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On μ-stability in dynamical systems

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THE GROUP μ-boundedness and group μ-convergence of the trajectories, where μ is the measure, for continuous time and discrete time dynamical systems (continuous and discrete time flows on topological spaces) are considered. The necessary and sufficient conditions for asymptotic group μ-convergence are given. The results are developed for differential dynamical systems defined by vector fields on Riemannian manifolds. The measurability and measure of the invariant subsets and of the limit sets of the set-trajectories are considered.

1. Introduction

THE TRAJECTORIES of a continuous dynamical system are group μ-bounded, where μ is a (positive) measure defined on a family of Borel subsets of the state-space [5, 9], if the volume $\mu(A)$ of any measurable and μ-bounded subset A in the state-space remains bounded along the trajectories of the system [11].

Asymptotic group μ-convergence of the trajectories ensures [15] that the measure $\mu(A)$ of each measurable and μ-bounded set A in the state-space tends to zero asymptotically, as $t \to \infty$.

The dynamical system $\mathcal{S}$ is defined as the quartet $(\mathbf{X}, T, \mathsf{U}, \varphi)$, where $\mathbf{X}$ is the state-space (the dynamic space), T is a group, U — a subset in $\mathbf{X} \times T$ enclosing $\mathbf{X} \times \{0\}$, and φ is a map $\mathsf{U} \to \mathbf{X}$ satisfying:

$$\varphi(x, 0) = x, \qquad \text{for each } x \in \mathbf{X}, \tag{1.1}$$

and

$$\varphi(x, t' + t'') = \varphi(\varphi(x, t'), t''), \tag{1.2}$$

for each $x \in \mathbf{X}$ and all $t', t'' \in T$, such that $(x, t') \in \mathsf{U}$, $(x, t' + t'') \in \mathsf{U}$ and $(\varphi(x, t'), t'') \in \mathsf{U}$ [4, 8, 14].

Additionally, if $\mathbf{X}$ is a topological space, where $\mathcal{T}$ is the topology on $\mathbf{X}$, T is a topological group, U is an open subset in $\mathbf{X} \times T$ and φ is a continuous map, then $\mathcal{S}$ is a continuous dynamical system, and φ is a continuous flow on $\mathbf{X}$ [8].

The dynamical systems are considered, where T is the space of reals or $T = I$, I — the set of integers.

The following assumptions are made:

ASSUMPTION 1. For each $x \in \mathbf{X}$, the motion $\varphi(x, \cdot)$ of the system $\mathcal{S}$ is defined for all t in T^+, where $T^+ = R^+$, $R^+ = [0, \infty)$, or $T^+ = I^+$, $I^+ = \{0, 1, 2, \ldots\}$, ([4], [6], [8]).

Thus, $\mathbf{X} \times T^+ \subseteq \mathsf{U}$. □

A dynamical semi-system $\mathcal{S}^+$ is defined as $(\mathbf{X}, T^+, \mathsf{U}^+, \varphi^+)$ where $T^+ = R^+$ or $T^+ = I^+$, U^+ is a subset in $\mathbf{X} \times T^+$ enclosing $\mathbf{X} \times \{0\}$, and φ^+ is a map $\mathsf{U}^+ \to \mathbf{X}$, where (1.1) is satisfied for all $x \in \mathbf{X}$ and (1.2) is satisfied for all $x \in \mathbf{X}$ and $t', t'' \in T^+$ such that $(x, t') \in \mathsf{U}^+$, $(x, t' + t'') \in \mathsf{U}^+$ and $(\varphi(x, t'), t'') \in \mathsf{U}^+$ [8].

The semi-system $\underline{\mathcal{S}}^+$ is continuous, if $\mathbf{X}$ is a topological space, $\mathsf{U}^+ \setminus \mathbf{X} \times \{0\}$ is an open subset of $\mathbf{X} \times T$, and φ^+ is a continuous map.

A (continuous) dynamical system $\mathcal{S} = (\mathbf{X}, T, \mathsf{U}, \varphi)$ corresponds to the uniquely-defined (continuous) semi-system $\mathcal{S}^+ = (\mathbf{X}, T^+, \mathsf{U}^+, \varphi^+)$, where $\mathsf{U}^+ = \mathsf{U} \cap (\mathbf{X} \times T^+)$ and $\varphi^+ = \varphi_{|\mathsf{U}^+}$.

ASSUMPTION 2. For each $x \in \mathbf{X}$, the motion $\varphi(x, \cdot)$ of the semi-system $\mathcal{S}^+ = (\mathbf{X}, T^+, \mathsf{U}^+, \varphi^+)$, where $T^+ = R^+$ or $T^+ = I^+$, is defined for all $t \in T^+$. Thus, $\mathsf{U}^+ = \mathbf{X} \times T^+$. □

The trajectories of the semi-system $\mathcal{S}^+ = (\mathbf{X}, T^+, \mathbf{X} \times T^+, \varphi^+)$ define the semi-group $(G_t)_{t \in T^+}$ of maps $G_t : \mathbf{X} \to \mathbf{X}$ [4, 6, 8],

$$\mathbf{X} \ni x \to G_t(x) = \varphi^+(x, t).$$

ASSUMPTION 3. The maps G_t, $t \in T^+$, defined by the trajectories of a continuous dynamical system $\mathcal{S}$ (a dynamical semi-system $\mathcal{S}^+$) are C^0-embeddings of $\mathbf{X}$ into $\mathbf{X}$ [1, 6, 8]. □

Let $\mathcal{S} = (\mathbf{X}, T, \mathsf{U}, \varphi)$ be a dynamical system, where the Assumptions are satisfied.

For a subset $A \subseteq \mathbf{X}$, ϕ_A is the map $T^+ \ni t \to 2^{\mathbf{X}}$ defined by:

$$\phi_A(t) = A_t, \qquad A_t = G_t(A), \qquad (G_t(\phi) = \phi),$$

($2^{\mathbf{X}}$ is the family of all subsets in $\mathbf{X}$).

DEFINITION 1. *A subset $A \subseteq \mathbf{X}$ is a strictly (positively) invariant subset for the system $\mathcal{S}^+$, if $G_t(A) = A$ for all $t \in T^+$.*

A subset A is a positively invariant subset for the system $\mathcal{S}^+$, if $G_t(A) \subseteq A$ for all $t \in T^+$, [6, 14]. □

Let $\phi^+_{\mathcal{S}} : 2^{\mathbf{X}} \times T^+ \to 2^{\mathbf{X}}$ be the map defined by

$$2^{\mathbf{X}} \times T^+ \ni (A, t) \to \phi^+_{\mathcal{S}}(A, t) = \phi_A(t).$$

$\phi^+_{\mathcal{S}}$ is the global semi-flow on $2^{\mathbf{X}}$, corresponding to the flow φ (the semi-flow φ^+) on $\mathbf{X}$;

$$\underline{\mathcal{S}}^+ = (2^{\mathbf{X}}, T^+, 2^{\mathbf{X}} \times T^+, \phi^+_{\mathcal{S}})$$

is the dynamical semi-system on $2^{\mathbf{X}} \times T^+$.

The (positive) half-trajectory of a subset $A \in 2^{\mathbf{X}}$ in the global semi-flow ϕ^+_s on $2^{\mathbf{X}}$ is denoted by Γ^+_A. $\operatorname{Im} \phi^+_{\mathcal{S}}(A, \cdot)$ is the trajectory curve of the system $\underline{\mathcal{S}}^+$, where $\operatorname{Im} \phi^+_{\mathcal{S}}(A, \cdot)$ encloses the set A.

The symbol γ^+_x is used for the half-trajectory of the point x in the semi-flow φ^+ on $\mathbf{X}$.

