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Steady-state plane Lamb's problem for a fluid-saturated poro-elastic medium

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IN THE PAPER the time harmonic plane Lamb's problem for a fluid-saturated porous elastic solid is investigated. The considerations are carried out on the basis of Biot's dynamical theory of consolidation. The problem is solved by means of the Fourier transformation technique. The solutions are derived in the form of improper integrals, which have been evaluated in a numerical way, described in the paper. In a limiting case of no pore fluid in the medium, the solutions obtained are shown to reduce to those known in the classical theory of elasticity. As an illustration, some results of numerical calculations performed for the material parameters corresponding to a water-filled coarse sand are presented.

1. Introduction

IN ENGINEERING practice we often deal with the dynamical interaction problems in which the motion of homogeneous half-space is induced by a structure vibrating at its free surface. Mathematically, these problems lead to mixed boundary value problems for which, as a rule, it is impossible to find closed analytical solutions. For this reason we usually apply approximate discrete methods. When constructing a discrete model of the problem, it is often necessary to solve the problem consisting in the determination of the dynamical response of the half-space to a load applied at a certain area of its surface. The latter problem, which may be regarded as one of the fundamental problems in the field of wave propagation phenomena, is known in the classical theory of elasticity as Lamb's problem. The solution to this problem for both harmonic time-dependent and impulsive loads as well as a method of evaluation of the obtained improper integrals were proposed by Lamb in 1904 (ACHENBACH [1], EWING *et al.* [4]). As concerns a fluid-saturated poro-elastic medium, the problem in question was treated by PAUL [8, 9]. In his first paper [8] PAUL considered the plane problem of propagation of disturbances excited by an impulsive line load applied at the free surface of the half-space. The second paper [9] dealt with the axisymmetric problem of deformation of a porous elastic half-space subjected to a suddenly applied uniform load within a circular area of its boundary. In both the papers considerations were confined to the case of non-dissipative medium (i.e. the internal friction between the skeleton and the pore fluid was neglected). The solutions for displacement and stress fields were obtained in the form of double improper integrals, resulting from the application of the integral transformation method. These integrals were evaluated by employing the contour integration method; in the paper [9] the displacements were determined with the help of the Cagniard–de Hoop technique. SEIMOV *et al.* [10] studied both plane and axisymmetric problems for dissipative two-phase poro-elastic solid. For these problems analytical expressions describing displacements of the medium were derived. For a particular case of non-dissipative medium and an impulsive line load, the method of numerical evaluation of the obtained improper integrals by means of contour integration was also presented.

In the present investigation the plane harmonic in time Lamb's problem for a fluid-filled porous elastic solid is discussed. The analysis is carried out on the basis of BIOT'S [2] dynamical theory of consolidation. In order to solve the problem at hand, the Fourier integral transformation method is applied. The improper integrals derived are evaluated in a numerical way, described in the paper. It is shown that, in a limiting case of no pore fluid, the results obtained agree with those for a purely elastic solid. The paper concludes with some results of numerical computations, performed for the material parameters pertaining to a water-saturated coarse sand.

2. Formulation of the problem

We consider the plane problem of motion of a fluid-saturated poroelastic solid, occupying the half-space $z \geq 0$ (Fig. 1). The motion of the medium is excited by a harmonic time-dependent load, applied at the free boundary of the half-space $z = 0$. Our aim is to determine dynamic displacements of the skeleton and the pore fluid due to the applied surface tractions $q_{zz}(x)\exp(i\omega t)$, $q_s(x)\exp(\omega t)$ and $q_{xz}(x)\exp(i\omega t)$, with ω being the angular frequency of oscillations.

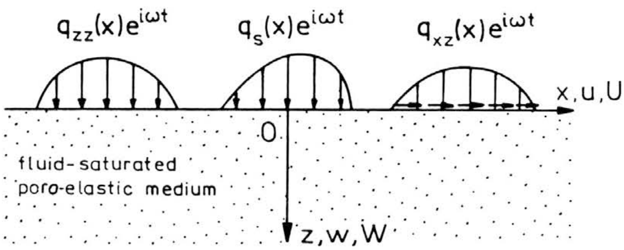

FIG. 1. Coordinate system and surface load components.

In the plane Oxz coordinate system the equations of motion of a fluid-saturated poro-elastic medium may be written, in accordance with BIOT [2], in the following form:

$$
\begin{aligned}
\nabla^2(P\operatorname{div}\mathbf{u} + Q\operatorname{div}\mathbf{U}) &= \frac{\partial^2}{\partial t^2}(\rho_{11}\operatorname{div}\mathbf{u} + \rho_{12}\operatorname{div}\mathbf{U}) + b\frac{\partial}{\partial t}\operatorname{div}(\mathbf{u}-\mathbf{U}),\\
\nabla^2(Q\operatorname{div}\mathbf{u} + R\operatorname{div}\mathbf{U}) &= \frac{\partial^2}{\partial t^2}(\rho_{12}\operatorname{div}\mathbf{u} + \rho_{22}\operatorname{div}\mathbf{U}) - b\frac{\partial}{\partial t}\operatorname{div}(\mathbf{u}-\mathbf{U}),\\
N\nabla^2\operatorname{rot}\mathbf{u} &= \frac{\partial^2}{\partial t^2}(\rho_{11}\operatorname{rot}\mathbf{u} + \rho_{12}\operatorname{rot}\mathbf{U}) + b\frac{\partial}{\partial t}\operatorname{rot}(\mathbf{u}-\mathbf{U}),\\
0 &= \frac{\partial^2}{\partial t^2}(\rho_{12}\operatorname{rot}\mathbf{u} + \rho_{22}\operatorname{rot}\mathbf{U}) - b\frac{\partial}{\partial t}\operatorname{rot}(\mathbf{u}-\mathbf{U}),
\end{aligned} \tag{2.1}
$$

