of Sec. 3 and Sec. 4, Eqs. (3.53)–(3.60) and (4.40). For instance

$$L_\omega \mathbf{b} = \dot{g}^{ij}\widehat{\mathbf{G}}_I \otimes \widehat{\mathbf{G}}_J, \qquad L_\omega \boldsymbol{\lambda} = \dot{\sigma}_{ij}\widehat{\mathbf{G}}^I \otimes \widehat{\mathbf{G}}^J \qquad \text{etc.} \tag{7.8}$$

We verify further an interesting property of L_ω. Repeat that $\widehat{\mathbf{C}} = \mathbf{R}\mathbf{C}\mathbf{R}^T$ which we interpreted as the physical equivalence between $\mathbf{C}$ and $\widehat{\mathbf{C}}$. Since the time derivative of $\mathbf{C}$ is trivially objective, the only meaningful rate for $\widehat{\mathbf{C}}$ is clearly $L_\omega\widehat{\mathbf{C}}$. We have then the isometric relation $\mathbf{R}\dot{\mathbf{C}}\mathbf{R}^T = L_\omega\widehat{\mathbf{C}}$. Thus the relations $\dot{\boldsymbol{\Sigma}} = \boldsymbol{\Pi}\dot{\mathbf{E}}$ and $L_\omega\widehat{\boldsymbol{\Sigma}} = \widehat{\boldsymbol{\Pi}}L_\omega\widehat{\mathbf{E}}$, with $\boldsymbol{\Pi}$ being the tangent operator and $\widehat{\boldsymbol{\Pi}}$ being the corresponding isometric tangent operator, are physically equivalent. Now we suggest that the same must also be true for the inverse tensors. Since we have $\mathbf{c} = \mathbf{R}\widehat{\mathbf{c}}\mathbf{R}^T$ which again stress the physical equivalence of $\widehat{\mathbf{c}}$ and $\mathbf{c}$, and if we conclude that $\mathbf{c}$ as the inverse of $\mathbf{C}$ is trivially objective, we see that the only meaningful rate preserving this equivalence in rate form is again L_ω. That is the relations $L_\omega\boldsymbol{\lambda} = \pi L_\omega\mathbf{e}$ and $\dot{\widehat{\boldsymbol{\lambda}}} = \widehat{\pi}\dot{\widehat{\mathbf{e}}}$ are physically equivalent, where π and $\widehat{\pi}$ are the corresponding isometric tangent operators. Moreover, it is easily verified that $L_\omega\widetilde{\boldsymbol{\lambda}}$ is a deviator for $\widetilde{\boldsymbol{\lambda}}$ itself being a deviator.

Our last argument lead even further. Is there any physical need to objective rates? Our answer is no, since all constitutive equations can always be defined equivalently in terms of tensors with time derivative being trivially objective.

As stated in Sec. 5.2, the Cauchy stress tensor does not have any conjugate strain measure. Hence, there cannot exist any objective rate which may be considered as naturally arising. In fact, in view of this remark it is not surprising that all attempts to consider an objective rate for the Cauchy stress tensor had failed to give a physically satisfactory response as applied to theories of hypo-elasticity and elasto-plasticity.

In formulating the free energy as a function of strain tensors with non objective rates, that is as a function of $\widehat{\mathbf{C}}$, $\widehat{\mathbf{U}}$, $\mathbf{c}$, or $\mathbf{u}$ or correspondingly as a function of $\widehat{\mathbf{E}}$, $\widehat{\mathbf{H}}$, $\mathbf{e}$, or $\mathbf{h}$, it is implicitly incorporated that only isotropic material behaviour is considered. This is due to the fact that Eqs. (5.24), (5.27), (5.29), and (5.32) does allow only for isotropic solutions because of the symmetry of $\boldsymbol{\sigma}$. On the contrary, Ψ may be an arbitrary anisotropic function of strain tensors taken with respect to the basis $\mathbf{G}_i$ or $\widehat{\mathbf{g}}_i$, that is tensors with time rates being trivially objective. Hence, in the general anisotropic case the time rate of the strain tensors in Eqs. (5.8), (5.41), (5.43), and (5.45) must be replaced by the objective rate L_ω since the free energy function is not formulated directly in terms of these tensors. As an example let us consider the *Almansi* strain tensor $\mathbf{e}$. We have $\rho\dot{\Psi} = \widehat{\boldsymbol{\lambda}} : \dot{\widehat{\mathbf{e}}} = \boldsymbol{\lambda} : L_\omega\mathbf{e}$. The last equation corresponds to a free energy formulated as $\Psi(\mathbf{R}^*(\mathbf{e}))$. That is, from a thermodynamic point of view the active variable is $\widehat{\mathbf{e}}$ and not $\mathbf{e}$ (see also SANSOUR [25]). These remarks concerning the anisotropic case confirm further what we have already noted out: All physical relations can be equivalently formulated by considering isometric tensors with time rates being trivially objective; these tensors allow, by their definition, for the incorporation of anisotropies as well.

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