The image set in $\mathbf{X}$ of the half-trajectory Γ^+_A of the set A in the semi-flow $\phi^+_{\mathcal{S}}$ on $2^{\mathbf{X}}$ is the subset

$$\widehat{\gamma}^+(A) = \bigcup_{t \in T^+} A_t.$$

$\widehat{\gamma}^+(A)$ is the invariant subset for the system $\mathcal{S}$.

Let $\mathcal{S} = (\mathbf{X}, T, \mathsf{U}, \varphi)$ be a continuous dynamical system, and let $\mathcal{M}$ be the family of Borel subsets in $\mathbf{X}$ corresponding to the given topology $\mathcal{T}$ on $\mathbf{X}$ [4, 5, 8, 9, 12]. By the Assumption 3, the maps G_t, $t \in T^+$, are the homeomorphisms of $\mathbf{X}$ onto $G_t(\mathbf{X})$.

This ensures that $G_t(A) \in \mathcal{M}$, for any set $A \in \mathcal{M}$ and $t \in T^+$.

By restricting the system $\underline{\mathcal{S}}^+$ to the subfamily $\mathcal{M}$ of $2^{\mathbf{X}}$, one obtains the dynamical semi-system $\mathcal{S}^+_{\mathcal{M}} = (\mathcal{M}, T^+, \mathcal{M} \times T^+, \widetilde{\phi}^+_{\mathcal{S}})$, where $\widetilde{\phi}^+_{\mathcal{S}}$ is the contraction of $\phi^+_{\mathcal{S}}$ to $\mathcal{M}$.

Let μ be a (positive) measure on σ-algebra $\mathcal{M}$ of Borel sets in $\mathbf{X}$. A set $A \in \mathcal{M}$ is called the null set, if $\mu(A) = 0$.

ASSUMPTION 4. The measure μ is complete on $\mathcal{M}$. This means that each subset of the null set is μ-measurable (is in $\mathcal{M}$). As a consequence, the measure of a subset enclosed in a null set has μ-measure equal to zero [9]. □

2. Group μ-boundedness of the trajectories

The group μ-boundedness of the trajectories of dynamical systems is considered.It is assumed that $\mathcal{S}$ is a continuous dynamical system, where the Assumptions 1 and 3 are satisfied. μ is a measure on σ-algebra $\mathcal{M}$ of Borel subsets of the topological space $\mathbf{X}$, the state-space of the system. The measure μ is complete (Assumption 4).

DEFINITION 2. *The trajectories of a continuous dynamical system* $\mathcal{S} = (\mathbf{X}, T, \mathsf{U}, \varphi)$, *where* $\mathbf{X} \times T^+ \subseteq \mathsf{U}$, *are group μ-bounded on* T^+, *if for each μ-bounded set* $A \in \mathcal{M}$

$$\sup_{t \in T^+} \mu(A_t) < \infty . \square$$

DEFINITION 3. *The trajectories of a continuous dynamical system $\mathcal{S}$ are strongly group μ-bounded on T^+, if for each μ-bounded set $A \in \mathcal{M}$ such that $\widehat{\gamma}^+(A) \in \mathcal{M}$, $\widehat{\gamma}^+(A)$ is a μ-bounded set.* □

For a subset $A \in \mathcal{M}$ such that $\widehat{\gamma}^+(A) \in \mathcal{M}$, the strong group μ-boundedness implies the group μ-boundedness. The inverse conclusion is not necessarily valid.

Denote by $\overline{R}^+$ the set $R^+ \cup \{\infty\}$.

THEOREM 1. *Let* $\mathcal{S} = (\mathbf{X}, R, \mathsf{U}, \varphi)$ *be a continuous dynamical system,* $\mathbf{X} \times R^+ \subset \mathsf{U}$.

Assume that the following is satisfied along the trajectories of $\mathcal{S}$.

For each μ-bounded subset $A \in \mathcal{M}$,

a. There exists an open interval $(0, \tau_A)$, $\tau_A > 0$, in R^+ where the function $\widetilde{\mu}_A(\cdot) = \mu(\phi^+_{\mathcal{S}}(A, \cdot))$ is continuously differentiable, and

b. $\dfrac{d}{dt}\widetilde{\mu}_A(t) \leq 0$, for $t \in (0, \tau_A)$.

Then the trajectories of the system $\mathcal{S}$ are group μ-bounded on R^+.

(The proof has been omitted). □

The following Theorem 2 concerns strong group μ-boundedness of the trajectories.

THEOREM 2. *Let the trajectories of a continuous dynamical system* $\mathcal{S} = (\mathbf{X}, R, \mathsf{U}, \varphi)$, *where* $\mathbf{X} \times R^+ \subset \mathsf{U}$, *be group μ-bounded on* R^+, *and let the function* $\widetilde{\mu}_A : R^+ \ni t \to \mu(A_t) \in R^+$ *be continuously differentiable, for any μ-bounded set* $A \in \mathcal{M}$.

Assume that for each μ-bounded set $A \in \mathcal{M}$,

$$\frac{d}{dt}\widetilde{\mu}_A(t) < -\lambda_1 \cdot e^{-\lambda_2 t} \quad \text{on } R^+ ,$$

for some positive constants λ_1, λ_2, where λ_1 and λ_2 depend on A and satisfy $\dfrac{\lambda_1(A)}{\lambda_2(A)} = \mu(A)$.

Then the trajectories of the system $\mathcal{S}$ are strongly group μ-bounded on R^+.

Proof. Let $A \in \mathcal{M}$ be a μ-bounded subset of $\mathbb{X}$. There exist the following estimates:

$$\mu(\widehat{\gamma}^+(A)) \leq \int_0^\infty \widetilde{\mu}_A(t) \cdot dt = \int_0^\infty \left[\int_0^t \frac{d}{d\tau}\widetilde{\mu}_A(\tau) \cdot d\tau + \mu(A) \right] dt$$

$$< \int_0^\infty \left[\int_0^t - \lambda_1 \cdot e^{-\lambda_2 \tau} \cdot d\tau + \mu(A) \right] dt$$

$$= \int_0^\infty \left[\frac{\lambda_1}{\lambda_2} \cdot e^{-\lambda_2 t} - \frac{\lambda_1}{\lambda_2} + \mu(A) \right] dt$$

$$= \frac{\lambda_1}{\lambda_2^2} + \int_0^\infty \left[\mu(A) - \frac{\lambda_1}{\lambda_2} \right] dt = \frac{\lambda_1}{\lambda_2^2} < \infty, \qquad \text{for} \quad \frac{\lambda_1}{\lambda_2} = \mu(A),$$

which yield the thesis. □

DEFINITION 4. ([10]). *Let $\mathbb{X}$ be a metric space. The trajectories of the dynamical system $\mathcal{S}$, where $\mathbb{X} \times T^+ \subseteq \mathbb{U}$, are uniformly equi-bounded on T^+, if for any constant $\varepsilon > 0$, there exists $\delta > 0$ such that* Dia $A < \delta$ *for a subset $A \subseteq \mathbb{X}$ implies* Dia $\widehat{\gamma}^+(A) < \varepsilon$. □

Let $\mathcal{S}$ be a dynamical system, where $\mathbb{X}$ is a metric space, $\mathbb{X} \times T^+ \subseteq \mathbb{U}$. Assume that the μ-measurable and bounded subsets of $\mathbb{X}$ are μ-bounded. Then the uniform equi-boundedness of the trajectories of the system $\mathcal{S}$ on T^+ implies the strong group μ-boundedness, assuming Dia $A < \infty$.

3. Asymptotic group μ-convergence of the trajectories

The limit behavior of the trajectories of a continuous dynamical system $\mathcal{S} = (\mathbb{X}, T, \mathbb{U}, \varphi)$ is considered. The system $\mathcal{S}$ satisfies the Assumptions 1 and 3. μ is a complete measure (Assumption 4) on σ-algebra $\mathcal{M}$ of Borel sets in the state-space $\mathbb{X}$.