where ∇^2 denotes the Laplace operator, t is time. In the latter equations $\mathbf{u} = [u, 0, w]^T$ and $\mathbf{U} = [U, 0, W]^T$ denote the displacement vectors of the skeleton and the pore fluid, respectively. N, A, Q and R are elastic moduli of the medium, $P = 2N + A$. The stress-strain relations have the form

$$
\begin{aligned}
\sigma_{ij} &= 2Ne_{ij} + \delta_{ij}(A\operatorname{div}\mathbf{u} + Q\operatorname{div}\mathbf{U}),\\
s &= Q\operatorname{div}\mathbf{u} + R\operatorname{div}\mathbf{U},
\end{aligned} \tag{2.2}
$$

with the strain tensor components in the skeleton

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where σ_{ij} denotes the stress tensor in the skeleton, s is the tension in the fluid, δ_{ij} is the Kronecker symbol, i and j take the values 1, 2, 3. The mass coefficients ρ_{11}, ρ_{12} and ρ_{22}, appearing in Eqs. (2.1), are related to the real densities ρ_s and ρ_w of the skeleton and the pore fluid, respectively, by means of the formulae

$$\begin{aligned} \rho_{11} &= (1-n)\rho_s + \rho_a, \\ \rho_{12} &= -\rho_a, \\ \rho_{22} &= n\rho_w + \rho_a, \end{aligned} \tag{2.3}$$

with n being the porosity of the solid matrix and ρ_a denoting the apparent mass coefficient, which expresses the effect of internal coupling of the motion of both the components of the medium. The damping parameter b is related to the commonly used filtration coefficient k_f; for the steady-state flow this parameter may be expressed by the formula

$$b = \frac{n^2 \rho_w g}{k_f},$$

where g is the gravitational acceleration. In a case of high frequency oscillations, the coefficient b should be corrected, for instance in a way proposed by BIOT [2].

Equations of motion (2.1) must be supplemented by the boudary conditions at the free surface $z = 0$. For the problem under consideration we express them in terms of the stress tensor components as follows:

$$\begin{aligned} \sigma_{zz}(x,0) &= -q_{zz}(x), \\ \sigma_{xz}(x,0) &= -q_{xz}(x), \\ s(x,0) &= -q_s(x), \end{aligned} \tag{2.4}$$

where for the sake of brevity we omitted the time factor $\exp(i\omega t)$. In the above equations the functions q_{zz}, q_{xz} and q_s denote the components of the surface load : $q_{zz}(x)$ is the load applied to the skeleton normally to the surface $z = 0$, $q_{xz}(x)$ is the load applied to the skeleton tangentially to the free surface, and, finally, $q_s(x)$ is the load applied to the pore fluid normally to the boundary $z = 0$. It seems to be somewhat artificial to split the total load, acting at the free surface, between both the components of the poro-elastic medium, because in practice it may be unrealistic to load one of the components without loading another one. But very often the partition of total stresses between the skeleton and the pore fluid cannot be determined a priori, for example, when the external pressure q_0 is applied to the surface of the two-phase medium by means of a rigid, impervious element (what is typical for engineering practice). In this case the boundary conditions at $z = 0$ should be written in the form

$$\begin{gathered} \sigma_{zz}(x,0) + s(x,0) = -q_0(x), \\ w(x,0) = W(x,0). \end{gathered}$$

The boundary value problem of this type was analysed in the paper by STAROSZCZYK [12], where the problem of harmonic vibrations of a rigid plate being in contact with the fluid-filled porous solid was considered. Generally, it seems that the approach consisting in splitting up the external loads is useful in all such cases in which we are interested in finding the distribution of the stresses between both components of the medium.

3. Solution of the steady-state Lamb's problem

In order to solve the boundary value problem, defined by Eqs. (2.1) and (2.4), it is expedient to carry out a decomposition of the displacement vectors of both the components of the medium by applying the Helmholtz resolution formula (ACHENBACH [1]) :

$$\tag{3.1} \begin{aligned} \mathbf{u} &= \operatorname{grad}\phi + \operatorname{rot}\boldsymbol{\psi}, \\ \mathbf{U} &= \operatorname{grad}\Phi + \operatorname{rot}\boldsymbol{\Psi}, \end{aligned}$$

where ϕ and Φ are the scalar potentials for the skeleton and the pore fluid; $\boldsymbol{\psi} = [0, \psi, 0]^T$ and $\boldsymbol{\Psi} = [0, \Psi, 0]^T$ are the vector potentials for both the constituents of the solid, respectively. The components of the displacement vectors are related to the potentials by means of

$$\tag{3.2} \begin{aligned} u &= \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, \qquad w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}, \\ U &= \frac{\partial\Phi}{\partial x} - \frac{\partial\Psi}{\partial z}, \qquad W = \frac{\partial\Phi}{\partial z} + \frac{\partial\Psi}{\partial x}. \end{aligned}$$

Substituting Eqs. (3.2) into Eqs. (2.2) we get the relevant components of the stress tensor in terms of potentials:

$$\tag{3.3} \begin{aligned} \sigma_{zz} &= 2N\left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\psi}{\partial x\partial z}\right) + A\nabla^2\phi + Q\nabla^2\Phi, \\ \sigma_{xz} &= N\left(2\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right), \\ s &= Q\nabla^2\phi + R\nabla^2\Phi. \end{aligned}$$