DEFINITION 5. *The trajectories of the dynamical system $\mathcal{S}$ are asymptotically group μ-convergent, as $t \to \infty$, if*

$$\mu(A_t) \to 0 \quad as \quad t \to \infty, \tag{3.1}$$

for each μ-bounded set $A \in \mathcal{M}$. □

DEFINITION 6. *The trajectories of the continuous dynamical system $\mathcal{S} = (\mathbb{X}, T, \mathbb{U}, \varphi)$, $\mathbb{X} \times T^+ \subset \mathbb{U}$ are asymptotically group μ-convergent in stable mode, as $t \to \infty$, if the trajectories of $\mathcal{S}$, are asymptotically group μ-convergent, and for any constant $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(A) < \delta$ for a subset $A \in \mathcal{M}$ ensures $\mu(A_t) < \varepsilon$, for all $t \in T^+$.* □

DEFINITION 7. *The trajectories of a continuous dynamical system $\mathcal{S}$ are monotone asymptotically group μ-convergent, if for any μ-bounded set $A \in \mathcal{M}$, $\mu(A_t) \to 0$ monotone as $t \to \infty$.* □

DEFINITION 8. *A subset $B \subseteq \mathbb{X}$ is the equilibrium state for the system $\underline{\mathcal{S}}^+$, if $B_t = B$, for all $t \in T^+$.* □

If the trajectories of a continuous dynamical system $\mathcal{S}$ are asymptotically group μ-convergent, as $t \to \infty$, then μ-bounded equilibrium states of the system $\underline{\mathcal{S}}^+$ corresponding to $\mathcal{S}$ are null sets.

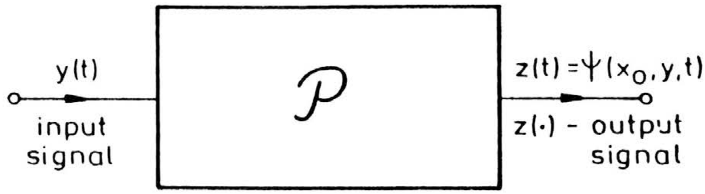

FIG. 1.

The following Example concerning information processing system illustrates the theory ([3]).

EXAMPLE. Consider the information processing system $\mathcal{P}$, as shown in Fig. 1.

The mathematical model of system $\mathcal{P}$ is designed as follows. $\mathcal{P} = (\mathbf{X} \times \mathbf{Y}, T^+, \mathbf{X} \times \mathbf{Y} \times T^+, \psi)$, where: $\mathbf{X}$ is the (topological) state-space of the system, x is a state vector, x_0 denotes the state at $t = 0$, $y(\cdot)$ is the input signal, and $\mathbf{Y}$ is the space of input values, $y(\cdot) : T^+ \to \mathbf{Y}$.

ψ is a map from $\mathbf{X} \times \mathbf{Y} \times T^+$ into $\mathbf{X}$.

It is assumed that for each value $y_0 \in \mathbf{Y}$, $\mathcal{P}_{|y_0} = (\mathbf{X} \times \{y_0\}, T^+, \mathbf{X} \times \{y_0\} \times T^+, \psi_{|y_0})$ is a continuous dynamical semi-system. In the simplified model, the value $z(t)$ of the output signal $z(\cdot)$ is assumed to be equal the value $x(t)$ of the state vector x at time t.

As an example, consider the system $\mathcal{S}_f$, where the non-autonomous differential equation

$$\frac{dx}{dt} = f(x, y), \tag{3.2}$$

$f \in C^1(R^n \times R^m, R^n)$, $x \in R^n$, $y \in R^m$, is the state equation (the dynamic equation).

It is assumed that the solutions of the equation exist for all $t \in R^+$, and that the system possesses the unique steady-state solution for each constant value y_0 of the input signal. Precisely, for each $y_0 \in \mathbf{Y}$, the system $\mathcal{S}_{f|y_0}$ possesses the globally asymptotically stable equilibrium point x^*, where x^* yields both the steady-state vector and the output state of the system $\mathcal{S}_f$, for a fixed value of the input signal.

In the physical model, the equilibrium state varies in time becoming a stochastic process $x^*(\cdot)$, $x^*(t) \in \mathbf{X}$. It is claimed that the average value of $x^*(\cdot)$ assumes the value given in the design of the system.

It is essential, from the engineering point of view, that the statistical displacement of the equilibrium state becomes negligible as $t \to \infty$. I.e., the output value is not sensitive to the dispersion in the equilibrium placement, under disturbance, in the steady-state as $t \to \infty$.

In this context, the quality of the information processing is formulated as follows:

It is assumed that the measure μ is given on the family $\mathcal{M}$ of measurable (Borel) sets in the state-space $\mathbf{X}$ of the system $\mathcal{P}$. It is required that for each input value y_0, $\lim_{t\to\infty} \mu(A_t) = 0$ for any μ-bounded set $A \in \mathcal{M}$. It is then assumed that the trajectories of the system $\mathcal{S}_{f|y_0}$, for a fixed value of the excitation, are asymptotically group μ-convergent, in the steady-state, for $t \to \infty$. □

In the following Theorems 3 and 4, the necessary and sufficient conditions for asymptotic group μ-convergence of the trajectories of a continuous dynamical system are proposed.

$\mathcal{S}^+ = (\mathbf{X}, T^+, \mathbf{X} \times T^+, \varphi^+)$ is a continuous dynamical semi-system, where the Assumptions 2 and 3 are satisfied. $\mathcal{M}$ is the σ-algebra of Borel sets in $\mathbf{X}$, and μ is a measure on $\mathcal{M}$.

THEOREM 3. *Assume that $\mu(\mathbf{X}) < \infty$ and let $\mathbf{X}_{t''} \subseteq \mathbf{X}_{t'}$ for $t'' \geq t'$, $t', t'' \in T^+$. Then, the trajectories of the system $\mathcal{S}^+$ are asymptotically group μ-convergent, as $t \to \infty$, if, and only if, the maximal strictly (positively) invariant subset Λ for the system $\mathcal{S}^+$ is μ-measurable and $\mu(\Lambda) = 0$.*

Proof

i. Assume that the trajectories of $\mathcal{S}^+$ are asymptotically group μ-convergent and let Λ be the maximal strictly (positively) invariant subset for the system $\mathcal{S}^+$.

For each $t \in T^+$, $\Lambda \subseteq \mathbf{X}_t = G_t(\mathbf{X})$. Hence, $\Lambda \subseteq \mathbf{X}_\infty$, where $\mathbf{X}_\infty = \bigcap_{t \in T^+} \mathbf{X}_t$, and by taking into account that $\mathbf{X}_\infty$ is the strictly invariant subset for the system $\mathcal{S}^+$, $\Lambda = \mathbf{X}_\infty$.

The following is used in the proof that $\mathbf{X}_\infty$ is μ-measurable and $\mu(\mathbf{X}_\infty) = 0$.

Let $\{A_\vartheta\}_{\vartheta \in \Theta}$ be a family of μ-measurable sets, where $\phi \neq \Theta \subseteq R$, and $\overline{\vartheta} \notin \Theta$, $\overline{\vartheta} = \sup \Theta$.

If

1. $A_{\vartheta_2} \subseteq A_{\vartheta_1}$ for all $\vartheta_1, \vartheta_2 \in \Theta$, where $\vartheta_2 > \vartheta_1$,

and

2. There is an index $\vartheta' \in \Theta$ such that $\mu(A_{\vartheta'}) < \infty$,

then

1) the set $\bigcap_{\vartheta \in \Theta} A_\vartheta$ is μ-measurable,

and

2) $\lim_{\vartheta \to \overline{\vartheta}} \mu(A_\vartheta) = \mu\Big(\bigcap_{\vartheta \in \Theta} A_\vartheta\Big)$.

Thus, by noting that $\mathbf{X}_{t''} \subseteq \mathbf{X}_{t'}$ for $t', t'' \in T^+$, $t'' > t'$, and $\mu(\mathbf{X}) < \infty$, one obtains that $\mathbf{X}_\infty \in \mathcal{M}$ and

$$\mu(\mathbf{X}_\infty) = \lim_{t \to \infty} \mu(\mathbf{X}_t), \tag{3.3}$$

where the assumed group μ-convergence of the trajectories of the system $\mathcal{S}^+$ yields $\lim_{t \to \infty} \mu(\mathbf{X}_t) = 0$.

ii. Let the maximal strictly invariant set Λ for the system $\mathcal{S}^+$ be μ-measurable and $\mu(\Lambda) = 0$. Using the arguments as in the Part i, one obtains that $\mathbf{X}_\infty = \Lambda$. Hence $\mathbf{X}_\infty \in \mathcal{M}$, $\mu(\mathbf{X}_\infty) = 0$, and $\mu(\mathbf{X}_\infty) = \lim_{t \to \infty} \mu(\mathbf{X}_\infty)$.

Noting that $0 \leq \mu(A_t) \leq \mu(\mathbf{X}_t)$, for each set $A \in \mathcal{M}$ and $t \in T^+$, the limit $\lim_{t \to \infty} \mu(A_t)$ exists and equals zero, for any set $A \in \mathcal{M}$. □

As a corollary, consider a continuous dynamical system $\mathcal{S}$, where Assumptions 1 and 3 are satisfied; μ is a measure on σ-algebra $\mathcal{M}$ of Borel sets in $\mathbf{X}$. For each subset $V \in \mathcal{M}$ such that $\widehat{\gamma}^+(V) \in \mathcal{M}$ and $\mu(\widehat{\gamma}^+(V)) < \infty$, the trajectories of the dynamical system $\mathcal{S}^+_{|\widehat{\gamma}^+(V)}$, $\mathcal{S}^+_{|\widehat{\gamma}^+(V)}$ being the restriction of $\mathcal{S}^+$ to the subset $\widehat{\gamma}^+(V)$, are asymptotically group μ-convergent as $t \to \infty$ if, and only if, the maximal strictly invariant subset for $\mathcal{S}^+_{|\widehat{\gamma}^+(V)}$ is μ-measurable and its μ-measure equals zero. □

THEOREM 4. *Suppose that $\mu(\mathbf{X}) = \infty$ and assume that the trajectories of the system $\mathcal{S}^+$, are strongly group μ-bounded on T^+* (Definition 3).

Let the set union Λ of all strictly (positively) invariant subsets for $\mathcal{S}^+$, being proper subsets of $\mathbf{X}$, be a μ-measurable set and let $\mu(\Lambda) < \infty$.

Assume that the measure μ is complete and assume that for each μ-bounded set $A \in \mathcal{M}$ there exists an open and μ-bounded (proper) subset $\mathbf{V} \subset \mathbf{X}$ enclosing the set A.

Then the trajectories of the system $\mathcal{S}^+$ are asymptotically group μ-convergent as $t \to \infty$ if, and only if, $\mu(\Lambda) = 0$.

Proof

i. Assume that the trajectories of the system $\mathcal{S}^+$ are group μ-convergent as $t \to \infty$. Then the μ-measurability and μ-boundedness of Λ and the strict invariance of the set Λ yield $\mu(\Lambda) = \lim_{t\to\infty} \mu(\Lambda_t) = 0$.

ii. Assume that $\mu(\Lambda) = 0$ and let A be a μ-bounded set in $\mathcal{M}$. We shall prove that $\lim_{t\to\infty} \mu(A_t) = 0$.

Let $\mathbf{V}$ be an open and μ-bounded subset in $\mathbf{X}$ enclosing the set A. $\widehat{\gamma}^+(\mathbf{V})$ is open, and hence a μ-measurable subset of $\mathbf{X}$, and by the assumed strong group μ-boundedness on T^+ of the trajectories of the system $\mathcal{S}^+$, $\widehat{\gamma}^+(\mathbf{V})$ is μ-bounded.

The maximal strictly (positively) invariant subset $\Lambda_{\widehat{\gamma}^+(\mathbf{V})}$ for the system $\mathcal{S}^+_{|\widehat{\gamma}^+(\mathbf{V})}$ is a subset of Λ, and $\mu(\Lambda) = 0$ ensures that $\Lambda_{\widehat{\gamma}^+(\mathbf{V})}$ is μ-measurable, and $\mu(\Lambda_{\widehat{\gamma}^+(\mathbf{V})}) = 0$. As in the Corollary following the Theorem 3, $\lim_{t\to\infty} \mu(\mathbf{V}_t) = 0$, and hence $0 \le \mu(A_t) \le \mu(\mathbf{V}_t)$ yields $\lim_{t\to\infty} \mu(A_t) = 0$. □

The measure of the ω-limit sets of the set-trajectories are considered. $\mathcal{S}^+$ is a continuous semi-system satisfying the Assumption 2.

The positive hull $H^+_{x_0}$ of a point $x_0 \in \mathbf{X}$ denotes the closure of the image set $\widehat{\gamma}^+(x_0)$ of the positive half-trajectory $\gamma^+_{x_0}$ [6, 8],

$$H^+_{x_0} = \operatorname{cl} \widehat{\gamma}^+(x_0),$$

and Ω_{x_0} is the ω-limit set of the motion $\varphi^+(x_0, \cdot)$ [4, 6, 8] (Ω_{x_0} is the ω-limit set of a point x_0 in the positive semi-flow φ^+ on $\mathbf{X}$),

$$\Omega_{x_0} \overset{\text{df}}{=} \bigcap_{\tau \in T^+} H^+_{\varphi^+(x_0,\tau)}.$$

For a subset $U \subseteq \mathbf{X}$,

$$H^+(U) \overset{\text{df}}{=} \bigcup_{x_0 \in U} H^+_{x_0}.$$

Equivalently,

$$H^+(U) = \widehat{\gamma}^+(U) \cup \Big\{ \bigcup_{x_0 \in U} \Omega_{x_0} \Big\}.$$

DEFINITION 9. *The ω-limit set $\Omega(U)$ of a subset U, in the semi-flow φ^+ defined by the trajectories of the semi-system $\mathcal{S}^+$, is given by*

$$\Omega(U) = \bigcap_{\tau \in T^+} H^+(U_\tau). \ \square$$

It is easily observed that $\Omega(U)$ is the set union of the maximal strictly invariant subset for $\mathcal{S}^+$ enclosed in $\widehat{\gamma}^+(U)$ and the strictly invariant subset $\bigcup_{x_0\in U} \Omega_{x_0}$,

$$\Omega(U) = \Big\{ \bigcap_{\tau\in T^+} \widehat{\gamma}^+(U_\tau)\Big\} \cup \Big\{ \bigcup_{x_0\in U} \Omega_{x_0}\Big\}.$$

$\Omega(U)$ is the strictly invariant subset for $\mathcal{S}^+$. For $U = \{x_0\}$, $\Omega(U) = \Omega_{x_0}$.

Let χ_A denote the characteristic function of a subset $A \subseteq \mathbf{X}$. Then,

$$\chi_{\Omega(U)} = \lim_{t\to\infty} \chi_{H^+(U_t)},$$

where the point-wise convergence is taken into considerations.