Insertion of (3.1) into the equations of motion (2.1) yields, for the variations harmonic in time, the following set of equations:

$$\tag{3.4} \begin{aligned} \tau_{11}\nabla^2\phi + \tau_{12}\nabla^2\Phi &= -k_c^2(\gamma_{11}\phi + \gamma_{12}\Phi) + k_c^2 if(\phi - \Phi), \\ \tau_{12}\nabla^2\phi + \tau_{22}\nabla^2\Phi &= -k_c^2(\gamma_{12}\phi + \gamma_{22}\Phi) - k_c^2 if(\phi - \Phi), \\ \nabla^2\psi &= -k_r^2(\gamma_{11}\psi + \gamma_{12}\Psi) + k_r^2 if(\psi - \Psi), \\ 0 &= -k_r^2(\gamma_{12}\psi + \gamma_{22}\Psi) - k_r^2 if(\psi - \Psi). \end{aligned}$$

In these equations, the following dimensionless elastic and dynamic coefficients appear:

$$\tag{3.5} \tau_{11} = \frac{P}{H}, \qquad \tau_{12} = \frac{Q}{H}, \qquad \tau_{22} = \frac{R}{H}, \qquad H = P + 2Q + R,$$

$$\tag{3.6} \gamma_{11} = \frac{\rho_{11}}{\rho}, \qquad \gamma_{12} = \frac{\rho_{12}}{\rho}, \qquad \gamma_{22} = \frac{\rho_{22}}{\rho}, \qquad \rho = \rho_{11} + 2\rho_{12} + \rho_{22},$$

with ρ being the total mass density of the skeleton-fluid aggregate. In Eqs. (3.4) also the dimensionless damping parameter f appears,

$$\tag{3.7} f = \frac{b}{\rho\omega}.$$

The parameters k_c and k_r are the wave numbers defined by

$$k_c = \frac{\omega}{V_c}, \qquad k_r = \frac{\omega}{V_r},$$

with V_c and V_r being the reference velocities of dilatational and rotational waves, respectively:

$$V_c = (H/\rho)^{\frac{1}{2}}, \qquad V_r = (N/\rho)^{\frac{1}{2}}. \tag{3.8}$$

The first two Equations (3.4) describe the propagation of dilatational waves, while the remaining two — that of rotational waves. It is known (BIOT [2]) that in the fluid-filled porous medium two dilatational and one shear wave may propagate. From the analysis of Eqs. (3.4) one can deduce that the potentials for the skeleton and the pore fluid are, for the case of time harmonic variations, coupled to each other by means of certain complex coefficients. Accordingly, we may write the general solutions of Eqs. (3.4) in the form

$$\begin{aligned} \phi &= \phi_1 + \phi_2, \\ \Phi &= \Phi_1 + \Phi_2 = \varepsilon_1\phi_1 + \varepsilon_2\phi_2, \\ \Psi &= \varepsilon_3\psi. \end{aligned} \tag{3.9}$$

Hereafter the subscripts 1 and 2 refer to the dilatational waves of the first and the second kind, respectively, and the subscript 3 refers to the shear wave. The coupling coefficients are given by

$$\begin{aligned} \varepsilon_j &= -(\tau_{11}\vartheta_j - \gamma_{11} + if)/(\tau_{12}\vartheta_j - \gamma_{12} - if), \qquad j = 1,2, \\ \varepsilon_3 &= -(\gamma_{12} + if)/(\gamma_{22} - if). \end{aligned} \tag{3.10}$$

The parameters ϑ_1 and ϑ_2 are the complex roots of the following quadratic equation:

$$\vartheta^2(\tau_{11}\tau_{22} - \tau_{12}^2) - \vartheta(\tau_{11}\gamma_{22} + \tau_{22}\gamma_{11} - 2\tau_{12}\gamma_{12} - if) + \gamma_{11}\gamma_{22} - \gamma_{12}^2 - if = 0. \tag{3.11}$$

The boundary value problem under consideration is solved by employing the exponential Fourier transformation over x, defined for a function $f(x, z)$ by the pair of equations:

$$f^*(\xi, z) = \int_{-\infty}^{\infty} f(x, z)e^{i\xi x}\, dx, \tag{3.12}$$

$$f(x, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(\xi, z)e^{-i\xi x}\, d\xi, \tag{3.13}$$

where ξ is the transformation parameter. The Fourier transforms of the displacements (3.2) are given by:

$$\begin{aligned} u^* &= -i\xi\phi^* - \frac{d\psi^*}{dz}, \qquad w^* = \frac{d\phi^*}{dz} - i\xi\psi^*, \\ U^* &= -i\xi\Phi^* - \frac{d\Psi^*}{dz}, \qquad W^* = \frac{d\Phi^*}{dz} - i\xi\Psi^*, \end{aligned} \tag{3.14}$$

and the transforms of the stress tensor components (3.3) have the form:

$$\begin{aligned}\sigma^*_{zz} &= H\left[\tau_{11}\frac{d^2\phi^*}{dz^2} - \xi^2(\tau_{11}-2\beta^2)\phi^* - 2i\beta^2\xi\frac{d\psi^*}{dz} + \tau_{12}\left(\frac{d^2\Phi^*}{dz^2} - \xi^2\Phi^*\right)\right],\\ \sigma^*_{xz} &= -N\left(2i\xi\frac{d\phi^*}{dz} + \frac{d^2\psi^*}{dz^2} + \xi^2\psi^*\right),\\ s^* &= H\left[\tau_{12}\left(\frac{d^2\phi^*}{dz^2} - \xi^2\phi^*\right) + \tau_{22}\left(\frac{d^2\Phi^*}{dz^2} - \xi^2\Phi^*\right)\right],\end{aligned} \tag{3.15}$$

with $\beta = (N/H)^{1/2}$.