REMARK 1. The limit set defined as the set union of the ω-limit sets Ω_x of the trajectories γ_x^+, for $x \in U$, and the ω-limit set $\Omega(U)$ in $\mathbf{X}$ of the set-trajectory Γ_U^+ are equal if, and only if, the maximal strictly invariant subset for $\mathcal{S}^+$ enclosed in $\widehat{\gamma}^+(U)$ is a subset of $\bigcup_{x\in U} \Omega_x$. □

The following Corollary concerns the measurability and measures of the ω-limit sets $\Omega(U)$, $U \subseteq \mathbf{X}$, $\mathbf{X}$ being state-space of the dynamical system.

COROLLARY. Consider the continuous dynamical system $\mathcal{S}^+ = (\mathbf{X}, T^+, \mathbf{X}\times T^+, \varphi^+)$. Assume that $\mathbf{X}_{t''} \subseteq \mathbf{X}_{t'}$, for any $t'' > t'$, $t', t'' \in T^+$, and $\mu(\mathbf{X}) < \infty$, where the measure μ is complete.

Then the asymptotic group μ-convergence of the trajectories of the system $\mathcal{S}^+$ implies that $\Omega(U) \in \mathcal{M}$ and $\mu(\Omega(U)) = 0$, for any subset U of the state-space $\mathbf{X}$.

Proof. Let U be a subset of X. The ω-limit set $\Omega(U)$ is the strictly invariant subset for $\mathcal{S}^+$, and hence $\Omega(U)$ is a subset of the maximal strictly invariant subset Λ for the system $\mathcal{S}^+$. By the Theorem 3, $\mu(\Lambda) = 0$. The assumption that the measure μ is complete ensures then that $\Omega(U) \in \mathcal{M}$, and $\mu(\Omega(U)) = 0$. □

Theorem 5 concerns the case $\mu(\mathbf{X}) = \infty$.

THEOREM 5. *A continuous dynamical system $\mathcal{S}^+ = (\mathbf{X}, T^+, \mathbf{X}\times T^+, \varphi^+)$ is considered, where Assumptions 2 and 3 are satisfied. Let μ be a complete measure on the σ-algebra $\mathcal{M}$ of Borel subsets of $\mathbf{X}$, and for each μ-bounded subset $A \in \mathcal{M}$ let us assume that there exists an open and μ-bounded subset $\mathbf{V}$ of $\mathbf{X}$ enclosing the set A.*

Assume that the trajectories of the system $\mathcal{S}^+$ are strongly group μ-bounded on T^+ and that they are asymptotically group μ-convergent as $t \to \infty$.

Additionally, let $\widehat{\gamma}^+(\mathbf{V})$ be a regular subset, for any open set $\mathbf{V} \subseteq \mathbf{X}$ [9]. This means that the boundary $\mathrm{cl}\,\widehat{\gamma}^+(\mathbf{V}) \setminus \widehat{\gamma}^+(\mathbf{V})$ of $\widehat{\gamma}^+(\mathbf{V})$ is a μ-measurable set and $\mu(\mathrm{cl}\,\widehat{\gamma}^+(\mathbf{V}) \setminus \widehat{\gamma}^+(\mathbf{V})) = 0$.

Once the above conditions are satisfied, $\Omega(U) \in \mathcal{M}$ and $\mu(\Omega(U)) = 0$, for any μ-bounded subset $U \in \mathcal{M}$.

Proof. Let $U \in \mathcal{M}$ be a μ-bounded set, and let $\mathbf{V}$ be an open and μ-bounded subset of $\mathbf{X}$ enclosing the set U. For each $t \in T^+$, $\widehat{\gamma}^+(\mathbf{V}_t) = G_t(\widehat{\gamma}^+(\mathbf{V})) \in \mathcal{M}$ and by the strong group μ-boundedness of the trajectories on T^+,

$$\mu(\widehat{\gamma}^+(\mathbf{V})) < \infty. \tag{3.4}$$

$H^+(\mathbf{V}_t) \setminus \widehat{\gamma}^+(\mathbf{V}_t)$ is a subset of $\mathrm{cl}\,\widehat{\gamma}^+(\mathbf{V}_t) \setminus \widehat{\gamma}^+(\mathbf{V}_t)$, where $\widehat{\gamma}^+(\mathbf{V}_t)$ is the regular set. The completeness of the measure μ ensures that $H^+(\mathbf{V}_t) \setminus \widehat{\gamma}^+(\mathbf{V}_t) \in \mathcal{M}$ and $\mu(H^+(\mathbf{V}_t) \setminus \widehat{\gamma}^+(\mathbf{V}_t)) = 0$.

Finally $\widehat{\gamma}^+(\mathbf{V}_t) \in \mathcal{M}$, $\mu(\widehat{\gamma}^+(\mathbf{V}_t)) < \infty$, and

$$\mu(H^+(\mathbf{V}_t)) = \mu(\widehat{\gamma}^+(\mathbf{V}_t)). \tag{3.5}$$

Noting that

$$\Omega(U) = \bigcap_{t\in T^+} H^+(U_t) \subseteq \Omega(\mathbf{V}) = \bigcap_{t\in T^+} H^+(\mathbf{V}_t), \tag{3.6}$$

it suffices to prove $\Omega(\mathbf{V}) \in \mathcal{M}$ and $\mu(\Omega(\mathbf{V})) = 0$. Then the completeness of the measure μ ensures that $\Omega(U) \in \mathcal{M}$, and $\mu(\Omega(U)) = 0$.

From (3.4) and (3.5), by taking into account that $H^+(\mathbf{V}_{t''}) \subseteq H^+(\mathbf{V}_{t'})$ for $t'' \geq t'$, $t', t'' \in T^+$, and using the fact appearing in the proof of the Theorem 3, $\Omega(\mathbf{V}) \in \mathcal{M}$ and $\mu(\Omega(\mathbf{V})) = \lim_{t\to\infty} \mu(H^+(\mathbf{V}_t)) = \lim_{t\to\infty} \mu(\widehat{\gamma}^+(\mathbf{V}_t))$, where the asymptotic group μ-convergence of the trajectories ensures that $\mu(\Omega(\mathbf{V})) = 0$.

By the assumed completeness of the measure μ, the relation (3.6) yields $\Omega(U) \in \mathcal{M}$ and $\mu(\Omega(U)) = 0$. □

4. Smooth dynamical systems on Riemann manifolds

The group μ-boundedness and asymptotic group μ-convergence of the trajectories of differential dynamical system is considered. The state-space $\mathbf{X}$ of the system $\mathcal{S}$ is a finite-dimensional second countable connected C^2-manifold, $n = \mathrm{Dim}\,\mathbf{X}$, with the structure of Riemann space defined by symmetric and positive definite covariant C^1-tensor field g of the degree two on $\mathbf{X}$. It is assumed that the manifold $\mathbf{X}$ is orientable, and that the chosen orientation on $\mathbf{X}$ [1, 7] has been assigned.

ASSUMPTION 5. The closure of a ρ_g-bounded subset of the manifold $\mathbf{X}$, ρ_g being a Riemann metric on $\mathbf{X}$, is a compact subset of $\mathbf{X}$. This is equivalent to the assumption that $\mathbf{X}$ is a complete metric space [1]. □

Let $\{(U_\xi, \psi_\xi(\cdot))\}_{\xi\in\Xi}$ be the maximal C^2-atlas on $\mathbf{X}$, in the chosen orientation on $\mathbf{X}$.

The σ-algebra $\mathcal{M}_{\mathbf{X}}$ of measurable sets in $\mathbf{X}$ is defined in the following way [1, 12]: a subset $A \subseteq \mathbf{X}$ is a measurable set, if $\psi_\xi(A \cap U_\xi) \in \mathcal{M}_n$, for each U_ξ, $\mathcal{M}_n$ being the σ-algebra of Borel sets in R^n.

The σ-algebra $\mathcal{M}_{\mathbf{X}}$ is indentical with the family of Borel subsets of $\mathbf{X}$, where the original topology on the manifold $\mathbf{X}$ is identical with the topology defined by the Riemann metric on $\mathbf{X}$ ([1]).

The measure μ_g is defined on $\mathcal{M}_{\mathbf{X}}$ in the following way: let $A \in \mathcal{M}_{\mathbf{X}}$ and let $\{A_\vartheta\}_{\vartheta\in\Theta}$ be any countable $\mathcal{M}_{\mathbf{X}}$-decomposition of the set A, subordinate to the cover $\{U_\xi\}_{\xi\in\Xi}$. This means that $A_\vartheta \in \mathcal{M}_{\mathbf{X}}$, for each $\vartheta \in \Theta$, $\bigcup_{\vartheta\in\Theta} A_\vartheta = A$, $A_{\vartheta'} \cap A_{\vartheta''} = \phi$ for $\vartheta', \vartheta'' \in \Theta$, where $\vartheta' \neq \vartheta''$, and each set A_ϑ is enclosed in a coordinate neighbourhood $U_{\xi(\vartheta)}$, $U_{\xi(\vartheta)}$ selected in $\{U_\xi\}_{\xi\in\Xi}$.

Set

$$\mu_g(A) = \sum_{\substack{\vartheta\in\Theta \\ \xi=\xi(\vartheta)}} \int_{\psi_\xi(A_\vartheta)} \sqrt{h^{(\xi)}(y)} \cdot d\nu_n, \tag{4.1}$$

where ν_n is the Lebesgue measure on R^n, $y = \psi_\xi(x)$, $g_{ij}^{(\xi)}(y)$, $i, j = 1, 2, \ldots, n$, are the coordinates of the metric tensor g in the map (in the coordinate system) $(U_\xi, \psi_\xi(\cdot))$, and $h^{(\xi)}(y) = \det[g_{ij}^{(\xi)}(y)]$.

The value of $\mu_g(A)$, which may become ∞, does not depend on a particular choice of $\mathcal{M}_{\mathbf{X}}$-decomposition $\{A_\vartheta\}$, $(A_\vartheta \subseteq U_{\xi(\vartheta)})$, for A [12]. The function $\mu_g : \mathcal{M}_{\mathbf{X}} \ni A \to [0, \infty]$, defined in (4.1), is the measure on $\mathcal{M}_{\mathbf{X}}$. The measure μ_g is complete.

The C^1-vector field is given on the Riemannian manifold $\mathbf{X}$. It is assumed that the trajectories of the vector field f exist for $t \in R^+$. Thus, the solutions of the differential equation $dx/dt = f(x)$ define a differentiable flow φ_f on $\mathbf{X}$; $\mathcal{S}_f = (\mathbf{X}, R, \mathsf{U}, \varphi_f)$ is the differential dynamical system on $\mathbf{X}$, where $\mathbf{X} \times R^+ \subset \mathsf{U}$.

REMARK 2. For a compact subset A of the Riemann (connected and second countable) manifold $\mathbf{X}$, where the metric tensor field g is continuous,

$$\mu_g(A) < \infty .$$

If the closure of any ρ_g-bounded subset of $\mathbf{X}$ is a compact subset of $\mathbf{X}$ (Assumption 5), then ρ_g-bounded subsets of $\mathbf{X}$ are μ_g-bounded.

Proof. Let $\{(W_\zeta, \psi_\zeta(\cdot))\}_{\zeta \in \mathcal{Z}}$ be a C^1-atlas on $\mathbf{X}$, where with no loss in generality for the cover $\{W_\zeta\}_{\zeta \in \mathcal{Z}}$, each set $\operatorname{cl} W_\zeta$ is a compact subset of $\mathbf{X}$. There exists a refinement $\{V_\zeta\}_{\zeta \in \mathcal{Z}}$ of $\{W_\zeta\}_{\zeta \in \mathcal{Z}}$ such that each $\operatorname{cl} V_\zeta$, $\zeta \in \mathcal{Z}$, is a compact subset of W_ζ. Each $\psi_\zeta(\operatorname{cl} V_\zeta)$, $\zeta \in \mathcal{Z}$, is a compact subset of R^n.

Because A is a compact subset of $\mathbf{X}$, there exists a finite cover $\{V_{\zeta(1)}, \ldots, V_{\zeta(r)}\}$, $V_{\zeta(1)}, \ldots, V_{\zeta(r)} \in \{V_\zeta\}$ for A. Let $\{A_1, \ldots, A_r\}$ be the subordinate $\mathcal{M}_{\mathbf{X}}$-decomposition of A, where $A_1 = A \cap V_{\zeta(1)}, \ldots, A_r = (A \cap V_{\zeta(r)}) \setminus \bigcup_{i=1}^{r-1} A_i$. Each set $\operatorname{cl} A_i$ is a compact subset of the neighbourhood $W_{\zeta(i)}$, and hence $\psi_{\zeta(i)}(\operatorname{cl} A_i)$ is a compact subset of R^n.

Write ζ_i for $\zeta(i)$.

For each $i = 1, 2, \ldots, r$, $\overline{h}^{(\zeta_i)} = \sup_{x \in A_i} h^{(\zeta_i)}(\psi_{\zeta_i}(x)) < \infty$, (the continuous function $h^{(\zeta_i)}(\psi_{\zeta_i}(x))$ attains the supreme on A_i at a point in a compact subset $\operatorname{cl} A_i$ of W_{ζ_i}). Hence,

$$\mu_g(A) = \sum_{i=1}^{r} \int_{\psi_{\zeta_i}(A_i)} \sqrt{h^{(\zeta_i)}(y)} \cdot d\nu_n \le \sum_{i=1}^{r} \sqrt{\overline{h}^{(\zeta_i)}} \cdot \nu_n(\psi_{\zeta_i}(A_i)),$$

where $\nu_n(\psi_{\zeta_i}(A_i)) < \infty$ ($\psi_{\zeta_i}(A_i)$ is a subset of the compact set $\operatorname{cl} \psi_{\zeta_i}(A_i) = \psi_{\zeta_i}(\operatorname{cl} A_i) \subset R^n$). □

By the Remark 2, the Assumption 5 ensures that $\mu_g(A) < \infty$, for each ρ_g-bounded set $A \in \mathcal{M}_{\mathbf{X}}$, where $\mathbf{X}$ is a Riemann (connected, second countable) oriented C^1-manifold, and the metric tensor field is continuous.

REMARK 3. Consider the differential dynamical system $\mathcal{S}_f = (\mathbf{X}, R, \mathsf{U}, \varphi_f)$, $\mathbf{X} \times R^+ \subset \mathsf{U}$, where $\mathbf{X}$ is a finite-dimensional (connected and second countable) Riemann oriented C^2-manifold and φ_f is the flow defined by the solutions of the differential equation $dx/dt = f(x)$, where f is a complete$^+$ C^1-vector field on $\mathbf{X}$ (that is, the domain of each of the solutions of the differential equation $dx/dt = f(x)$ encloses R^+).

For each μ_g-measurable and ρ_g-bounded subset A of the manifold $\mathbf{X}$, the domain of the function $\widetilde{\mu}_{g,A} : t \to \mu_g(A_t)$ encloses an open interval in R containing R^+, and $\widetilde{\mu}_{g,A}$ takes values in R^+ and is C^1-differentiable.

Let $\{U_{\xi(1)}, \ldots, U_{\xi(r)}\}$ be a finite cover of the set A, where the sets $U_{\xi(i)}$ are selected as coordinate neighbourhoods of the maximal C^2-atlas $\{(U_\xi, \Psi_\xi(\cdot))\}_{\xi \in \Xi}$ defining orientation on $\mathbf{X}$ (the maximal C^2-atlas in the given orientation on $\mathbf{X}$). Assumption 5 ensures that a finite cover exists, for each ρ_g-bounded subset of the manifold $\mathbf{X}$.