In view of Eqs. (3.9), the appropriate expressions for the transformed potential functions are

$$\begin{aligned}\phi^*(\xi,z) &= A_1(\xi)e^{-\nu_1 z} + A_2(\xi)e^{-\nu_2 z},\\ \Phi^*(\xi,z) &= \varepsilon_1 A_1(\xi)e^{-\nu_1 z} + \varepsilon_2 A_2(\xi)e^{-\nu_2 z},\\ \psi^*(\xi,z) &= A_3(\xi)e^{-\nu_3 z},\\ \Psi^*(\xi,z) &= \varepsilon_3 A_3(\xi)e^{-\nu_3 z},\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\nu_k^2 &= \xi^2 - k_c^2\vartheta_k, \qquad k = 1,2,\\ \nu_3^2 &= \xi^2 - k_r^2\vartheta_3, \qquad \vartheta_3 = (\gamma_{11}\gamma_{22} - \gamma_{12}^2 - if)/(\gamma_{22} - if).\end{aligned} \tag{3.17}$$

The Sommerfeld radiation conditions require that ϕ, Φ, ψ, Ψ and their first derivatives with respect to x and z should tend to zero as $|x|$ and z approach infinity. Thus, the following inequalities must be satisfied:

$$\operatorname{Re}\nu_k \geq 0, \qquad k = 1,2,3.$$

On inserting the functions (3.16) into Eqs. (3.15) and then into the transformed boundary conditions (2.4), we arrive at the following set of three algebraic equations with unknown functions $A_1(\xi)$, $A_2(\xi)$ and $A_3(\xi)$

$$\begin{bmatrix}\eta_1 - 2\zeta^2, & \eta_2 - 2\zeta^2, & -2i\zeta\mu_3\\ -2i\zeta\mu_1, & -2i\zeta\mu_2, & 2\zeta^2 - \vartheta_3\\ \eta_3, & \eta_4, & 0\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\\ A_3\end{Bmatrix} = \frac{1}{Nk_r^2}\begin{Bmatrix}q^*_{zz}\\ q^*_{xz}\\ q^*_s\end{Bmatrix}. \tag{3.18}$$

In the above set of equations the following dimensionless parameters appear:

$$\begin{cases}\zeta = \xi/k_r, \quad \mu_k = \nu_k/k_r = (\zeta^2 - \beta^2\vartheta_k)^{1/2}, \quad \operatorname{Re}\mu_k \geq 0, \quad k = 1,2,\\ \mu_3 = \nu_3/k_r = (\zeta^2 - \vartheta_3)^{1/2}, \quad \operatorname{Re}\mu_3 \geq 0;\end{cases} \tag{3.19}$$

$$\begin{cases}\eta_1 = \vartheta_1(\tau_{11} + \varepsilon_1\tau_{12}), \qquad \eta_3 = \vartheta_1(\tau_{12} + \varepsilon_1\tau_{22}),\\ \eta_2 = \vartheta_2(\tau_{11} + \varepsilon_2\tau_{12}), \qquad \eta_4 = \vartheta_2(\tau_{12} + \varepsilon_2\tau_{22}).\end{cases} \tag{3.20}$$

By expanding the main determinant of the matrix of coefficients of Eqs. (3.18) we obtain the function

$$F(\zeta) = (\eta_4 - \eta_3)(2\zeta^2 - \vartheta_3)^2 + 4\zeta^2\mu_3(\eta_3\mu_2 - \eta_4\mu_1). \tag{3.21}$$

This function is called the Rayleigh function for the fluid-filled porous solid, since if we make it equal to zero, we get the dispersion equation, which determines the parameters of the surface waves propagating along the free boundary of the half-space filled with the two-phase medium.

Substitution of the functions $A_j(\zeta)$, obtained from Eqs. (3.18), into Eqs. (3.16) and then into Eqs. (3.14) yields the Fourier transforms of the displacements of the medium. By carrying out the inverse transformation (3.13) we arrive at the solution of the problem at hand in the form of improper integrals. For convenience, we write the derived displacement functions separately for each of the components of the loading function $q(x)$:

a) $q_{zz} \neq 0$, $q_{xz} = q_s = 0$

$$u = \frac{i}{2\pi N} \int_{-\infty}^{\infty} \zeta[(2\zeta^2 - \vartheta_3)(\eta_4 E_1 - \eta_3 E_2) + 2\mu_3(\eta_3\mu_2 - \eta_4\mu_1)E_3] \frac{q_{zz}^* e^{-i\zeta\widetilde{x}}}{F(\zeta)}\, d\zeta\,, \tag{3.22}$$

$$\begin{aligned} w = \frac{1}{2\pi N} \int_{-\infty}^{\infty} [(2\zeta^2 - \vartheta_3)&(\eta_4\mu_1 E_1 - \eta_3\mu_2 E_2) \\ &+ 2\zeta^2(\eta_3\mu_2 - \eta_4\mu_1)E_3] \frac{q_{zz}^* e^{-i\zeta\widetilde{x}}}{F(\zeta)}\, d\zeta\,; \end{aligned} \tag{3.23}$$

b) $q_{xz} \neq 0$, $q_{zz} = q_s = 0$

$$u = \frac{1}{2\pi N} \int_{-\infty}^{\infty} \mu_3[2\zeta^2(-\eta_4 E_1 + \eta_3 E_2) + (2\zeta^2 - \vartheta_3)(\eta_4 - \eta_3)E_3] \frac{q_{xz}^* e^{-i\zeta\widetilde{x}}}{F(\zeta)}\, d\zeta\,, \tag{3.24}$$

$$w = \frac{i}{2\pi N} \int_{-\infty}^{\infty} \zeta[2\mu_3(\eta_4\mu_1 E_1 - \eta_3\mu_2 E_2) - (2\zeta^2 - \vartheta_3)(\eta_4 - \eta_3)E_3] \frac{q_{xz}^* e^{-i\zeta\widetilde{x}}}{F(\zeta)}\, d\zeta\,; \tag{3.25}$$