Let $\{A_1, \ldots, A_r\}$ be the (standard) $\mathcal{M}_{\mathbf{X}}$-decomposition of the set A, subordinate to the cover $\{U_{\xi(1)}, \ldots, U_{\xi(r)}\}$, $(A_1 = A \cap U_{\xi(1)}, \ldots, A_r = (A \cap U_{\xi(r)}) \setminus \bigcup_{i=1}^{r-1} A_i$, where the sets $A_1, \ldots, A_r$ are disjoint).

The following expression is found for $d/dt\widetilde{\mu}_{g,A}(t = 0)$, $(\xi_i = \xi(i))$,

$$\frac{d}{dt}\widetilde{\mu}_{g,A}(t = 0) = \sum_{i=1}^{r} \int_{\psi_{\xi_i}(A_i)} \operatorname{div}(\sqrt{h^{(\xi_i)}(y)} \cdot f_{(\xi_i)}(y))\, d\nu_n\,, \tag{4.2}$$

where $f_{(\xi_i)} = (\psi_{\xi_i})^*(f_{|U_{\xi_i}})$, $\psi^*_{\xi_i}$ being the induced map from $T_x\mathbf{X}$ onto $T_{\psi_{\xi_i}(x)}R^n$.

Proof. Let A be a μ_g-measurable and ρ_g-bounded subset of $\mathbf{X}$. For each $t \in R^+$, the set $\mathrm{cl}(A_t) = G_t(\overline{A})$, where $\overline{A} = \mathrm{cl}\, A$, is a compact subset of $\mathbf{X}$ (Assumption 5 ensures that $\overline{A}$ is a compact subset of $\mathbf{X}$). The domain of the function $\widetilde{\mu}_{g,A}$ encloses an open interval in R containing R^+ (the C^1-vector field f is complete$^+$).

Let $\{(U_{\xi'}, \psi_{\xi'}(\cdot))\}_{\xi' \in \Xi'}$ be a C^2-atlas for $\mathbf{X}$, in given orientation on $\mathbf{X}$. With no loss in generality, $\mathrm{cl}\, U_{\xi'}$ is a compact subset of $\mathbf{X}$, for each $\xi' \in \Xi'$. There is a refinement $\{U'_{\xi'}\}_{\xi' \in \Xi'}$ of $\{U_{\xi'}\}_{\xi' \in \Xi'}$, such that each set $\mathrm{cl}\, U'_{\xi'}$ is a compact subset of $U_{\xi'}$, and there is a shrinkage $\{V_{\xi'}\}_{\xi' \in \Xi'}$ of $\{U'_{\xi'}\}_{\xi' \in \Xi'}$ such that each set $\mathrm{cl}\, V_{\xi'}$ is a compact subset of $U'_{\xi'}$.

By the defined construction of the covers

$$\inf_{\substack{x' \in V_{\xi'} \\ x'' \in U_{\xi'} \setminus U'_{\xi'}}} \rho_g(x', x'') > 0\,.$$

Let $\{V_{\xi'(1)}, \ldots, V_{\xi'(r)}\}$ be a finite cover for the set A, where each $V_{\xi'(i)}$ is selected in $\{V_{\xi'}\}_{\xi' \in \Xi'}$. $\{A_1, \ldots, A_r\}$ is the subordinate to the cover $\{V_{\xi'(1)}, \ldots, V_{\xi'(r)}\}$ (standard) disjoint $\mathcal{M}_{\mathbf{X}}$-decomposition of A.

The constructions of the covers ensure that there exists such an open interval Δt in R containing zero, that $G_t(A_i) \subset U'_{\xi'(i)}$ for $t \in \Delta t$ and each $i = 1, 2, \ldots, r$. This yields, $(\xi'_i = \xi'(i))$,

$$\mu_g(A_t) = \sum_{i=1}^{r} \int_{\psi_{\xi'_i}((A_i)_t)} \sqrt{h^{(\xi'_i)}(y)} \cdot d\nu_n, \qquad ((A_i)_t = G_t(A_i)) \tag{4.3}$$

for $t \in \Delta t$. Thus ([4]), $\mu_g(A_t)$ is a C^1-function on Δt.

Because the solutions of the differential equation $dx/dt = f(x)$ (the trajectories of $\mathcal{S}_f$) exists for all $t \geq 0$, the expression (4.3) remains valid for each $t_0 \in R^+$ and all t in an open neighbourhood Δt of t_0, where the cover $\{V_{\xi'(i)}\}$ is selected for $(A)_{t_0}$.

Finally, for any μ_g-measurable ρ_g-bounded subset A, the function $\widetilde{\mu}_{g,A}$ is C^1-differentiable on an open interval in R containing R^+, and ([4, 8]),

$$(4.4)\qquad \frac{d}{dt}\widetilde{\mu}_{g,A}(t=0) = \sum_{i=1}^{r} \frac{d}{dt} \int\limits_{\psi_{\xi_i'}((A_i)_t)} \sqrt{h^{(\xi_i')}(y)} \cdot d\nu_n|_{t=0}$$

$$= \sum_{i=1}^{r} \int\limits_{\psi_{\xi_i'}(A_i)} \operatorname{div}(\sqrt{h^{(\xi_i')}(y)} \cdot f_{(\xi_i')}(y))\, d\nu_n\,,$$

($\xi_i' = \xi'(i)$), where $\{A_i\}$ is the subordinate to the cover $\{V_{\xi'(i)}\}$ (standard) disjoint $\mathcal{M}_{\mathbf{X}}$-decomposition of the set A, and the value of $(d/dt)\widetilde{\mu}_{g,A}(t=0)$ in (4.4) does not depend on the particular choice of the finite cover $\{V_{\xi'(i)}\}$ for A, selected from the family of coordinate neighbourhoods of the oriented C^2-atlas on $\mathbf{X}$. □

In the coordinate system $(U_\xi, \psi_\xi(\cdot))$, $U_\xi \ni x \to y = \psi_\xi(x) \in R^n$, of the maximal C^2-atlas in a chosen (fixed) orientation on $\mathbf{X}$:

$$(4.5)\qquad \operatorname{div}(\sqrt{h^{(\xi)}(y)} \cdot f_{(\xi)}(y))$$

$$= \sqrt{h^{(\xi)}(y)} \cdot \left[(\operatorname{div} f_{(\xi)})_{|y} + \frac{1}{\sqrt{h^{(\xi)}(y)}} \cdot (\operatorname{grad} \sqrt{h^{(\xi)}} \cdot f_{(\xi)}(y))\right].$$

The expression $(\operatorname{div} f_{(\xi)})_{|y} + \dfrac{1}{\sqrt{h^{(\xi)}(y)}} \cdot \operatorname{grad}(\sqrt{h^{(\xi)}})_{|y} \cdot f_{(\xi)}(y)$ defines the divergence $\operatorname{div}_{\mathbf{X}} f$ of the vector field f, in terms of local coordinates on the oriented Riemann manifold $(\mathbf{X}, g)$ [1]. Thus,

$$\frac{d}{dt}\widetilde{\mu}_{g,A}(t=0) = \int\limits_A \operatorname{div}_{\mathbf{X}} f \cdot d\mu_g\,,$$

for a μ_g-measurable ρ_g-bounded subset A of $\mathbf{X}$.

The following Theorem 6 concerns the group μ_g-boundedness [15] of the trajectories of the system $\mathcal{S}_f$.