c) $q_s \neq 0$, $q_{zz} = q_{xz} = 0$

$$\begin{aligned} u = \frac{i}{2\pi N} \int_{-\infty}^{\infty} \zeta\{[(2\zeta^2 - \vartheta_3)&(2\zeta^2 - \eta_2) - 4\zeta^2\mu_2\mu_3]E_1 \\ &-[(2\zeta^2 - \vartheta_3)(2\zeta^2 - \eta_1) - 4\zeta^2\mu_1\mu_3]E_2 \\ &-2\mu_3[\mu_1(2\zeta^2 - \eta_2) - \mu_2(2\zeta^2 - \eta_1)]E_3\} \frac{q_s^* e^{-i\zeta\widetilde{x}}}{F(\zeta)}\, d\zeta\,, \end{aligned} \tag{3.26}$$

$$\begin{aligned} w = \frac{1}{2\pi N} \int_{-\infty}^{\infty} \{\mu_1[(2\zeta^2 - \vartheta_3)&(2\zeta^2 - \eta_2) - 4\zeta^2\mu_2\mu_3]E_1 \\ &-\mu_2[(2\zeta^2 - \vartheta_3)(2\zeta^2 - \eta_1) - 4\zeta^2\mu_1\mu_3]E_2 \\ &-2\zeta^2[\mu_1(2\zeta^2 - \eta_2) - \mu_2(2\zeta^2 - \eta_1)]E_3\} \frac{q_s^* e^{-i\zeta\widetilde{x}}}{F(\zeta)}\, d\zeta\,; \end{aligned} \tag{3.27}$$

where $E_k = \exp(-\mu_k\widetilde{z})$, $k = 1, 2, 3$. In the above relations the dimensionless parameters appear

$$\widetilde{x} = k_r x\,, \qquad \widetilde{z} = k_r z\,.$$

The expressions for the pore fluid displacement functions U and W may be obtained from Eqs. (3.22)–(3.27) by replacing E_k by $\varepsilon_k E_k$ ($k = 1, 2, 3$), respectively, in the corresponding equations.

The solutions (3.22)–(3.27) are valid for any load function $q(x)\exp(i\omega t)$, for which the Fourier transform $q^*(\zeta)\exp(i\omega t)$ exists. The latter holds if the functions $q_{zz}(x)$,

$q_{xz}(x)$ and $q_s(x)$ satisfy Dirichlet's conditions and are absolutely integrable in the interval $(-\infty, \infty)$ (SNEDDON [11]).

Now let us consider a limiting case of no pore fluid in the medium. To this end we assume that elastic constants $Q = R = 0$, the real pore fluid density $\rho_w = 0$ (what implies $\rho_{12} = \rho_{22} = 0$) and the damping parameter $b = 0$. By substituting these constants into Eqs. (3.5)–(3.7) we get the following values of dimensionless parameters:

$$\begin{aligned} &\tau_{11} = 1, \quad \tau_{12} = \tau_{22} = 0, \\ &\gamma_{11} = 1, \quad \gamma_{12} = \gamma_{22} = 0, \\ &f = 0. \end{aligned} \tag{3.28}$$

Moreover, from Eq. (3.5) it follows that $H = 2N + A$ and, by virtue of (3.8), the reference velocity of dilatational wave $V_c = [(2N + A)/\rho]^{1/2}$ is the same as for the case of purely elastic solid, provided that N and A are Lame's constants. Accordingly, the parameter $\beta = [N/(2N + A)]^{1/2}$ is the ratio between the velocities of ideally elastic shear and dilatational waves. On putting Eqs. (3.28) into the set of Eqs. (3.4) we see that the second and fourth equations disappear. Thus, the set (3.4) reduces to two equations with two unknown functions ϕ and ψ — the potential functions Φ and Ψ for the pore fluid become indeterminate. Therefore, the relations (3.9) and (3.10) can no longer be applied. From the dispersion equation (3.11), which describes the parameters of two dilatational waves propagating in the two-phase medium, it now follows that $\vartheta_1 = 1$ and ϑ_2 is indeterminate. So, the formulae obtained cannot be applied directly in the case of single-phase medium. In order to avoid this disadvantage it is useful to suppose in further analysis that both dilatational waves are the same, i.e. to assume that also $\vartheta_2 = 1$. In addition, from the last of Eq. (3.17) it is seen that $\vartheta_3 = 1$, too. Accordingly, we can write

$$\vartheta_1 = \vartheta_2 = \vartheta_3 = 1. \tag{3.29}$$

Substituting the above relations into Eqs. (3.19) we can see that now the functions μ_k $(k = 1, 2, 3)$ are real-valued and may be expressed as follows:

$$\begin{aligned} \mu_1 = \mu_2 &= (\zeta^2 - \beta^2)^{1/2} = \alpha_1, \\ \mu_3 &= (\zeta^2 - 1)^{1/2} = \alpha_2. \end{aligned} \tag{3.30}$$

The latter relations yield the following form of the function (3.21):

$$F(\zeta) = (\eta_4 - \eta_3)[(2\zeta^2 - 1)^2 - 4\zeta^2(\zeta^2 - \beta^2)^{1/2}(\zeta^2 - 1)^{1/2}] = (\eta_4 - \eta_3)F_e(\zeta), \tag{3.31}$$

where $F_e(\zeta)$ is the Rayleigh function for an ideally elastic medium (ACHENBACH [1], EWING *et al.* [4]). By insertion of Eqs. (3.29)–(3.31) into Eqs. (3.22)–(3.25) we arrive at the following expressions for the displacement functions:

a) $q_{zz} \neq 0$, $q_{xz} = 0$

$$u = \frac{i}{2\pi N} \int_{-\infty}^{\infty} \zeta[(2\zeta^2 - 1)\exp(-\alpha_1 \tilde{z}) - 2\alpha_1\alpha_2 \exp(-\alpha_2 \tilde{z})] \frac{q^*_{zz} \exp(-i\zeta\tilde{x})}{F_e(\zeta)}\, d\zeta, \tag{3.32}$$