THEOREM 6. *x_0 is a point in the state-space $\mathbf{X}$ of the dynamical system $\mathcal{S}_f$, where f is a complete$^+$ C^1-vector field on C^2-(connected, second countable and complete) Riemann manifold $\mathbf{X}$. Assume that there exist constants $\lambda > 0$, $\delta > 0$ such that*

$$\operatorname{div}_{\mathbf{X}} f(x) < -\lambda$$

for all $x \in \mathbf{X} : \rho_g(x, x_0) > \delta$, ρ_g being the Riemann metric on $\mathbf{X}$.

Then the trajectories of the system $\mathcal{S}_f$ are group μ_g-bounded on R^+, for each *ρ_g-bounded set $A \in \mathcal{M}_{\mathbf{X}}$.*

Proof. Let $\overline{B}_\delta$ denote the closed ball $\{x \in \mathbf{X} : \rho_g(x, x_0) \le \delta\}$. $\overline{B}_\delta$ is a compact subset of $\mathbf{X}$.

For any ρ_g-bounded set $E \in \mathcal{M}_{\mathbf{X}}$,

$$(4.6)\qquad \frac{d}{dt}\widetilde{\mu}_{g,E}(t=0) \le \int\limits_{\overline{B}_\delta} |\operatorname{div}_{\mathbf{X}} f|\, d\mu_g + \int\limits_{E\setminus\overline{B}_\delta} \operatorname{div}_{\mathbf{X}} f \cdot d\mu_g$$

(4.6) [cont.]
$$\le \sup_{x\in\overline{B}_\delta} |\mathrm{div}_{\mathbf{X}} f| \cdot \mu_g(\overline{B}_\delta) + \int_{E\setminus\overline{B}_\delta} \mathrm{div}_{\mathbf{X}} f \cdot d\mu_g$$
$$\le \Big(\sup_{x\in\overline{B}_\delta} |\mathrm{div}_{\mathbf{X}} f| \cdot \mu_g(\overline{B}_\delta) + \lambda \cdot \mu_g(\overline{B}_\delta)\Big) - \lambda \cdot \mu_g(E).$$

From the estimate (4.6), $(d/dt)\widetilde{\mu}_{g,E}(t=0) < 0$, for each ρ_g-bounded subset $E \in \mathcal{M}_{\mathbf{X}}$ having the measure
$$\mu_g(E) > \frac{1}{\lambda} \cdot \sup_{x\in\overline{B}_\delta} |\mathrm{div}_{\mathbf{X}} f| \cdot \mu_g(\overline{B}_\delta) + \mu_g(\overline{B}_\delta).$$

Thus $\sup_{t\in R^+} \widetilde{\mu}_g(A_t) < \infty$, for each ρ_g-bounded set $A \in \mathcal{M}_{\mathbf{X}}$. □

f is a C^1-vector field on C^2-Riemann manifold $\mathbf{X}$, as in the Theorem 6.

THEOREM 7. *Assume that the trajectories of the system* $\mathcal{S}_f$ *are uniformly equi-bounded on* R^+ (Definition 4). *When for each* ρ_g*-bounded set* $B \in \mathcal{M}_{\mathbf{X}}$

$$\frac{d}{dt}\widetilde{\mu}_{g,B}(t=0) \le 0 \tag{4.7}$$

and

$$\frac{d}{dt}\widetilde{\mu}_{g,B}(t=0) = 0 \tag{4.8}$$

if, and only if, $\mu_g(B) = 0$, *then the trajectories of the system* $\mathcal{S}_f$ *are monotone asymptotically group* μ_g*-convergent as* $t\to\infty$, *for each* ρ_g*-bounded set* $A \in \mathcal{M}_{\mathbf{X}}$.

Proof. Let A be a ρ_g-bounded μ_g-measurable subset of $\mathbf{X}$, and let $\mathbf{V}$ be an open and ρ_g-bounded subset in $\mathbf{X}$ enclosing A. The image set $\widehat{\gamma}^+(\mathbf{V})$ is a ρ_g-bounded (the trajectories are uniformly equi-bounded) and μ_g-measurable subset of $\mathbf{X}$.

In order to prove that $\lim_{t\to\infty} \mu_g(A_t) = 0$, it suffices to show that the maximal strictly invariant subset $\Lambda_{\widehat{\gamma}^+(\mathbf{V})}$ for the system $\mathcal{S}^+_{f|\widehat{\gamma}^+(\mathbf{V})}$ is μ_g-measurable and $\mu_g(\Lambda_{\widehat{\gamma}^+(\mathbf{V})}) = 0$.

The μ_g-measurability of $\Lambda_{\widehat{\gamma}^+(\mathbf{V})}$ is proved in the same way as the μ-measurability of the maximal strictly invariant subset Λ for the system $\mathcal{S}^+$ has been proved in the part i of the proof of Theorem 3.

$\mu_g(\Lambda_{\widehat{\gamma}^+(\mathbf{V})}) = 0$ and monotone convergence follows immediately from (4.7) and (4.8). □

REMARK 4. If $\mathrm{div}_{\mathbf{X}} f(x) < 0$, μ_g — almost everywhere on $\mathbf{X}$, then the conditions (4.7) and (4.8) are satisfied,for each ρ_g-bounded set $B \in \mathcal{M}_{\mathbf{X}}$. □

The assumptions for the manifold $\mathbf{X}$ and a vector field f are the same as in the Theorem 6, where f is not necessarily complete$^+$.

COROLLARY. Let x^* be a (locally) asymptotically stable equilibrium point (a critical point) of the vector field f, $f(x^*) = 0$. Then ([13]),

$$\mathrm{div}_{\mathbf{X}} f(x^*) \le 0. \tag{4.9}$$

Proof. Let x^* be a (locally) asymptotically stable equilibrium point of the vector field f. Suppose that the relation (4.9) is not satisfied at x^*.

Let $\{(U_\xi, \Psi_\xi(\cdot))\}_{\xi\in\Xi}$ be (maximal) C^2-atlas on $\mathbf{X}$, in a chosen orientation on $\mathbf{X}$, and let $U_{\xi'}$ be a coordinate neighbourhood enclosing x^*. There is such an open neighbourhood $B_\varepsilon(x^*) = \{x \in U_{\xi'} : \rho_g(x, x^*) < \varepsilon\}$, $0 < \varepsilon < \infty$, of x^* in $U_{\xi'}$, that

$$\text{div}_{\mathbf{X}}\, f(x^*) > 0\,, \tag{4.10}$$

for all $x \in B_\varepsilon(x^*)$. For ε sufficiently small, x^* is the maximal invariant subset of the system $\mathcal{S}_f$ in $B_\varepsilon(x^*)$.

Because x^* is the stable equilibrium point of the system $\mathcal{S}_f$ (of the differential equation $(dx/dt) = f(x)$ on $\mathbf{X}$), then there exists such an open neighbourhood $B_\delta(x^*)$ of x^*, that

$$G_\tau(B_\delta(x^*)) \subseteq B_\varepsilon(x^*), \qquad \text{for all } \tau \in R^+\,. \tag{4.11}$$

As in the Corollary following the Theorem 3, where $B_\delta(x^*)$ is set for V and x^* is the maximal invariant subset for the system $\mathcal{S}^+_{f|\widehat{\gamma}^+(B_\delta(x^*))}$,

$$\lim_{\tau\to\infty} \mu_g(G_\tau(B_\delta(x^*))) = 0\,. \tag{4.12}$$

But from (4.10) and (4.11),

$$\frac{d}{dt}\widetilde{\mu}_{g,G_\tau(B_\delta(x^*))}(t=0) = \int\limits_{G_\tau(B_\delta(x^*))} \text{div}_{\mathbf{X}}\, f \cdot d\mu_g > 0$$

for each $\tau \in R^+$, which contradicts (4.12). □

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