$$w = \frac{1}{2\pi N} \int_{-\infty}^{\infty} \alpha_1[(2\zeta^2 - 1)\exp(-\alpha_1 \tilde{z}) - 2\zeta^2 \exp(-\alpha_2 \tilde{z})] \frac{q^*_{zz} \exp(-i\zeta\tilde{x})}{F_e(\zeta)}\, d\zeta; \tag{3.33}$$

b) $q_{xz} \neq 0$, $q_{zz} = 0$

$$(3.34) \qquad u = \frac{1}{2\pi N} \int_{-\infty}^{\infty} \alpha_2[-2\zeta^2 \exp(-\alpha_1 \tilde{z}) + (2\zeta^2 - 1)\exp(-\alpha_2 \tilde{z})] \frac{q^*_{xz} \exp(-i\zeta\tilde{x})}{F_e(\zeta)} \, d\zeta ,$$

$$(3.35) \qquad w = \frac{i}{2\pi N} \int_{-\infty}^{\infty} \zeta[2\alpha_1\alpha_2 \exp(-\alpha_1 \tilde{z}) - (2\zeta^2 - 1)\exp(-\alpha_2 \tilde{z})] \frac{q^*_{xz} \exp(-i\zeta\tilde{x})}{F_e(\zeta)} \, d\zeta .$$

One can check that for a particular case of concentrated force Q_0, normal to the boundary $z = 0$ at $x = 0$, the integrals (3.32) and (3.33) coincide with those known in the classical theory of elasticity (ACHENBACH [1]). To prove this it suffices to assume $q_{zz}(x) = Q_0\delta(x)$, with $\delta(x)$ being the Dirac delta function, and to remember that, in view of Eq. (3.19), $\zeta = \xi/k_r$ and $\mu_k = \nu_k/k_r$. Similarly, it can also be shown that for a case of the load applied tangentially to the free surface $z = 0$ the integrals (3.34) and (3.35) are equivalent to the solutions, which can be easily derived on the basis of the classical theory of elasticity. Thus, we can ascertain that in the limiting case of no pore fluid the derived solutions (3.22)–(3.25) for the fluid-saturated poro-elastic medium agree with those for the purely elastic solid.

4. Numerical calculations

The integrands of the derived solutions (3.22)–(3.27) of the problem discussed are multi-valued functions of complex variable. Because of their complicated form it is not possible to evaluate the improper integrals (3.22)–(3.27) analytically. Therefore, it remains to compute them in an approximate way, by the use of numerical integration. In this case there are, generally, two possible approaches. The first one consists in carrying out the contour integration over the complex ζ-plane around a closed path. Before the Cauchy integral theorem is applied, the integrands must be made uniform functions by introducing suitable branch cuts in the complex plane. The advantage of this rather complicated method of integration is that it enables us to obtain separate information about each of the waves propagating in the half-space (EWING *et al.* [4]).

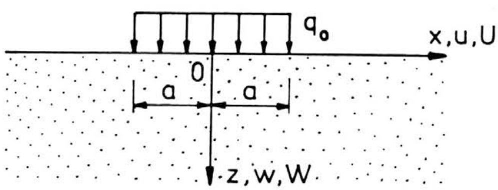

FIG. 2. Surface load in a form of strip function.

In the present investigation we apply the second approach that consists in carrying out the numerical integration along the real axis $\mathrm{Im}\,\zeta = 0$. In order to present this method, we confine our attention to the particular case of the surface load in the form of a strip function of constant pressure q_0, acting over a segment $-a \le x \le a$ and applied normally to the boundary $z = 0$ (Fig. 2). Accordingly, we write the boundary condition at the free surface as the function

$$q_{zz}(x) = \begin{cases} q_0, & |x| \le a, \\ 0, & |x| > a, \end{cases} \tag{4.1}$$

for which the Fourier transform (3.12) is

$$q^*_{zz}(\zeta) = \frac{2q_0 \sin\zeta\widetilde{a}}{k_r\zeta}, \tag{4.2}$$

with $\widetilde{a} = k_r a$.

By substituting Eq. (4.2) into Eq. (3.23) we get the vertical displacement of the skeleton at the point $x = 0$ and $z = 0$ in the form

$$\widetilde{w}(0,0) = -\frac{2q_0}{\pi N}\int\limits_0^\infty \vartheta_3(\eta_4\mu_1 - \eta_3\mu_2)\frac{\sin\zeta\widetilde{a}}{\zeta F(\zeta)}\,d\zeta = \int\limits_0^\infty I(\zeta)\,d\zeta, \tag{4.3}$$

with $\widetilde{w} = k_r w$ being the nondimensional displacement. The integrand of Eq. (4.3) has in the half-plane $\mathrm{Re}\,\zeta \ge 0$ four singular points: three branch points ζ_{0k} $(k = 1, 2, 3)$,

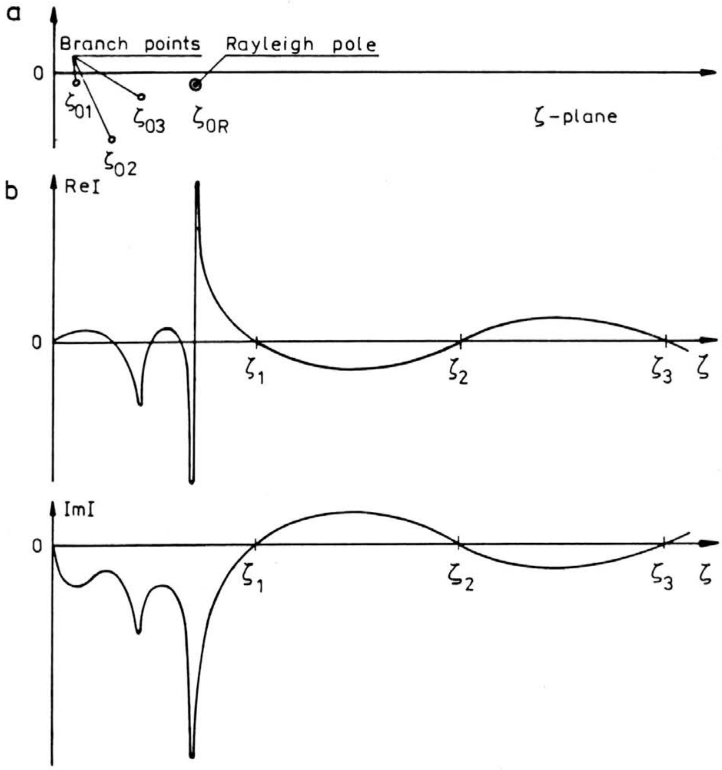

FIG. 3. a. Distribution of the singular points of the integrand (4.3) in the half-plane $\mathrm{Re}\,\zeta \ge 0$. b. Variation of the integrand (4.3) (real and imaginary parts) along the half-axis $\mathrm{Re}\,\zeta \ge 0$, $\mathrm{Im}\,\zeta = 0$.

introduced by the roots μ_k (3.19), and one simple pole ζ_{0R} being the zero of the Rayleigh function $F(\zeta)$, given by (3.21). The distribution of these points in the complex half-plane $\mathrm{Re}\,\zeta \geq 0$ is shown in Fig. 3a. In the same figure we sketch the variation of the integrand I of the function (4.3) with ζ along the positive real half-axis $\mathrm{Im}\,\zeta = 0$; $\zeta_1, \zeta_2, \zeta_3, \ldots$ are the successive zeros of the function $\sin \zeta\widetilde{a}$. It is seen that in the first subinterval $0 \leq \mathrm{Re}\,\zeta \leq \zeta_1$ the integrand considered is a very irregular function. This is due to the location of the singular points very close to the path of integration. Especially distinct is the influence of the Rayleigh pole ζ_{0R}; in the neighbourhood of this point the integrand can assume very large values (in the limiting case of nondissipative medium — infinite ones). Because of this strong irregularity in the first subinterval, a special numerical technique should be applied in order to ensure satisfactory accuracy of evaluation of the integrals discussed. The application of the methods which use interpolation formulae (e.g. KRYLOV and SKOBLYA [5]) has met only moderate success, because in this case it is difficult to estimate the error of computations. It was found out that more effective is to employ the iterative Romberg's algorithm combined with the Richardson's extrapolation scheme (DAHLQUIST and BJORCK [3]), because this method allows us to attain the desired accuracy of calculations in a simple manner, namely through the densification of nodal points in a region of strong irregularity of the integrand. In the interval $\mathrm{Re}\,\zeta \geq \zeta_1$ the integrated function varies much more regularly and familiar numerical quadrature formulae (e.g. those of Cotes or Gauss–Legendre) may be used successfully. For $\mathrm{Re}\,\zeta \geq \zeta_1$ the integrand oscillates around zero in such a way that both the real and imaginary parts of the integrals over the successive subintervals $\zeta_k \leq \mathrm{Re}\,\zeta \leq \zeta_{k+1}$ tend monotonously to zero in their absolute values. In addition, they alternate in sign for each two neighbouring half-cycles. Thus, the values of the integrals may be treated as terms of a certain convergent alternating series. In the case of slow convergence of this series it is very effective to employ either a procedure of multiplied averaging of the partial sums of the series (DAHLQUIST and BJORCK [3]) or to carry out Euler's transformation of the series (LONGMAN [6]).

For the general case $x \neq 0$ and $z \neq 0$ and for the functions (3.22) and (3.24)–(3.27) the behaviour of the integrands is more complicated. However, the method of computations described above is still useful, since the most dominant influence on the integrands of Eqs. (3.22)–(3.27) is exerted by the function $F(\zeta)$, which occurs in the denominators of all the integrals considered.

In accordance with the analytical solutions (3.22)–(3.27) of the problem discussed, some numerical calculations were performed for water-saturated coarse sand. As an example, we present the results obtained for the case of surface load in the form of the strip function (4.1), applied to the skeleton normally to the free surface $z = 0$. By substituting Eq. (4.2) into Eq. (3.22) and (3.23) we can write the skeleton displacement functions u and w in the following dimensionless form:

$$\text{(4.4)} \qquad \frac{\pi N k_r u}{2q_0} = \int\limits_0^\infty K(\zeta)\, d\zeta = |\overline{u}| \exp(i\delta_u)\,,$$

$$\text{(4.5)} \qquad \frac{\pi N k_r w}{2q_0} = \int\limits_0^\infty L(\zeta)\, d\zeta = |\overline{w}| \exp(i\delta_w)\,,$$

where $K(\zeta)$ and $L(\zeta)$ are the functions of the dimensionless dynamic and elastic parameters (3.5)–(3.7) and of the dimensionless variables $\widetilde{x} = k_r x$, $\widetilde{z} = k_r z$, $\widetilde{a} = k_r a$. The quantities $|\overline{u}|$ and $|\overline{w}|$ are the complex moduli of the dimensionless displacements, δ_u and

δ_w denote complex arguments (phase angles) of the horizontal and vertical displacements, respectively. Relations similar to Eqs. (4.4) and (4.5) may be written for the pore fluid displacements U and W. In the numerical computations the following data were used:

$$N = 3.75 \times 10^8\,\mathrm{Pa}, \qquad A = 2.82 \times 10^9\,\mathrm{Pa}, \qquad Q = 1.38 \times 10^9\,\mathrm{Pa},$$
$$R = 9.2 \times 10^8\,\mathrm{Pa}, \qquad n = 0.40, \qquad k_f = 0.01\,\mathrm{m/s},$$
$$\rho_s = 2.65 \times 10^3\,\mathrm{kg/m^3}, \qquad \rho_w = 10^3\,\mathrm{kg/m^3}, \qquad \omega = 4.341\,\mathrm{s^{-1}}.$$

The latter data correspond to the following dimensionless parameters:

$$\tau_{11} = 0.4924, \qquad \tau_{12} = 0.1903, \qquad \tau_{22} = 0.1269,$$
$$\gamma_{11} = 0.7990, \qquad \gamma_{12} = 0.0, \qquad \gamma_{22} = 0.2010, \qquad f = 18.163.$$

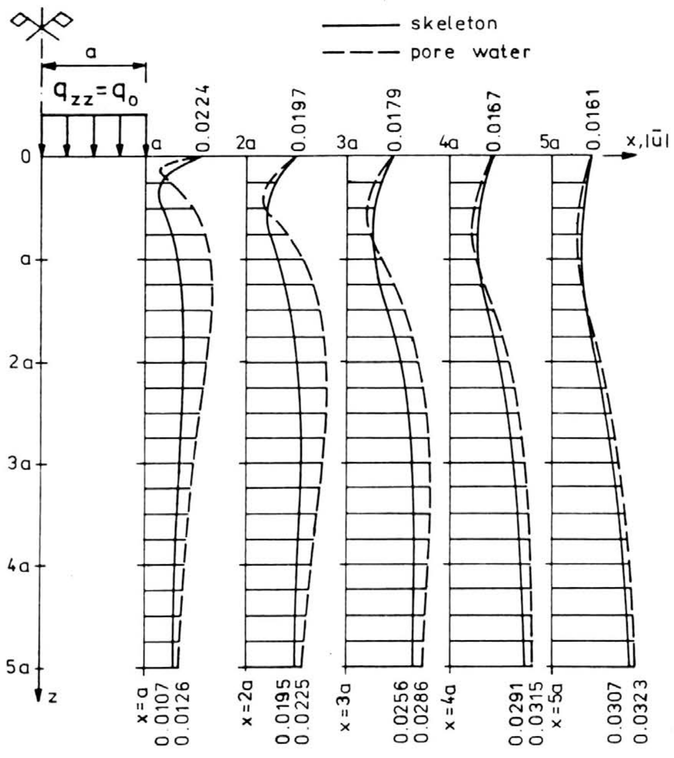

FIG. 4. Amplitudes of dimensionless horizontal displacements (coarse sand, $k_r a = 0.1$).

Figures 4 and 5 illustrate the variation of the amplitudes of the dimensionless displacements (4.4) and (4.5) of the skeleton and the pore water for the case of dimensionless frequency parameter $\tilde{a} = k_r a = 0.1$. It is seen that the displacements of both the components of the medium differ significantly only in the relatively small region in the vicinity of the zone of excitation. For the data taken in the computations we may consider the displacements mentioned above to be approximately the same at the distances greater than about $2a - 3a$ from the origin of the coordinate system (the latter correspond to about 1/30–1/20 of the length of rotational wave). So, in many applications we may treat

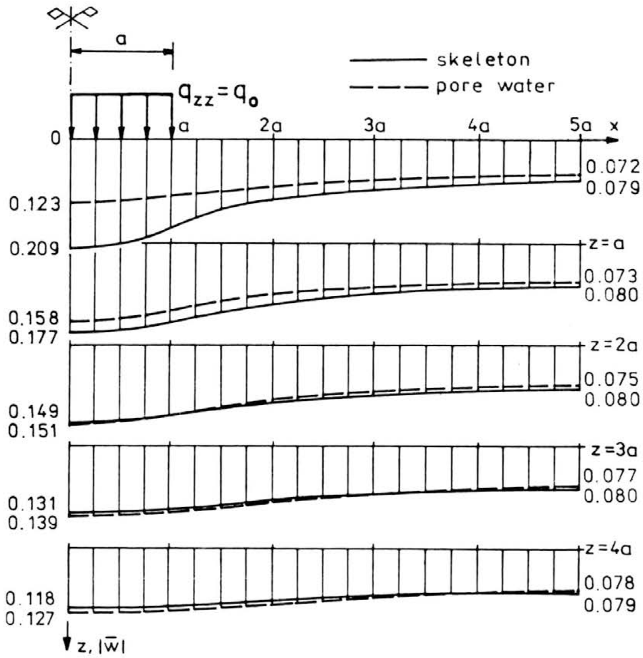

FIG. 5. Amplitudes of dimensionless vertical displacements (coarse sand, $k_r a = 0.1$).

the medium as a single-phase one; especially when we are interested in wave phenomena occurring far away from the sources of disturbances. Such an approach forms a framework of some practical methods — for instance the boundary layer theory, formulated by MEI and FODA [7].

5. Conclusions

In the paper the steady-state, time harmonic plane Lamb's problem for the Biot's medium is considered. The analytical solution to the problem at hand is derived in the form of improper integrals of complex variables. It is shown that for a limiting case of no pore fluid in the solid the solutions obtained reduce to those known in the theory of elasticity. Because of a complicated form of the integrals derived, their evaluation is possible only in an approximate way — but the accuracy of calculations may be easily controlled. Numerical analysis shows that the displacements of the skeleton and the pore fluid differ significantly only in the relatively small domain close to the excited zone.

The analytical solutions obtained are useful in the construction of discrete models of the steady-state structure-soil interaction problems, in which we deal with poro-elastic half-space. Since the solutions derived fulfil the radiation conditions for regular semi-infinite region, it is sufficient then to carry out the discretization only in the contact zone between the vibrating structure and the fluid-saturated subsoil. This enables us to reduce significantly the number of discrete points of a numerical model of the problem solved, as compared to the case of application of such discrete methods as FEM or FDM.

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